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ESTIMATES FOR ELECTRIC FIELDS BLOWN UP BETWEEN CLOSELY ADJACENT CONDUCTORS WITH ARBITRARY SHAPE*

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Abstract. It may be well known in practice that high stress concentrations occur in fiber-reinforced composites. There have been several works by analysis to estimate for the stresses between closed spaced fibers. However, the known results on stiff fibers have until now been restricted to the particular case of circular cross-sections. Thus, we extend the blow-up results on the stresses specialized only for disks to the general case of arbitrary shapes. Moreover, we prove that the blow-up rate of the general case is exactly the same as that of disks. Nevertheless, from the viewpoint of methodology, the technique we use is significantly different from the previous one restricted to the case of disks. Referring to antiplane shear problems, these works are reduced to the gradient estimates for the solution to the conductivity problem containing two closely spaced conductors which represent the cross-sections of fibers. We establish a novel representation for the solution on conductors by a probability function. Based on this, the general blow-up results are derived by a simpler method.

Key words. gradient estimates, blow-up, arbitrary shape, conductivity problems, stresses, composite materials

AMS subject classifications. 35J25, 73C40

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1. Introduction. This paper is concerned with high stress concentrations between closely spaced stiff fibers in an infinite matrix. According to Budiansky and Carrier [8], unexpectedly low strengths in longitudinal shear have been reported for brittle-matrix, fiber-reinforced composites, and it has been suggested that this might be explained by high stress concentrations between neighboring fibers (see also [5, 9]). However, according to Keller [12], it is difficult to calculate numerically the stresses in a narrow region because the stresses are much larger than elsewhere. Several approaches by analysis have been developed, but the blow-up results on the stresses are restricted to the particular case where fibers have circular cross-sections. Until now there has not been any established result associated with a large class of shapes. This paper presents the blow-up result for a class of shapes which is general enough. Moreover, the blow-up rate is exactly the same as the one for disks.

We consider two parallel elastic fibers embedded in an infinite elastic matrix. We suppose that D_1 and D_2 are very closely spaced inclusions in $\mathbb{R}^2$ which are ϵ apart, representing the cross-sections of the fibers, and the shear moduli of the inclusions are constants a_1 and a_2, different from the constant outside shear modulus 1. Referring to a problem of antiplane shear, we get the following conductivity equation for a given harmonic function H in $\mathbb{R}^2$:

$$(1.1)\qquad \begin{cases} \nabla \cdot \left\{ \left(1 + \sum_{i=1,2} (a_i - 1)\chi(D_i)\right) \nabla u \right\} = 0, \\ u(x) - H(x) = O(|x|^{-1}) \quad \text{as } |x| \to \infty, \end{cases}$$

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where the function u represents the out-of-plane elastic displacement. For applications to the composite materials, our work focuses on the stresses, represented by ∇u, particularly in the case when ∇H is a uniform field, i.e., $H(x) = A \cdot x$ for some constant vector A. The question of interest is to establish the optimal estimate on $|\nabla u|$ as the separation distance ϵ approaches 0.

We give a brief description of remarkable works by analysis on gradient estimates for solutions. For finite and strictly positive shear moduli (or conductivities) a_1 and a_2, it has been shown by Bonnetier and Vogelius in [7] that $|\nabla u|$ remains bounded for circular touching inclusions with comparable radii. Li and Vogelius derived in [15] a uniform upper bound on $|\nabla u|$ that is independent of the distance ϵ between D_1 and D_2, assuming that the moduli a_1 and a_2 stay away from 0 and ∞. It may be noted that this result of [15] holds for an arbitrary number of inclusions with arbitrary shape and is not restricted to two-dimensional space. Moreover, this result has been extended to elliptic systems by Li and Nirenberg in [14].

However, to explain high stresses occurring between stiff fibers, we should pay attention to the case of the extreme valued shear moduli. For two circular inclusions with $a_1 = a_2 = \infty$ or $a_1 = a_2 = 0$, it has been shown by Ammari, Kang, and Lim [2] and Ammari et al. [4] (see also [8]) that $|\nabla u|$ blows up as the distance ϵ approaches 0 for a special uniform field ∇H. Moreover, the optimal rate of blow-up is $\epsilon^{-1/2}$. These results on blow-up are specialized only for the case of circular inclusions. Thus there has been a strong need for a result that is not only associated with a large class of shapes but also has the same blow-up rate as circular inclusions.

In this paper we present the desirable result: for two inclusions D_1 and D_2 whose shapes are arbitrary enough, $|\nabla u|$ blows up as the distance ϵ approaches 0 for a special uniform field ∇H, and the blow-up rate is exactly $\epsilon^{-1/2}$, which is the known rate in the circular cases.

We now proceed to state the main results of this paper. To do so we need to make our notation and assumption more precise. Let D_{right} be a bounded domain in $\mathbb{R}^+ \times \mathbb{R}$ that is strictly convex at the unique left endpoint $(0, 0)$ of this domain, and let D_{left} be a bounded domain in $\mathbb{R}^- \times \mathbb{R}$ that has a right endpoint at $(0, 0)$ and a C^2 boundary. In addition, we assume that $\varphi : \mathbb{C}\backslash B_1(0) \to \mathbb{R}^2\backslash D_{\text{right}}$ is a conformal mapping such that $\varphi \in C^2(\mathbb{C}\backslash B_1(0))$ and $\varphi'(z) \neq 0$ for $z \in \partial B_1(0)$ (refer to the Riemann mapping theorem in [1]). We shall not distinguish between $\mathbb{R}^2$ and $\mathbb{C}$ in this paper. Let the domain D_1 and D_2 be as follows:

$$D_1 = D_{\text{right}} + \frac{1}{2}\epsilon \quad \text{and} \quad D_2 = D_{\text{left}} - \frac{1}{2}\epsilon.$$

To consider the case when $a_1 = a_2 = \infty$, given any harmonic function H in $\mathbb{R}^2$, (1.1) is rewritten in the following form:

$$(1.2) \qquad \begin{cases} \triangle u = 0 \quad \text{in } \mathbb{R}^2\backslash\overline{(D_1 \cup D_2)}, \\ u(x) - H(x) = O(|x|^{-1}) \quad \text{as } |x| \to \infty, \\ u|_{\partial D_i} = C_i \text{ (constant), and} \\ \int_{\partial D_i} \partial_\nu u \, ds = 0 \quad \text{for } i = 1, 2. \end{cases}$$

The solution u can also be interpreted physically as the voltage potential outside uncharged conductors D_1 and D_2 under the action of applied electric field ∇H (see [10]).

THEOREM 1.1. *Assume the conditions above. Let u be the solution to* (1.2) *for $H(x_1, x_2) = x_1$. If the distance ϵ is sufficiently small, then there exists a strictly* *positive constant C_* independent of ϵ such that*

$$|u|_{\partial D_1} - u|_{\partial D_2}| \geq C_* \sqrt{\epsilon} \tag{1.3}$$

and, owing to the mean value theorem,

$$\max_{-\frac{\epsilon}{2} < x_1 < \frac{\epsilon}{2}} |\nabla u(x_1, 0)| \geq C_* \left(\frac{1}{\sqrt{\epsilon}} \right). \tag{1.4}$$

As mentioned earlier, we extend the result known only for disks to the general case of inclusions with arbitrary shape. Nevertheless, the technique we use is significantly different from the previous one. In the case of disks, the authors took advantage of Kelvin transform and properties of layer potential which are specialized only for circles in [2, 4, 8]. It is difficult to apply this method to other shaped inclusions even though they are ellipses. Thus, we need to provide a new method for the general case. To do so, we establish a new and easy representation for the difference $u|_{\partial D_1} - u|_{\partial D_2}$ in Lemma 2.3 by a probability function $\partial_\nu w$. Based on this, the blow-up result (1.4) can be derived by a method simpler than the previous ones such as asymptotic expansions related to discontinuous conductivity in [2, 3, 4, 6, 11, 16]. On the other hand, applying the new representation to the recent result of Ammari et al. in [4], we can derive the optimal upper bound of $|\nabla u|$ in Theorem 1.2.

