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Optimal bound on high stresses occurring between stiff fibers with arbitrary shaped cross-sections ☆

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ABSTRACT

We consider high stresses in stiff-fiber reinforced materials, which increase rapidly as fibers approximate to one another. This paper presents the optimal blow-up rate of the stresses with respect to the distance between a pair of stiff fibers in $\mathbb{R}^3$. The blow-up result plays an important role in our understanding of low strengths of fiber-reinforced composites. Referring to a problem of anti-plane shear, the stresses can be interpreted as the electric fields outside closely spaced perfect conductors in $\mathbb{R}^2$, under the action of applied electric field ∇H. It has been shown by Ammari, Kang et al. that in the particular case of circular inclusions, the electric field blows up at the optimal rate $\epsilon^{-1/2}$ as $\epsilon \to 0$, where ϵ is the distance between conductors. Recently, Yun has extended the blow-up result to pairs of conductors associated with a large class of shapes whose complements can be transformed conformally to the outside of a circle with C^2 mapping. However, it presented a suboptimal result that only for a special uniform field $\nabla H = (1, 0)$, the electric fields blow up at the exact rate $\epsilon^{-1/2}$. In this paper, an upper bound with the rate $\epsilon^{-1/2}$ of electric field for any harmonic function H is established. This yields the optimal blow-up rate $\epsilon^{-1/2}$ for the inclusions in the same class of shapes as Yun.

1. Introduction

This paper is concerned with high stress concentrations in stiff-fiber reinforced composites, which causes low strengths of materials. There is a strong correlation between the stress and the distance among fibers such that the stress becomes arbitrarily large as the distance approaches zero; refer to Budiansky and Carrier [7]. In this paper, we present the optimal blow-up rate of the stress with respect to the distance between a pair of stiff fibers closely spaced in an infinite matrix.

We suppose that D_1 and D_2 are closely spaced inclusions in $\mathbb{R}^2$ which are ϵ apart, representing the cross-sections of two parallel elastic fibers in infinite elastic matrix, and the shear moduli of the inclusions are the constants a_1 and a_2, distinguished from the outside shear modulus 1. Referring to a problem of anti-plane shear, for a given harmonic function H in $\mathbb{R}^2$, the out-of-plane elastic displacement u satisfies the following conductivity equation

$$\begin{cases} \nabla \cdot \left\{ \left(1 + \sum_{i=1,2} (a_i - 1) \chi(D_i) \right) \nabla u \right\} = 0, \\ u(\mathbf{x}) - H(\mathbf{x}) = O\left(|\mathbf{x}|^{-1}\right) \quad \text{as } |\mathbf{x}| \to \infty. \end{cases} \tag{1}$$

Here, the gradient of potential ∇u represents the stress.

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The purpose of this paper is to establish an upper bound on $|\nabla u|$ for small distance $\epsilon > 0$, which would indicate the optimal blow-up rate of ∇u with respect to ϵ. The conductivity models such as (1) have been used for the mathematical studies of stress in fiber-reinforced compositions (see [4,5,7–9]). For the case that the inclusions have finite shear moduli a_1 and a_2, with the effort of Bonnetier and Vogelius [6] as the beginning point, it has been shown by Li and Vogelius [12] that the stress ∇u does not blow up even though the distance ϵ goes to 0. Moreover, the nonblow-up result has been extended to elliptic systems by Li and Nirenberg in [11].

However, the main interest of our work lies in the blow-up of stresses (or electric fields). In this respect, we consider the case of extreme valued shear moduli $a_1 = a_2 = \infty$. For two circular inclusions, Ammari, Kang and Lim et al. have shown in [2,3] that $|\nabla(u - H)|$ blows up at the optimal rate $\epsilon^{-1/2}$ as the distance $\epsilon \to 0$. Here, $|\nabla(u - H)|$ was estimated instead of $|\nabla u|$, because $|\nabla H|$ can be unbounded.