THEOREM 1.2. *Let H and u be the same as in Theorem* 1.1. *In addition, we assume that D_{left} also has a conformal mapping $\psi : \mathbb{C} \backslash B_1(0) \to \mathbb{R}^2 \backslash D_{\text{left}}$ with the same regularity conditions as φ. If the distance ϵ is sufficiently small, then we have constants C_1^* and C_2^* independent of ϵ such that*

$$|u|_{\partial D_1} - u|_{\partial D_2}| \leq C_1^* \sqrt{\epsilon} \tag{1.5}$$

and

$$\|\nabla u\|_{L^\infty(\mathbb{R}^2 \backslash (D_1 \cup D_2))} \leq C_2^* \left(\frac{1}{\sqrt{\epsilon}} \right). \tag{1.6}$$

To summarize the two theorems, Theorem 1.1 provides that $|\nabla u|$ blows up as the distance $\epsilon \to 0$ for a special uniform field and the blow-up rate is not less than $\epsilon^{-1/2}$. It follows from the bound (1.6) in Theorem 1.2 that the rate is exactly $\epsilon^{-1/2}$.

We now consider the case when $a_1 = a_2 = 0$. It is probably relevant to consider the case when $a_1 = a_2 = 0$, because the fibers are there for reinforcement. However, the solution to this case can be interpreted physically as the voltage potential outside nonconductors D_1 and D_2 under the action of applied electric field $\nabla \widetilde{H}$ (see [13]). For any given harmonic function $\widetilde{H}$ in $\mathbb{R}^2$, let $\widetilde{u}$ be the unique solution to the following Neumann problem:

$$\begin{cases} \triangle \widetilde{u} = 0 \quad \text{in } \mathbb{R}^2 \backslash \overline{(D_1 \cup D_2)}, \\ \widetilde{u}(x) - \widetilde{H}(x) = O(|x|^{-1}) \quad \text{as } |x| \to \infty, \\ \partial_\nu \widetilde{u} = 0 \quad \text{on } \partial D_i \quad \text{for } i = 1, 2, \end{cases} \tag{1.7}$$

where $\partial_\nu \widetilde{u}$ is the normal derivative of $\widetilde{u}$.

THEOREM 1.3. *Assume that D_{left} is the same as in Theorem* 1.2, *and that C_* and C_2^* are the constants used in Theorems* 1.1 *and* 1.2, *respectively. Let $\widetilde{u}$ be the* *solution to* (1.7) *for* $\widetilde{H}(x_1, x_2) = x_2$. *If the distance* ϵ *is small enough, then we have the estimates*

$$\max_{-\frac{\epsilon}{2}<x_1<\frac{\epsilon}{2}} |\nabla \widetilde{u}(x_1, 0)| \geq C_* \left(\frac{1}{\sqrt{\epsilon}}\right) \tag{1.8}$$

and

$$\|\nabla \widetilde{u}\|_{L^\infty(\mathbb{R}^2 \setminus (D_1 \cup D_2))} \leq C_2^* \left(\frac{1}{\sqrt{\epsilon}}\right). \tag{1.9}$$

Proof. Let u be the voltage potential for $H = x_1$ in Theorem 1.1. By Lemma 2.1, we have

$$u = H + V_1 + V_2,$$

where V_i is a harmonic function in $\mathbb{R}^2 \setminus D_i$ and $V_i(x) = O(|x|^{-1})$ as $|x| \to \infty$ for $i = 1, 2$. Owing to Poincaré's theorem, we have a well-defined conjugate harmonic function of V_i, denoted by $\widetilde{V_i}$, for $i = 1, 2$ such that

$$\widetilde{V_1}(x) + \widetilde{V_2}(x) = O(|x|^{-1}) \quad \text{as } |x| \to \infty.$$

Let $\widetilde{u} = \widetilde{H} + \widetilde{V_1} + \widetilde{V_2}$. Then $\widetilde{u}$ satisfies (1.7) and is also a harmonic conjugate function of u. Hence, we have

$$|\nabla u| = |\nabla \widetilde{u}| \quad \text{in } \mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)}.$$

Therefore, we have completed the proof. $\square$

2. Proof of Theorem 1.1. In this section we will give a proof of the inequality (1.3). The proof is based on (2.6) and Lemma 2.3 which present an interesting representation for the difference $u|_{\partial D_1} - u|_{\partial D_2}$ by a probability function $\partial_\nu w$. Thus we choose a constant C satisfying the inequality $\partial_\nu w \geq C\sqrt{\epsilon}$ (2.15). This inequality completes the proof.

We start by representing the voltage potential u as a function related to H. To do so, we define the operator $R_1 : C^\infty(\mathbb{R}^2 \setminus \overline{D_2}) \to C^\infty(\mathbb{R}^2 \setminus \overline{D_1}) \cap C(\mathbb{R}^2 \setminus D_1)$ as

$$\begin{cases} \triangle R_1(v) = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D_1}, \\ R_1(v)(x) = O(|x|^{-1}) \quad \text{as } |x| \to \infty, \\ (v - R_1(v))|_{\partial D_1} = C \text{ (constant)}, \end{cases} \tag{2.1}$$

where C is a constant dependent on v, and we also define $R_2 : C^\infty(\mathbb{R}^2 \setminus \overline{D_1}) \to C^\infty(\mathbb{R}^2 \setminus \overline{D_2}) \cap C(\mathbb{R}^2 \setminus D_2)$ similarly. It follows from Green's theorem that

$$\int_{\partial_{D_1}} \frac{\partial R_1(v)}{\partial \nu} \, ds = 0.$$

By definition, $H - R_1(H)$ can be interpreted physically as the voltage potential due only to the presence of D_1, under the action of applied electric field ∇H. Since the voltage potential u is due not only to D_1 but also to D_2, we take advantage of R_2. We thus expect $u \sim H - R_1(H) - R_2(H)$. But since $H - R_1(H) - R_2(H)$ is not constant on the boundaries ∂D_1 and ∂D_2, we expect $u \sim H - R_1(H) - R_2(H) + R_2R_1(H) + R_1R_2(H)$ again. These steps can proceed inductively. The process provides the following lemma.

Lemma 2.1. *For a harmonic function H defined in $\mathbb{R}^2$, the voltage function u is represented as follows:*

$$\begin{aligned} u &= H - R_1(H) - R_2(H) + R_1R_2(H) + R_2R_1(H) \\ &\quad - R_1R_2R_1(H) - R_2R_1R_2(H) + \cdots \\ &= H - R_1(H) - R_2(H) + R_1R_2(H) + R_2R_1(H) \\ &\quad + \sum_{n=1}^{\infty}(R_1R_2)^n(-R_1(H) + R_1R_2(H)) + (R_2R_1)^n(-R_2(H) + R_2R_1(H)). \end{aligned} \tag{2.2}$$

Proof. We choose an interior point p of D_1. Let $\Omega = \{\frac{x-p}{|x-p|^2} \mid x \in \mathbb{R}^2 \backslash D_1\} \cup \{0\}$ and $\Omega_\epsilon = \{\frac{x-p}{|x-p|^2} \mid x \in D_2\} \cup \{0\}$. Then we have $\Omega_\epsilon \subset \Omega$ and the distance $\mathrm{d}(\Omega_\epsilon, \partial\Omega) > 0$. Hence, by the maximum principle and standard estimates, one can choose a positive constant $c < 1$ such that

$$\max_{\Omega_\epsilon} h - \min_{\Omega_\epsilon} h \leq c\left(\max_{\Omega} h - \min_{\Omega} h\right)$$

for any harmonic function h defined in Ω. It follows that

$$\max_{\partial D_2} R_2R_1(v) - \min_{\partial D_2} R_2R_1(v) \leq c\left(\max_{\partial D_1} R_1 v - \min_{\partial D_1} R_1 v\right) \tag{2.3}$$

and, since $R_1(v)(x) = O(|x|^{-1})$ as $|x| \to \infty$, we have

$$\|R_1(v)\|_{L^\infty} \leq \max_{\partial D_1} R_1(v) - \min_{\partial D_1} R_1(v) \tag{2.4}$$

for any $v \in C^\infty(\mathbb{R}^2 \backslash D_2)$. And we can obtain similar results for D_2 and R_2. Since $0 < c < 1$, the expansion (2.2) is well defined and satisfies (1.2). $\square$

In the particular case of circular inclusions, it follows from Lemma 2.1 and Kelvin transforms that the main result (1.4) holds (see [2]). However, it is not easy to apply the asymptotic expansion (2.2) directly to the general case of arbitrary shape. On this account, we would make the expansion (2.2) simpler. We define the operator $K_1 : C^\infty(\mathbb{R}^2 \backslash \overline{D_2}) \to C^\infty(\mathbb{R}^2 \backslash \overline{D_1}) \cap C(\mathbb{R}^2 \backslash D_1)$ as follows:

$$\begin{cases} \triangle K_1(v) = 0 \quad \text{in } \mathbb{R}^2 \backslash \overline{D_1}, \\ K_1(v)(x) \text{ converges to some constant as } |x| \to \infty, \\ v = K_1(v) \text{ on } \partial D_1. \end{cases}$$

And we also define K_2 similarly. By the definitions of R_i and K_i $(i = 1, 2)$, we have

$$R_i(v) = K_i(v) + \text{some constant},$$

where the constant is dependent on v.

Lemma 2.2. *We have*

$$\begin{aligned} u|_{\partial D_1} &= C_0 + \lim_{n\to\infty} (K_2K_1)^n(H)|_{\partial D_1}, \\ u|_{\partial D_2} &= C_0 + \lim_{n\to\infty} (K_1K_2)^n(H)|_{\partial D_2}, \end{aligned} \tag{2.5}$$

where C_0 is a constant.

Proof. Since $K_i(v) = v$ on ∂D_i for $i = 1, 2$, we have

$$
\begin{aligned}
&(R_1R_2)^n(-R_1(H) + R_1R_2(H)) + (R_2R_1)^n(-R_2(H) + R_2R_1(H)) \\
&= c_n + (K_1K_2)^n(-K_1(H) + K_1K_2(H)) + (K_2K_1)^n(-K_2(H) + K_2K_1(H))
\end{aligned}
$$

for $n = 0, 1, 2, 3, \ldots$, where

$$
\begin{aligned}
c_n = &\{-(R_1R_2)^n R_1(H) + (R_2R_1)^n R_2R_1(H)\}\,|_{\partial D_2} \\
&+ \{-(R_2R_1)^n R_2(H) + (R_1R_2)^n R_1R_2(H)\}\,|_{\partial D_1}.
\end{aligned}
$$

By (2.3) and (2.4), we have $\sum_{n=0}^{\infty} |c_n| < \infty$. We thus set $C_0 = \sum_{n=0}^{\infty} c_n$. Then

$$
\begin{aligned}
u &= H - R_1(H) - R_2(H) + R_1R_2(H) + R_2R_1(H) \\
&\quad + \sum_{n=1}^{\infty} (R_1R_2)^n(-R_1(H) + R_1R_2(H)) + (R_2R_1)^n(-R_2(H) + R_2R_1(H)) \\
&= C_0 + H - K_1(H) - K_2(H) + K_1K_2(H) + K_2K_1(H) \\
&\quad + \sum_{n=1}^{\infty} (K_1K_2)^n(-K_1(H) + K_1K_2(H)) + (K_2K_1)^n(-K_2(H) + K_2K_1(H)) \\
&= C_0 + \{H - K_1(H)\} + \{-K_2(H) + K_1K_2(H)\} + K_2K_1(H) + \cdots \\
&= C_0 + \lim_{n\to\infty} (K_2K_1)^n(H) \text{ on } \partial D_1. \qquad \square
\end{aligned}
$$

Hence we conclude that

$$
u|_{\partial D_1} - u|_{\partial D_2} = \lim_{n\to\infty} (K_2K_1)^n(H)|_{\partial D_1} - \lim_{n\to\infty} (K_1K_2)^n(H)|_{\partial D_2}. \tag{2.6}
$$

In what follows, we consider

$$
\lim_{n\to\infty} (K_2K_1)^n(H)|_{\partial D_1}.
$$

Now we present an interesting representation for $\lim_{n\to\infty}(K_2K_1)^n(H)|_{\partial D_1}$ in the following lemma. Based on this, the main result would be derived without any asymptotic analysis and layer potentials.

LEMMA 2.3. *Let w be the solution of the following problem:*

$$
\begin{cases}
\triangle w = 0 \quad \text{in } \mathbb{R}^2 \backslash \overline{(D_1 \cup D_2)}, \\
w(x) = O(|x|^{-1}) \quad \text{as } \ |x| \to \infty, \\
w|_{\partial D_1} = c_{1\epsilon} \ \text{(constant)}, \\
w|_{\partial D_2} = c_{2\epsilon} \ \text{(constant)}, \\
\int_{\partial D_1} \partial_\nu w \; ds = 1.
\end{cases} \tag{2.7}
$$

Then we have

$$
\lim_{n\to\infty} (K_2K_1)^n(H)|_{\partial D_1} = \int_{\partial D_1} (\partial_\nu w) H ds.
$$

Proof. By definition, we have

$$
H = K_1(H) \quad \text{on } \partial D_1.
$$

Since $K_1(H)$ is harmonic in D_2 and w is constant on ∂D_2, we have

$$\int_{\partial D_2} w\partial_\nu K_1(H)ds = 0.$$

And since $K_1(H)$ converges to some constant as $|x| \to \infty$ and w is constant on ∂D_1, we also have

$$\int_{\partial D_1} w\partial_\nu K_1(H)ds = 0.$$

It follows from Green's theorem that

$$\int_{\partial D_1} (\partial_\nu w)Hds + \int_{\partial D_2} (\partial_\nu w)K_1(H)ds = 0.$$

Similarly we have

$$\int_{\partial D_2} (\partial_\nu w)K_1(H)ds + \int_{\partial D_1} (\partial_\nu w)K_2K_1(H)ds = 0.$$