Recently, Yun [13] has extended the blow-up result originally known only for circular inclusions to inclusions associated with a sufficiently general class of shapes whose complements can be transformed conformally to the outside of a circle with C^2 mapping (refer to the next section). More precisely, it has been shown in [13] that $|\nabla(u - H)|$ blows up at the exact rate $\epsilon^{-1/2}$ only for a special harmonic function $H(x_1, x_2) = x_1$ when D_1 and D_2 are separated by the line $x_1 = 0$ and $\mathbf{x} = (x_1, x_2)$.

However, the method used in [13] is not applicable to the case of $\partial_{x_2} H(0,0) \neq 0$, especially when $H = x_2$. The reason is because it is based on the integral identity (8) for $u|_{D_1} - u|_{D_2}$ in Lemma 2 and in the case of $\partial_{x_2} H(0,0) \neq 0$, two integral values of (8) can be too large to get the blow-up rate of $\epsilon^{-1/2}$. After all, the result for a special function $H = x_1$ in [13] indicates only the existence of the blow-up at the rate of $\epsilon^{-1/2}$, but does not mean that the optimal rate is $\epsilon^{-1/2}$.

In this paper, the method used in [13] is modified for the general harmonic function H even with $\partial_{x_2} H(0,0) \neq 0$. Based on this, an upper bound of $|\nabla(u - H)|$ for an arbitrary entire harmonic function H is established in terms of ϵ where the inclusions are also associated with the same class of shapes as Yun [13]. It leads to an optimal conclusion that the blow-up rate of $|\nabla(u - H)|$ is exactly $\epsilon^{-1/2}$, which has been known only for circular inclusions [2,3].

2. Governing equation and the main result

We now make the notations and assumptions more precise. To define a pair of inclusions D_1 and D_2 approaching each other, we consider two domains D_{right} and D_{left} in $\mathbb{R}^+ \times \mathbb{R}$ and $\mathbb{R}^- \times \mathbb{R}$, respectively, which are strictly convex at the unique left (or right) endpoint $(0,0)$ of these domains. In addition, we assume that $\varphi_{\text{right}} : \mathbb{C} \setminus B_1(0) \to \mathbb{R}^2 \setminus D_{\text{right}}$ and $\varphi_{\text{left}} : \mathbb{C} \setminus B_1(0) \to \mathbb{R}^2 \setminus D_{\text{left}}$ are conformal mappings such that $\varphi_{\text{right}}, \varphi_{\text{left}} \in C^2(\mathbb{C} \setminus B_1(0))$, $\varphi'_{\text{right}}(z) \neq 0$ and $\varphi'_{\text{left}}(z) \neq 0$ for $z \in \partial B_1(0)$. For convenience, we will not distinguish $\mathbb{R}^2$ from $\mathbb{C}$ for the rest of this paper. Let the domains D_1 and D_2 be as follows

$$D_1 = D_{\text{right}} + \frac{1}{2}\epsilon \quad \text{and} \quad D_2 = D_{\text{left}} - \frac{1}{2}\epsilon,$$

and

$$\varphi_1 := \varphi_{\text{right}} + \frac{\epsilon}{2} \quad \text{and} \quad \varphi_2 := \varphi_{\text{left}} - \frac{\epsilon}{2}.$$

The C^2 regularity condition of these conformal mappings doses not allow non-smooth inclusions such as polygons, but Riemann mapping theorem yields a sufficiently general class of shapes: refer to Ahlfors [1].

In the case of the extreme valued shear moduli $a_1 = a_2 = \infty$, given any harmonic function H in $\mathbb{R}^2$, Eq. (1) can be rewritten as follows

$$\begin{cases} \triangle u = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)}, \\ u(\mathbf{x}) - H(\mathbf{x}) = O\left(|\mathbf{x}|^{-1}\right) \quad \text{as } |\mathbf{x}| \to \infty, \\ u|_{\partial D_i} = C_i \text{ (constant)} \quad \text{and} \\ \displaystyle\int_{\partial D_i} \partial_\nu u \, ds = 0 \quad \text{for } i = 1, 2, \end{cases} \tag{2}$$

where $\partial_\nu u$ is the normal derivative of u.