Hence we obtain

$$\begin{aligned}\int_{\partial D_1} (\partial_\nu w)Hds &= \int_{\partial D_1} (\partial_\nu w)K_2K_1(H)ds \\ &= \int_{\partial D_1} (\partial_\nu w)(K_2K_1)^n(H)ds \quad \text{for} \quad n = 1, 2, 3, \dots \\ &= \int_{\partial D_1} (\partial_\nu w) \lim_{n\to\infty} (K_2K_1)^n(H)ds \\ &= \lim_{n\to\infty} (K_2K_1)^n(H). \qquad \square\end{aligned}$$

Thus we focus on $\partial_\nu w$. We define $\varphi_1 : \mathbb{C}\backslash B_1(0) \to \mathbb{R}^2\backslash D_1$ as $\varphi_1 = \varphi + \frac{\epsilon}{2}$ and the conformal mapping $\Phi : \overline{B_1(0)}\backslash\{0\} \to \mathbb{R}^2\backslash D_1$ as

$$\Phi(z) = \varphi_1\left(\frac{z}{|z|^2}\right),$$

where $\varphi : \mathbb{C}\backslash B_1(0) \to \mathbb{R}^2\backslash D_{\text{right}}$ is the conformal mapping defined in the introduction. Then we have

$$\Phi^{-1}\left(\mathbb{R}^2\backslash\overline{(D_1 \cup D_2)}\right) \subset B_1(0)$$

and

$$\Phi^{-1}(\partial D_1) = \partial B_1(0).$$

We consider the solution W to the following Dirichlet problem:

$$\begin{cases} \triangle W = 0 \quad \text{in } B_1(0)\backslash\overline{\Phi^{-1}(D_2)}, \\ W = 1 \quad \text{on } \Phi^{-1}(\partial D_1) = \partial B_1(0), \\ W = -1 \quad \text{on } \Phi^{-1}(\partial D_2). \end{cases} \tag{2.8}$$

Let $M = \frac{1}{2}(c_{1\epsilon} - c_{2\epsilon})$ and $c_* = \frac{1}{2}(c_{1\epsilon} + c_{2\epsilon})$. Then, even though $w(\Phi(z))$ is not defined at 0, it follows from the decreasing behavior of w at infinity that

$$\begin{cases} w(\Phi(z)) = MW(z) + c_* \quad \text{for } z \in \Phi^{-1}\left(\mathbb{R}^2 \backslash \overline{(D_1 \cup D_2)}\right), \\ \partial_{\nu_{(x)}} w(x) = \left(\frac{M}{|\varphi'(z)|}\right) \partial_{\nu_{(z)}} W(z) \quad \text{for } z \in \partial B_1(0), \end{cases} \tag{2.9}$$

where $x = \Phi(z)$.

Without loss of generality, we assume that

$$B_{r_2}(r_2 + \epsilon - 1) \subset \Phi^{-1}(D_2) \subset B_{r_1}(r_1 + \epsilon - 1), \tag{2.10}$$

where r_1 and r_2 are independent of ϵ for any sufficiently small $\epsilon \geq 0$. (See Lemma 4.1 in the appendix for details.) Then we consider the solution U_1 and U_2 to the following equations:

$$\begin{cases} \triangle U_i = 0 \quad \text{in } B_1(0) \backslash \overline{B_{r_i}(r_i + \epsilon - 1)}, \\ U_i = 1 \quad \text{on } \Phi^{-1}(\partial D_1) = \partial B_1(0), \\ U_i = -1 \quad \text{on } \partial B_{r_i}(r_i + \epsilon - 1) \ \text{ for } i = 1, 2. \end{cases} \tag{2.11}$$

By the maximum principle, we have

$$U_1 \leq W \leq U_2 \ \text{ in } B_1(0) \backslash B_{r_1}(r_1 + \epsilon - 1),$$

and by Hopf's lemma, we have

$$\partial_\nu U_2 \leq \partial_\nu W \leq \partial_\nu U_1 \ \text{ on } \partial B_1(0). \tag{2.12}$$

Lemma 2.4. *We have the conformal mappings Ψ_1 and Ψ_2 such that*

$$\begin{cases} \Psi_1(B_1(0)) = \Psi_2(B_1(0)) = B_1(0), \\ B_{r_1}(r_1 + \epsilon - 1) = \Psi_1\left(B_{1 - \alpha_1\sqrt{\epsilon} + o(\sqrt{\epsilon})}(0)\right), \\ B_{r_2}(r_2 + \epsilon - 1) = \Psi_2\left(B_{1 - \alpha_2\sqrt{\epsilon} + o(\sqrt{\epsilon})}(0)\right), \end{cases} \tag{2.13}$$

and for $i = 1, 2$,

$$\Psi_i^{-1}(z) = \left(\beta_i\sqrt{\epsilon} + o\left(\sqrt{\epsilon}\right)\right) \frac{1}{z - (-1 - \gamma_i\sqrt{\epsilon} + o(\sqrt{\epsilon}))} + \kappa_i \quad \text{as } \epsilon \to 0, \tag{2.14}$$

where α_i, β_i, and γ_i are strictly positive constants.

Proof. See section 4.1 in the appendix. ∎

Moreover, by (2.11) and (2.13), we have

$$U_i(\Psi_i(t)) = -2(\log(1 - \alpha_i\sqrt{\epsilon} + o(\sqrt{\epsilon})))^{-1} \log(|t|) + 1$$

for $t \in B_1(0)$ and

$$\begin{aligned} \int_{\partial B_1(0)} \partial_{\nu(z)} U_i(z) ds(z) &= \int_{\partial B_1(0)} \partial_{\nu(t)} U_i(\Psi_i(t)) ds(t) \\ &= -4\pi(\log(1 - \alpha_i\sqrt{\epsilon} + o(\sqrt{\epsilon})))^{-1} \quad \text{for } i = 1, 2. \end{aligned}$$

Therefore, by (2.9) and (2.12), we have

$$\int_{\partial B_1(0)} M\partial_{\nu_{(z)}} W(z)ds(z) = \int_{\partial B_1(0)} \partial_{\nu(z)} w(\Phi(z))ds(z) = \int_{\partial D_1} \partial_\nu w(x)ds(x) = 1$$

and

$$M \geq -\frac{\log(1-\alpha_1\sqrt{\epsilon}+o(\sqrt{\epsilon}))}{4\pi}.$$

It follows that

$$\begin{aligned}
\partial_{\nu_{(z)}} w(\Phi(z)) &= M\partial\nu_{(z)}W(z)\\
\text{by (2.12)}\quad &\geq -\frac{\log(1-\alpha_1\sqrt{\epsilon}+o(\sqrt{\epsilon}))}{4\pi}\partial\nu_{(z)}U_2(z)\\
&= -\frac{\log(1-\alpha_1\sqrt{\epsilon}+o(\sqrt{\epsilon}))}{4\pi}\partial\nu_{(t)}U_2(\Psi_2(t))|(\Psi_2^{-1})'(z)|\\
&\geq \frac{\log(1-\alpha_1\sqrt{\epsilon}+o(\sqrt{\epsilon}))}{2\pi\log(1-\alpha_2\sqrt{\epsilon}+o(\sqrt{\epsilon}))}|(\Psi_2^{-1})'(z)|\\
&\geq \frac{\log(1-\alpha_1\sqrt{\epsilon}+o(\sqrt{\epsilon}))}{2\pi\log(1-\alpha_2\sqrt{\epsilon}+o(\sqrt{\epsilon}))}\min_{|z|=1}|(\Psi_2^{-1})'(z)|\\
\text{by (2.14)}\quad &\geq C\sqrt{\epsilon}, \qquad (2.15)
\end{aligned}$$

where $z = \Psi_2(t) \in \partial B_1(0)$. And since φ_1' is independent of ϵ, when $H(x_1, x_2) = x_1$, we have

$$\lim_{n\to\infty}(K_2K_1)^n(H)|_{\partial D_1} \geq C_*\sqrt{\epsilon}$$

and, by definition,

$$\lim_{n\to\infty}(K_1K_2)^n(H)|_{\partial D_2} \leq 0.$$

These bounds complete the proof of (1.3). ◻

REMARK 2.5. *We assumed that D_2 has a C^2 boundary. This regularity condition is used only for choosing r_2 of* (2.10) *in Lemma* 4.1. *We observe that one can prove our estimates on a relaxed regularity condition. For example, even when $D_2 = (-1, 0) \times (-1, 1) - \epsilon/2$, the estimates* (1.3), (1.4), *and* (1.8) *hold.*

3. The proof of Theorem 1.2. We first prove that the inequality (1.5) holds. This proof is the continuation of the proof of Theorem 1.1. By the argument similar to (2.15), we have

$$\begin{aligned}
\partial_{\nu_{(z)}} w(\Phi(z)) &= M\partial\nu_{(z)}W(z)\\
&\leq -\frac{\log(1-\alpha_2\sqrt{\epsilon}+o(\sqrt{\epsilon}))}{4\pi}\partial\nu_{(z)}U_1(z)\\
&= -\frac{\log(1-\alpha_2\sqrt{\epsilon}+o(\sqrt{\epsilon}))}{4\pi}\partial\nu_{(t)}U_1(\Psi_1(t))|(\Psi_1^{-1})'(z)|\\
&\leq \frac{\log(1-\alpha_2\sqrt{\epsilon}+o(\sqrt{\epsilon}))}{2\pi\log(1-\alpha_1\sqrt{\epsilon}+o(\sqrt{\epsilon}))}|(\Psi_1^{-1})'(z)|\\
\text{by } \overline{z} = \frac{1}{z} \text{ on } \partial B_1(0),\quad &\leq CP(p_\epsilon\overline{z}), \qquad (3.1)
\end{aligned}$$

where $P(x, y)$ is a Poisson kernel and

$$\begin{aligned} p_\epsilon &= -1 + \gamma_1\sqrt{\epsilon} + o(\sqrt{\epsilon}) \\ &= -1 + O(\sqrt{\epsilon}) \in B_1(0) \quad \text{as } \epsilon \to \infty. \end{aligned}$$

One can prove by assuming $\varphi(-1) = 0$ and the regularity conditions of φ on the boundary, instead of the assumption (2.10), that this bound (3.1) holds and $p_\epsilon = -1 + O(\sqrt{\epsilon}) \in B_1(0)$ as $\epsilon \to 0$.

From the definition of φ, $(\varphi(\frac{1}{z}) + c)^{-1}$ is extended to a conformal mapping defined in $B(0)$ for some constant c that attains zero value at $z = 0$. Hence $\varphi(z)$ can be rewritten as follows:

$$\varphi(z) = a_1 z + h\left(\frac{1}{z}\right) \quad \text{for } z \in \mathbb{C} \setminus B_1(0), \tag{3.2}$$

where h is analytic in $B_1(0)$ and a_0 is a nonzero constant. Then for $H(x_1, x_2) = x_1$, we have

$$H(\varphi_1(z)) = \Re\left(a_1 z + h\left(\frac{1}{z}\right)\right) + \frac{\epsilon}{2}.$$

Then we define $\mathbf{H}$ on $\overline{B}_1(0)$ as follows:

$$\mathbf{H}(z) = \Re\left(a_1 \overline{z} + h(z)\right) + \frac{\epsilon}{2} \quad \text{for } z \in \overline{B}_1(0). \tag{3.3}$$

It is easy to see that $\mathbf{H}(z) = H(\varphi_1(\overline{z}))$ on $\partial B_1(0)$, and that $\mathbf{H}$ is a harmonic function in $B_1(0)$ and belongs to $C^1(\overline{B_1(0)})$. Then it follows from Lemma 2.3 that

$$\begin{aligned} 0 \le \lim_{n\to\infty} (K_2K_1)^n(H)|_{\partial D_1} &= \int_{\partial D_1} \partial_\nu w(x) H(x) ds(x) \\ &\le C_1 \int_{\partial B_1(0)} P(p_\epsilon, \overline{z}) H(\varphi_1(z)) ds(z) \\ &= C_1\ \mathbf{H}(\overline{p_\epsilon}) \\ \text{by } \mathbf{H}(\varphi_1(-1)) = \frac{\epsilon}{2},\ &\le C_2\sqrt{\epsilon}. \end{aligned} \tag{3.4}$$

Applying the same argument to ψ and D_{left}, we also have

$$0 \ge \lim_{n\to\infty} (K_1K_2)^n(H)|_{\partial D_2} \ge C_2{}'\sqrt{\epsilon}.$$

These bounds are reduced to the inequality (1.5), that is,

$$|u|_{\partial D_1} - u|_{\partial D_2}| \le C_1^*\sqrt{\epsilon}.$$

REMARK 3.1. *We suggest another method to get the inequalities* (3.4) *and* (1.5). *We divide the integration in* (3.4) *into two parts as follows:*

$$\begin{aligned} \int_{\partial B_1(0)} P(p_\epsilon, \overline{z}) H(\varphi_1(z)) ds(z) = &\int_{\partial B_1(0)\ \text{and}\ |z+1| \le \sqrt[4]{\epsilon}} P(p_\epsilon, \overline{z}) H(\varphi_1(z)) ds(z) \\ &+ \int_{\partial B_1(0)\ \text{and}\ |z+1| > \sqrt[4]{\epsilon}} P(p_\epsilon, \overline{z}) H(\varphi_1(z)) ds(z). \end{aligned}$$

Then the inequality (3.4) *can also be proved directly without* (3.2). *Moreover, one can prove by using the same partition of integration that the bound* (1.5) *still holds for any harmonic function H with $\partial_{x_2}H(0) = 0$.*

We now derive the inequality (1.6). We divide u into four parts as follows:

$$u = x_1 + u_0 + u_1 + u_2$$

such that for $i = 0, 1, 2$, $\Delta u_i = 0$ in $\mathbb{R}^2 \backslash \overline{D_1 \cup D_2}$ and $u_i = O(1)$ as $|x| \to \infty$ with the boundary conditions

$$\begin{cases} u_0 &= u \text{ (constants)} \quad \text{on } \partial D_1 \cup \partial D_2, \\ u_1 &= -x_1 \text{ on } \partial D_1 \quad \text{and } u_1 = 0 \text{ on } \partial D_2, \\ u_2 &= -x_1 \text{ on } \partial D_2 \quad \text{and } u_2 = 0 \text{ on } \partial D_1. \end{cases}$$

Hence, we would estimate them separately.