Theorem 2.1. *Let H be an entire harmonic function and u be the solution to Eq.* (2)*. If the distance ϵ is sufficiently small, then we have a constant C^* independent of ϵ such that*

$$\left\|\nabla(u - H)\right\|_{L^\infty(\mathbb{R}^2 \setminus (D_1 \cup D_2))} \leqslant C^* \frac{1}{\sqrt{\epsilon}}. \tag{3}$$

Remark 2.2. It has been shown by Yun in [13] that the blow-up rate is not less than $\epsilon^{-1/2}$. This means that Theorem 2.1 provides the optimal blow-up rate $\epsilon^{-1/2}$.

The solution u to (2) can also be interpreted as the voltage outside perfect conductors D_1 and D_2 under the action of applied electric field ∇H; see Jackson [10]. We now consider the voltage v outside perfect nonconductors D_1 and D_2 under the action of applied electric field $\nabla\widetilde{H}$. Then the voltage v is the unique solution to the following Neumann problem

$$\begin{cases} \triangle v = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)}, \\ v(\mathbf{x}) - \widetilde{H}(\mathbf{x}) = O\big(|\mathbf{x}|^{-1}\big) \quad \text{as } |\mathbf{x}| \to \infty, \\ \partial_\nu v = 0 \quad \text{on } \partial D_i \text{ for } i = 1, 2, \end{cases} \tag{4}$$

where $\partial_\nu v$ is the normal derivative of v. The optimal estimate for $|\nabla v|$ is also derived by Theorem 2.1. We find an entire harmonic conjugate function H of $\widetilde{H}$. We consider the solution u to (3) for H. Then u can be decomposed into U_1 and U_2 such that

$$u = H + U_1 + U_2 \tag{5}$$

and U_i is defined in $\mathbb{R}^2 \setminus D_i$ for $i = 1, 2$; for more details, see the proof of theorem below. By Poincaré's Theorem, we have a conjugate harmonic function v of u. Drawing from the Cauchy–Riemann equations, v is the desirable solution to (4), and we present the following corollary.

Corollary 1. *Assume the above. If the distance ϵ is sufficiently small, then we have a constant C^* independent of ϵ such that*

$$\big\|\nabla(v - \widetilde{H})\big\|_{L^\infty(\mathbb{R}^2 \setminus (D_1 \cup D_2))} \leqslant C^* \frac{1}{\sqrt{\epsilon}}.$$

3. Proof of Theorem 2.1

As mentioned above, we can decompose the solution u to (2) into three functions H, U_1 and U_2 such that U_i is a harmonic function defined in $\mathbb{R}^2 \setminus D_i$ and satisfies $\lim_{\mathbf{x}\to\infty} U_i = 0$ for $i = 1, 2$. The decomposition (5) can be derived as follows. In electrostatics, the voltage $u - H$ is actually determined by the charge distributions on the conductors D_1 and D_2 under the action of applied field ∇H. In this respect, if U_i is the potential only due to the distribution on D_i ($i = 1, 2$), then we can obtain the desirable decomposition

$$u - H = U_1 + U_2. \tag{6}$$

However, this paper gives a mathematical explanation of the decomposition (6). We choose an interior point p_0 of D_2. Let the conformal mapping $T(x_1, x_2) := \frac{1}{(x_1 + x_2 i - p_0)}$. Then we can assume that $\{z \mid R_1 < |z| < R_2\} \subset T(\mathbb{R}^2 \setminus (D_1 \cup D_2))$ and $T(D_1) \subset \{z \mid |z| < R_1\}$. Since $\int_{\partial D_i} \partial_\nu u = 0$ for $i = 1, 2$, we have

$$(u - H)T^{-1}(z) = \gamma_0 + \sum_{n=1}^{\infty} \gamma_n(\theta) r^n + \sum_{n=1}^{\infty} \widetilde{\gamma}_n(\theta) \frac{1}{r^n} \quad \text{for } R_1 < |z| < R_2$$

where $\arg z = \theta$ and $r = |z|$. Since $(u - H)T^{-1}(z)$ and $\sum_{n=1}^{\infty} \widetilde{\gamma}_n(\theta) \frac{1}{r^n}$ are well defined in $\mathbb{R}^2 \setminus (T^{-1}(D_2) \cup B_{R_1}(0))$, $\gamma_0 + \sum_{n=1}^{\infty} \gamma_n(\theta) r^n$ can be extended harmonically to $\mathbb{R}^2 \setminus T(D_2)$, and similarly $\sum_{n=1}^{\infty} \widetilde{\gamma}_n(\theta) \frac{1}{r^n}$ is also extended to $\mathbb{R}^2 \setminus T(D_1)$. Therefore, we can obtain the decomposition (6) above.