Estimate for u_0. It follows from the maximum principle for analytic functions that

$$\|\nabla u_0\|_{L^\infty(\mathbb{R}^2 \backslash (D_1 \cup D_2))} \le \|\partial_\nu u_0\|_{L^\infty(\partial D_1 \cup \partial D_2)}.$$

Thus we estimate $\|\partial_\nu u_0\|_{L^\infty(\partial D_1)}$. There exists a constant C independent of ϵ such that

$$\|\partial_\nu u_0\|_{L^\infty(\partial D_1)} \le C \|\partial_{\nu(z)} u_0(\Phi(z))\|_{L^\infty(\partial B_1(0))},$$

where Φ is as defined in the proof of Theorem 1.1. By the argument similar to that of (2.12) and (3.1), we can choose a constant C such that for $z \in \partial B_1(0)$

$$\begin{aligned} \left|\partial_{\nu(z)} u_0(\Phi(z))\right| &\le \left|\frac{1}{2}(u|_{\partial D_1} - u|_{\partial D_2}) \partial_{\nu(z)} U_1(z)\right| \\ \text{by (1.5) and the Poisson kernel } P(p_\epsilon, \cdot), \quad &\le C \frac{1}{\sqrt{\epsilon}}, \end{aligned}$$

where U_1 is as defined in (2.11). Since u_0 is constant on ∂D_i for $i = 1, 2$, we conclude that

$$\|\nabla u_0\|_{L^\infty(\mathbb{R}^2 \backslash (D_1 \cup D_2))} \le C_a \frac{1}{\sqrt{\epsilon}}, \tag{3.5}$$

where C_a is a constant independent of ϵ.

Estimate for u_1. Let $\psi_2 : \mathbb{C} \backslash B_1(0) \to \mathbb{R}^2 \backslash D_2$ be the conformal mapping defined by $\psi_2 = \psi - \frac{\epsilon}{2}$ as φ_1. Since $u_1(\psi_2(z)) = 0$ on $\partial B_1(0)$, $u_1(\psi_2(z))$ can be extended harmonically to $\mathbb{C} \backslash (\mathbf{B} \cup \psi_2^{-1}(D_1))$ as follows:

$$-u_1 \left(\psi_2 \left(\frac{z}{|z|^2}\right)\right) \quad \text{for } z \in B_1(0) \backslash \mathbf{B},$$

where $\mathbf{B} = \{z \mid \frac{z}{|z|^2} \in \psi_2^{-1}(D_1)\}$. The symmetry of the extended $u_1(\psi_2(z))$ occurs on $\partial \mathbf{B}$ and $\partial \psi_2^{-1}(D_1)$. This yields

$$\|\nabla u_1\|_{L^\infty(\mathbb{R}^2 \backslash (D_1 \cup D_2))} \le C \|\nabla u_1\|_{L^\infty(\partial D_1)},$$

where C is independent of ϵ. We now estimate $\|\partial_\nu u_1\|_{L^\infty(\partial D_1)}$, because the tangential derivative of u_1 on ∂D_1 is not only fixed by $H(x_1,x_2)=x_1$ but also independent of ϵ. Using a linear fractional transform, without loss of generality, we can also assume that

$$B_{r_4}(-1-\epsilon-r_4)\subset\varphi_1^{-1}(D_2)\subset B_{r_3}(-1-\epsilon-r_3),$$

where r_3 and r_4 are independent of ϵ. Then we consider a harmonic function V_3 and V_4 as follows:

$$\begin{cases}\Delta V_i=0 \quad \text{in } \mathbb{R}^2\backslash\overline{(B_1(0)\cup B_{r_i}(-1-\epsilon-r_i))},\\ V_i(z)=O(1) \quad \text{as } |z|\to\infty,\\ V_i(z)=u_1(\varphi_1(z))=-H(\varphi_1(z)) \quad \text{for } z\in\partial B_1(0),\\ V_i(z)=0 \quad \text{for } z\in\partial B_{r_i}(-1-\epsilon-r_i)\end{cases}$$

for $i=3,4$. Since $u_1(\varphi_1(z))\le 0$ for $z\in\partial B_1(0)$, it follows from Hopf's lemma that

$$\partial_\nu V_3(z)\le\partial_{\nu(z)}u_1(\varphi_1(z))\le\partial_\nu V_4(z) \quad \text{for } z\in\partial B_1(0). \tag{3.6}$$

Therefore, we now estimate $\partial_\nu V_3(z)$ and $\partial_\nu V_4(z)$. By a definition similar to K_1 and K_2 in the previous proof, we define the operator $K_\alpha : C^\infty(\mathbb{R}^2\backslash\overline{B_{r_3}(-1-\epsilon-r_3)})\to C^\infty(\mathbb{R}^2\backslash\overline{B_1(0)})\cap C(\mathbb{R}^2\backslash B_1(0))$ as

$$\begin{cases}\triangle K_\alpha(v)=0 \quad \text{in } \mathbb{R}^2\backslash\overline{B_1(0)},\\ K_\alpha(v)(z)=O(1) \quad \text{as } |z|\to\infty,\\ (v-K_\alpha(v))|_{\partial B_1(0)}=0,\end{cases}$$

and $K_\beta : C^\infty(\mathbb{R}^2\backslash\overline{B_1(0)})\to C^\infty(\mathbb{R}^2\backslash\overline{B_{r_3}(-1-\epsilon-r_3)})\cap C(\mathbb{R}^2\backslash B_{r_3}(-1-\epsilon-r_3))$ is defined similarly. Indeed K_α is the Kelvin transform for $B_1(0)$ and K_β is the Kelvin transform for $B_{r_3}(-1-\epsilon-r_3)$ simply. Then we define a harmonic function U on $\mathbb{R}^2\backslash(B_{r_3}(-1-\epsilon-r_3)\cup B_1(0))$ as follows:

$$U=-K_\alpha\mathbf{H}(\overline{z})+K_\beta K_\alpha\mathbf{H}(\overline{z})-\sum_{n=1}^{\infty}(I-K_\beta)(K_\alpha K_\beta)^n K_\alpha\mathbf{H}(\overline{z}),$$

where $\mathbf{H}$ is defined at (3.3). By an argument similar to Lemma 2.2 or [2, 4], we have

$$\begin{cases}\triangle U=0 \quad \text{in } \mathbb{R}^2\backslash\overline{(B_{r_3}(-1-\epsilon-r_3)\cup B_1(0))},\\ U(z)=O(1) \quad \text{as } |z|\to\infty,\\ U=0 \quad \text{for } z\in\partial B_{r_3}(-1-\epsilon-r_3),\\ (U+H(\varphi(z)))\,|_{B_1(0)}=\text{a constant with order } O(\sqrt{\epsilon}) \quad \text{as } \epsilon\to 0.\end{cases}$$

We note that U is a solution to the equation with the circular inclusions. Thus, we can apply Theorem 1.1 of Kang et al. [4] in U. Then we have

$$\|\nabla U\|_{L^\infty\left(\mathbb{R}^2\backslash(B_{r_3}(-1-\epsilon-r_3)\cup B_1(0))\right)}\le C\left(\frac{1}{\sqrt{\epsilon}}\right).$$

On the other hand, we have

$$\begin{cases} \triangle(V_3-U)=0 \quad \text{in } \mathbb{R}^2\backslash\overline{(B_{r_3}(-1-\epsilon-r_3)\cup B_1(0))}, \\ (V_3-U)(z)=O(1) \quad \text{as } \ |z|\to\infty, \\ (V_3-U)(z)=0 \quad \text{for } z\in\partial B_{r_3}(-1-\epsilon-r_3), \\ (V_3-U)|_{\partial B_1(0)} = \text{a constant with order } O(\sqrt{\epsilon}) \quad \text{as } \epsilon\to 0. \end{cases}$$

By the same argument as *Estimate for* u_0, we have

$$\|\nabla(V_3-U)\|_{L^\infty\left(\mathbb{R}^2\backslash(B_{r_3}(-1-\epsilon-r_3)\cup B_1(0))\right)} \le C\left(\frac{1}{\sqrt{\epsilon}}\right).$$

These bounds are reduced to

$$\|\nabla V_3\|_{L^\infty\left(\mathbb{R}^2\backslash(B_{r_3}(-1-\epsilon-r_3)\cup B_1(0))\right)} \le C\left(\frac{1}{\sqrt{\epsilon}}\right).$$

Similarly, we also obtain

$$\|\nabla V_4\|_{L^\infty\left(\mathbb{R}^2\backslash(B_{r_4}(-1-\epsilon-r_4)\cup B_1(0))\right)} \le C\left(\frac{1}{\sqrt{\epsilon}}\right).$$

By (3.6), these bounds yield

$$\|\nabla u_1\|_{L^\infty(\mathbb{R}^2\backslash(D_1\cup D_2))} \le C_\beta\left(\frac{1}{\sqrt{\epsilon}}\right), \tag{3.7}$$

where C_β is a constant independent of ϵ.

Estimate for u_2. This estimate is derived in the same way as u_1.

Therefore, by (3.5) and (3.7) we have completed the proof of (1.6). □

4. Appendix. In this appendix we make up the parts omitted in the proof of Theorem 1.1. We first consider Lemma 2.4.

4.1. How to construct the conformal mapping Ψ_1^{-1}. We now prove Lemma 2.4; that is, we present a method for constructing Ψ_1^{-1}. For convenience, we use two steps to derive it.

- *Step* 1. To make $\Psi_1^{-1}(B_1(0))$ and $\Psi_1^{-1}(B_{r_1}(r_1+\epsilon-1))$ concentric balls, we find

$$f_1(z) = \frac{1}{z-(-1-\tau(\epsilon)\sqrt{\epsilon})}$$

with $f_1(-1)-f_1(-1+\epsilon) = f_1(-1+\epsilon+2r_1)-f_1(1)$. As ϵ approaches 0, we have

$$\begin{aligned} \lim_{\epsilon\to 0}\left(\frac{1}{\tau(\epsilon)}\right)^2 &= \lim_{\epsilon\to 0} f_1(-1+\epsilon+2r_1)-f_1(1) \\ &= \left(\frac{1}{\gamma_1}\right)^2. \end{aligned}$$

Therefore, we have

$$f_1(z) = \frac{1}{z-(-1-\gamma_1\sqrt{\epsilon}+o(\sqrt{\epsilon}))}.$$

• *Step* 2. To make $\Psi_1^{-1}(B_1(0))$ a unit disk, we find

$$f_2(z) = \lambda(\epsilon)\sqrt{\epsilon} f_1(z)$$

with $f_2(-1) - f_2(1) = 2$. Then, as ϵ approaches 0, we have

$$\begin{aligned} 2 &= \lim_{\epsilon\to 0} \lambda(\epsilon)\ \sqrt{\epsilon}\ f_1(-1) \\ &= \lim_{\epsilon\to 0} \lambda(\epsilon)\frac{1}{\gamma_1} = \frac{\beta_1}{\gamma_1}. \end{aligned}$$

Therefore, we have

$$f_2(z) = (\beta_1\sqrt{\epsilon} + o(\sqrt{\epsilon}))\frac{1}{z - (-1 - \gamma_1\sqrt{\epsilon} + o(\sqrt{\epsilon}))}$$

and

$$f_2(B_1(0)) = B_1(0) - k_1 \quad \text{for some constant } k_1.$$

Hence, we obtain

$$\Psi_1^{-1}(z) = f_2(z) + k_1,$$

satisfying

$$\Psi_1^{-1}(B_1(0)) = B_1(0). \qquad \square$$

4.2. How to construct the radii r_1 and r_2 satisfying (2.10). To construct them, we provide the variable ϵ to the definitions used in the main proof. Thus D_1 and D_2 are rewritten in the form $D_1(\epsilon) = D_{\text{right}} + \frac{\epsilon}{2}$ and $D_2(\epsilon) = D_{\text{left}} - \frac{\epsilon}{2}$. In addition, we assume that $\varphi : \mathbb{C}\backslash B_1(0) \to \mathbb{R}^2\backslash D_{\text{right}}$ is a bijective conformal mapping with $\varphi(-1) = 0$. Then φ_1 and Φ are rewritten as $\varphi_{1\epsilon} = \varphi + \epsilon$ and $\Phi_\epsilon = \varphi_{1\epsilon}(\frac{z}{|z|^2})$.

For any r and z with $0 < r < 1$ and $r \le |z| \le 1$, we define the ball with radius r, denoted by $B_r[z]$, in $B_1(0)$ whose boundary attains the minimal distance r to $\partial B_1(0)$ at z, i.e.,

$$\mathrm{d}(B_r[z],\ \partial B_1(0)) = \mathrm{d}(z,\ \partial B_1(0)) = r \quad \text{and} \quad z \in \partial B_r[z],$$

where $\mathrm{d}(A, B)$ is the distance between A and B.

LEMMA 4.1. *Let $\epsilon' = d(\partial B_1(0), \partial\Phi_\epsilon^{-1}(D_2(\epsilon)))$, that is, the distance between $\partial B_1(0)$ and $\partial\Phi_\epsilon^{-1}(D_2(\epsilon))$. Then we have a constant C such that*

$$\epsilon < C\epsilon' \qquad \textit{for}\ \ \epsilon < 1. \tag{4.1}$$

We suppose that for each $\epsilon > 0$, z_ϵ is the point at which $\partial\Phi_\epsilon^{-1}(D_2(\epsilon))$ attains the minimal distance ϵ' to $\partial B_1(0)$, i.e., $z_\epsilon \in \partial\Phi_\epsilon^{-1}(D_2(\epsilon))$ and $\epsilon' = d(z_\epsilon, \partial B_1(0))$. Then we have r_1, r_2, and $\epsilon_0 > 0$ such that

$$B_{r_2}[z_\epsilon] \subset \Phi_\epsilon^{-1}(D_2(\epsilon)) \subset B_{r_1}[z_\epsilon] \quad \textit{for}\ \epsilon < \epsilon_0. \tag{4.2}$$

If we use this result (4.2) instead of the assumption (2.10), then we can prove by the same derivation as (2.15) that $\partial_{\nu_{(z)}} w(\Phi(z)) \ge C\sqrt{\epsilon'}$. It follows from (4.1) that (2.15) holds.

Proof. To prove the inequality (4.1), we suppose that p_ϵ is the closest point at which $\partial B_1(0)$ attains the minimal distance ϵ' to $\partial\Phi_\epsilon^{-1}(D_2(\epsilon))$, i.e., $\epsilon' = \mathrm{d}(p_\epsilon, z_\epsilon)$ and $p_\epsilon \in \partial B_1(0)$. Then we obtain

$$\begin{aligned}\epsilon &\le \mathrm{d}(\Phi_\epsilon(p_\epsilon), \Phi_\epsilon(z_\epsilon))\\ &= \mathrm{d}(\Phi_0(p_\epsilon), \Phi_0(z_\epsilon))\\ &\le C\mathrm{d}(p_\epsilon, z_\epsilon) = C\epsilon' \quad \text{for } \epsilon < 1,\end{aligned}$$

where C is a strictly positive constant. Hence, we have completed the proof of (4.1).