We now introduce the solution w to the following problem

$$\begin{cases} \triangle w = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)}, \\ w(\mathbf{x}) = O\big(|\mathbf{x}|^{-1}\big) \quad \text{as } |\mathbf{x}| \to \infty, \\ w|_{\partial D_1} = c_{1\epsilon} \text{ (constant)}, \\ w|_{\partial D_2} = c_{2\epsilon} \text{ (constant)} \quad \text{and} \\ \displaystyle\int_{\partial D_1} \partial_\nu w \, ds = -\int_{\partial D_2} \partial_\nu w \, ds = 1. \end{cases} \tag{7}$$

Drawing from Hopf's Lemma, $\partial_\nu w$ is either positive or negative on each boundary. Thus, $(-1)^{i+1}\partial_\nu w$ can be a probability density function on each boundary ∂D_i. It follows from the equality (6) above that

$$\int_{\partial D_1} \partial_\nu (u - H) \, dS = \int_{\partial D_2} \partial_\nu (u - H) \, dS = 0.$$

Then we have

$$\begin{aligned} 0 &= \int_{\partial D_1} (\partial_\nu w)(u - H) \, dS + \int_{\partial D_2} (\partial_\nu w)(u - H) \, dS \\ &= (u|_{\partial D_1} - u|_{\partial D_2}) - \int_{\partial D_1} (\partial_\nu w) H \, dS - \int_{\partial D_2} (\partial_\nu w) H \, dS. \end{aligned}$$

Therefore we obtain the following lemma.

Lemma 2. *(See Yun [13].) Assume the above.*

$$u|_{\partial D_1} - u|_{\partial D_2} = \int\limits_{\partial D_1} (\partial_\nu w) H\, dS + \int\limits_{\partial D_2} (\partial_\nu w) H\, dS. \tag{8}$$

This equality (8) is originally from Lemma 2.3 by Yun in [13].

Remark 3.1. Differently from the original lemma in [13], it can be pointed out that even though H is not defined in the interiors of D_1 or D_2, the solution u to the main equation (2) is well defined, because we use only the Dirichlet data of H on $\partial D_1 \cup \partial D_2$ to get the solution u. Furthermore, it follows from the same derivation as Lemma 2 that the equality (8) holds, even though H is not defined in the interiors of D_1 or D_2.

For convenience, we assume from now on that $\varphi_{\mathrm{right}}(-1) = 0$; that is, $\varphi_1(-1) = \frac{1}{2}\epsilon$. Then it has been shown in [13] that

$$0 \leqslant \partial_{\nu_{(z)}} w\big(\varphi_1(z)\big) \leqslant C P(p_\epsilon, z) \quad \text{for } z \in \partial B_1(0) \tag{9}$$

where $P(\mathbf{x}, \mathbf{y})$ is a Poisson kernel and $p_\epsilon = -1 + O(\sqrt{\epsilon}) \in B_1(0)$ as $\epsilon \to 0$. Based on this estimate, we get the following two lemmas that were proved as a byproduct of Theorem 1.2 in [13].

Lemma 3.2. *We assume $|H(\mathbf{x})| \leqslant |x_1|$ and H need not be defined in the interior of $D_1 \cup D_2$. Let u be the solution to (2). If ϵ is sufficiently small, then there is a constant C_0 independent of ϵ such that*

$$|u|_{\partial D_1} - u|_{\partial D_2}| \leqslant C_0 \sqrt{\epsilon}.$$

This estimate can be derived by using the Poisson kernel or direct integration (8) as follows

$$\begin{aligned} \int\limits_{\partial B_1(0)} P(p_\epsilon, z) H\big(\varphi_1(z)\big)\, ds(z) &= \int\limits_{\partial B_1(0) \text{ and } |z+1| \leqslant \sqrt[4]{\epsilon}} P(p_\epsilon, z) H\big(\varphi_1(z)\big)\, ds(z) \\ &\quad + \int\limits_{\partial B_1(0) \text{ and } |z+1| > \sqrt[4]{\epsilon}} P(p_\epsilon, z) H\big(\varphi_1(z)\big)\, ds(z). \end{aligned}$$

Please, refer to Remark 3.1 in [13] for details.

Lemma 3.3. *Let W be the bounded harmonic function with $W|_{\partial D_1} = 1$ and $W|_{\partial D_2} = -1$ that is defined in $\mathbb{R}^2 \setminus (D_1 \cup D_2)$, we then have*

$$\|\nabla W\|_{L^\infty(\mathbb{R}^2 \setminus (D_1 \cup D_2))} \leqslant C_1 \epsilon^{-1} \tag{10}$$

where the constant C_1 is independent of sufficiently small ϵ.

This inequality (10) can be derived from the proof of the main theorem in Yun [13]. It follows from (2.12) and Lemma 2.4 in [13] that

$$\big|\partial_{\nu_z}\big(W\big(\varphi_1(z)\big)\big)\big| \leqslant C \big|\sqrt{\epsilon} \log\big(1 - \alpha\sqrt{\epsilon} + o(\sqrt{\epsilon})\big)\big|^{-1} \quad \text{on } \partial B_1(0)$$

where α and C are non-zero constants independent of ϵ. By the non-zero condition of $\varphi_1'(z)$ on $\partial B_1(0)$ and the maximum principle for ∇W, we can obtain Lemma 3.3.

Without loss of generality, we may assume that

$$H(\mathbf{x}) = a_1 x_1 + a_2 x_2 + O\big(|\mathbf{x}|^2\big) \quad \text{as } \mathbf{x} \to 0$$

and

$$B_1\left(1 + \frac{\epsilon}{2}\right) \subset D_1.$$

Remark 3.4. Speaking of the difference between this paper and Yun [13], the method used in [13] cannot be applied to the case of a harmonic function H with $\partial_{x_2} H(0) \neq 0$, that is, $a_2 \neq 0$. The integral expression for $u|_{D_1} - u|_{D_2}$ in Lemma 2 and the bound in Lemma 3.2 are essential for the method in [13]. In the case of $a_2 \neq 0$, because $|x_2|$ is comparable to $\sqrt{|x_1|}$ up to ϵ for any small $|(x_1, x_2)|$ on $\partial D_1 \cup \partial D_2$, the integral values in Lemma 2 may be too large to get the estimate for $u|_{D_1} - u|_{D_2}$ with order $\sqrt{\epsilon}$ such as Lemma 3.2, which implies the gradient estimate with order $\epsilon^{-1/2}$.

In this paper, we solve the problem of $a_2 \neq 0$ by neutralizing $a_2 x_2$ with

$$h_\epsilon := -a_2 \frac{x_2}{(x_1 - 1 - \frac{\epsilon}{2})^2 + x_2^2}.$$

If ϵ is sufficiently small, then we have a constant M independent of ϵ such that

$$\left|(H + h_\epsilon)(x_1, x_2)\right| \leqslant M|x_1| \quad \text{for } (x_1, x_2) \in \partial D_1 \cup \partial D_2 \tag{11}$$

and

$$\|\nabla h_\epsilon\|_{L^\infty(\mathbb{R}^2 \setminus (D_1 \cup D_2))} \leqslant M. \tag{12}$$

We now decompose u into four parts as follows

$$\begin{aligned} u &= (H + h_\epsilon) + (u - H - h_\epsilon) \\ &= (H + h_\epsilon) + u_0 + u_1 + u_2 \end{aligned} \tag{13}$$

where for $i = 0, 1, 2$, u_i is the bounded harmonic function in $\mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)}$ with the boundary conditions

$$\begin{cases} u_0 = u \text{ (two constants)} \quad \text{on } \partial D_1 \cup \partial D_2, \\ u_1 = -H - h_\epsilon \quad \text{on } \partial D_1 \quad \text{and} \quad u_1 = 0 \quad \text{on } \partial D_2, \\ u_2 = -H - h_\epsilon \quad \text{on } \partial D_2 \quad \text{and} \quad u_2 = 0 \quad \text{on } \partial D_1. \end{cases}$$

We will estimate u_0, u_1 and u_2 separately.