To construct the radius r_1, we may assume that the boundary $\partial(\varphi^{-1}(D_{\mathrm{left}}))$ near 0 is the graph of equation $z_1 = f(z_2)$, defined on I, as follows:

$$\begin{cases} f(z_2) + z_2 i \in \partial(\varphi^{-1}(D_{\mathrm{left}})) \quad \text{for} \quad z_2 \in I,\\ f(0) = -1 \quad \text{and} \quad f'(0) = 0,\end{cases}$$

where I is a sufficient small open interval containing 0. Owing to $\varphi \in C^2(\mathbb{C}\backslash B_1(0))$ and the strict convexity of D_{right} at 0, we obtain

$$\frac{d^2 f}{d{z_2}^2}(0) < \left[-\frac{d^2\sqrt{1 - z_2^2}}{d{z_2}^2}\right]_{z_2=0}.$$

Hence, one can choose $R_0 > 0$ such that

$$\Phi_0^{-1}\left(\cup_{0\le\epsilon\le 2} D_2(\epsilon)\right) \subset B_{R_0}(-1 + R_0) \subset B_1(0).$$

It may be noted that

$$\begin{cases} \Phi_\epsilon^{-1}(D_2(\epsilon)) = \Phi_0^{-1}(D_2(2\epsilon)) \subset B_{R_0}(-1 + R_0) \quad \text{for } \epsilon < 1,\\ R_0 < 1 \quad \text{and} \quad -1 \in \partial B_{R_0}(-1 + R_0).\end{cases} \tag{4.3}$$

Let $\mathbf{S} = \{z \in B_1(0) \mid \Phi_\epsilon(z) \in \partial D_1(\epsilon) \text{ for some } \epsilon \in [0, 1]\}$. Then there is a neighborhood N_1 of 0 such that for each $z \in \mathbf{S}\cap N_1$, ϵ is uniquely determined by $\Phi_\epsilon(z) \in \partial D_1(\epsilon)$, and for each ϵ, the boundary $\partial\Phi_\epsilon^{-1}(D_1(\epsilon))$ is connected in N_1. And we define $\kappa(z)$ as the curvature of $\partial\Phi_\epsilon^{-1}(D_1(\epsilon))$ at $z \in \mathbf{S}\cap N_1$. By the continuity of $\kappa(z)$ and $\kappa(0) > R_0^{-1}$, there is a neighborhood N_2 of 0 such that

$$\begin{cases} \frac{2}{R_0+1} < \kappa(z) \quad \text{for } z \in \mathbf{S} \cap N_2,\\ \text{the boundary} \ \ \partial\Phi_\epsilon^{-1}(D_1(\epsilon)) \text{ is connected in } N_2.\end{cases} \tag{4.4}$$

Let $r_1 = \frac{R_0+1}{2}$. Then we have

$$\cup_{0\le\epsilon\le 1}\Phi_\epsilon^{-1}\left(D_1(\epsilon)\right) \subset B_{R_0}(-1 + R_0) \subset B_{r_1}[-1]$$

and can choose a small constant $\delta > 0$ such that

$$\begin{cases} \cup_{0\le\epsilon\le 1}\, \Phi_\epsilon^{-1}\left(D_1(\epsilon)\right) \subset B_{r_1}[z] \cup N_2 \quad \text{for any } z \in B_\delta(-1) \cap B_1(0),\\ B_\delta(-1) \cap B_1(0) \subset N_2.\end{cases}$$

By (4.3) and the relation of $\partial B_1(0)$ and $B_{R_0}(-1 + R_0)$, we can obtain a sufficiently small constant $\epsilon_0 > 0$ such that

$$z_\epsilon \in B_\delta(-1) \quad \text{for each } \epsilon < \epsilon_0,$$

where z_ϵ is mentioned above. It follows that

$$\begin{aligned}\Phi_\epsilon^{-1}(D_1(\epsilon)) &\subset (\Phi_\epsilon^{-1}(D_1(\epsilon)) \cap N_2) \cup (\Phi_\epsilon^{-1}(D_1(\epsilon)) \cap N_2^c)\\ &\subset (\Phi_\epsilon^{-1}(D_1(\epsilon)) \cap N_2) \cup B_{r_1}[z_\epsilon]\\ \text{by (4.4)} &\subset B_{r_1}[z_\epsilon].\end{aligned}$$

This means that r_1 is the desirable radius.

To choose the radius r_2, we define $\kappa : [0, \epsilon_0] \times \partial D_{\text{left}} \to \mathbb{R}^+$ by

$$\kappa(\epsilon, z) = \text{the curvature of } \Phi_\epsilon^{-1}(\partial D_2(\epsilon)) \ \text{ at } \Phi_\epsilon^{-1}\left(z - \frac{\epsilon}{2}\right).$$

We set $r_2 = (\sup\{\kappa(\epsilon, z) \mid (\epsilon, z) \in [0, \epsilon_0] \times \partial D_{\text{left}}\})^{-1}$. It is easy to prove that

$$B_{r_2}[z_\epsilon] \subset \Phi_\epsilon^{-1}(D_2(\epsilon)) \ \text{ for } \epsilon < \epsilon_0.$$

Therefore we have completed the proof. □

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REFERENCES

[1] L. Ahlfors, *Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1979.

[2] H. Ammari, H. Kang, and M. Lim, *Gradient estimates for solutions to the conductivity problem*, Math. Ann., 332 (2005), pp. 277–286.

[3] H. Ammari, H. Kang, E. Kim, and M. Lim, *Reconstruction of closely spaced small inclusions*, SIAM J. Numer. Anal., 42 (2005), pp. 2408–2428.

[4] H. Ammari, H. Kang, H. Lee, J. Lee, and M. Lim, *Optimal Estimates for the Electric Field in Two Dimensions*, preprint, 2006; available online from http://www.arxiv.org/abs/math.AP/0610653.

[5] I. Babuška, B. Andersson, P. Smith, and K. Levin, *Damage analysis of fiber composites.* I. *Statistical analysis on fiber scale*, Comput. Methods Appl. Mech. Engrg., 172 (1999), pp. 27–77.

[6] M. F. Ben Hensen and E. Bonnetier, *Asymptotic formulas for the voltage potential in a composite medium containing closer or touching disks of small diameter*, Multiscale Model. Simul., 4 (2005), pp. 250–277.

[7] E. Bonnetier and M. Vogelius, *An elliptic regularity result for a composite medium with "touching" fibers of circular cross-section*, SIAM J. Math. Anal., 31 (2000), pp. 651–677.

[8] B. Budiansky and G. F. Carrier, *High shear stresses in stiff-fiber composites*, J. Appl. Mech., 51 (1984), pp. 733–735.

[9] J. G. Goree and A. B. Wilson, Jr., *Transverse shear loading in an elastic matrix containing two circular cylindrical inclusions*, Trans. ASEM J. Appl. Mech., 34 (1967), pp. 511–513.

[10] J. D. Jackson, *Classical Electrodynamics*, 3rd ed., Wiley, New York, 1999.

[11] H. Kang and J. K. Seo, *Layer potential technique for the inverse problems*, Inverse Problems, 12 (1996), pp. 267–278.

[12] J. B. Keller, *Stresses in narrow regions*, Trans. ASME J. Appl. Mech., 60 (1993), pp. 1054–1056.

[13] J. B. Keller, *Conductivity or a medium containing a dense array of perfectly conducting spheres or cylinders or nonconducting cylinders*, J. Appl. Phys., 34 (1963), pp. 991–993.

[14] Y. Y. Li and M. Nirenberg, *Estimates for elliptic systems from composite material*, Comm. Pure Appl. Math., 56 (2003), pp. 892–925.

[15] Y. Y. Li and M. Vogelius, *Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients*, Arch. Ration. Mech. Anal., 153 (2000), pp. 91–151.
[16] M. Vogelius and D. Volkov, *Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities*, M2AN Math. Model. Numer. Anal., 34 (2000), pp. 723–748.