Estimate for u_0. We have $u = (H + h_\epsilon) + (u - H - h_\epsilon)$. Then $(u - H - h_\epsilon)$ is a bounded harmonic function in $\mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)}$ with the infinite behavior $(u - H - h_\epsilon)(\mathbf{x}) = O(|\mathbf{x}|^{-1})$ as $|\mathbf{x}| \to \infty$. Thus, u can be the solution to (2) under the action of applied $(H + h_\epsilon)$ instead of H. Moreover, by (11), the applied $(H + h_\epsilon)$ is bounded by $M|x_1|$. Lemma 3.2 thus yields to

$$|u_0|_{\partial D_1} - u_0|_{\partial D_2}| = |u|_{\partial D_1} - u|_{\partial D_2}| \leqslant C\sqrt{\epsilon}.$$

Comparing u_0 and W in Lemma 3.3, there are two constant coefficients c_1 and c_2 such that $u_0 = c_1 W + c_2$ and $c_1 = O(\sqrt{\epsilon})$ as $\epsilon \to 0$. Owing to the bound of W in Lemma 3.3, we obtain

$$\|\nabla u_0\|_{\mathbb{R}^2 \setminus (D_1 \cup D_2)} \leqslant C \frac{1}{\sqrt{\epsilon}} \tag{14}$$

for any sufficiently small $\epsilon > 0$.

Estimate for u_1. We consider the conformal mapping $\varphi_2 : \mathbb{C} \setminus B_1(0) \to \mathbb{R}^2 \setminus D_2$ defined in the previous section. From definition, $u_1(\varphi_2(z)) = 0$ on $\partial B_1(0)$ and $u_1(\varphi_2(z))$ is defined in $\mathbb{C} \setminus (B_1(0) \cup \varphi_2^{-1}(D_1))$. By the Kelvin transform with respect to $\partial B_1(0)$, $u_1(\varphi_2(z))$ can be extended harmonically to $\mathbb{C} \setminus (\mathbf{B} \cup \varphi_2^{-1}(D_1))$ as follows

$$\begin{cases} u_1\big(\varphi_2(z)\big) & \text{for } z \in \mathbb{C} \setminus \big(B_1(0) \cup \varphi_2^{-1}(D_1)\big), \\ -u_1\left(\varphi_2\left(\frac{z}{|z|^2}\right)\right) & \text{for } z \in B_1(0) \setminus \mathbf{B}, \\ \lim\limits_{\mathbf{x} \to \infty} -u_1(\mathbf{x}) & \text{for } z = 0 \end{cases}$$

where $\mathbf{B} = \{z \mid \frac{z}{|z|^2} \in \varphi_2^{-1}(D_1)\}$. By definition, the extended $u_1(\varphi_2(z))$ above is an odd function of $r = |z|$ with respect to $|z| = 1$ in $\mathbb{C} \setminus (\varphi_2^{-1}(D_1) \cup \mathbf{B})$. The maximum principle for the analytic function $\overline{(\nabla u_1)}$ thus yields the inequality

$$\begin{aligned} \|\nabla u_1\|_{L^\infty(\mathbb{R}^2 \setminus (D_1 \cup D_2))} &\leqslant C\big(\|\nabla u_1\|_{L^\infty(\partial D_1)} + \|\nabla u_1\|_{L^\infty(\partial D_2)}\big) \\ &\leqslant C\|\nabla u_1\|_{L^\infty(\partial D_1)} \quad \text{by the odd function of } |z|, \end{aligned} \tag{15}$$

where $\overline{\nabla u_1}$ means the complex conjugate of ∇u_1.

We now estimate $\|\partial_\nu u_1\|_{L^\infty(\partial D_1)}$ instead of $\|\nabla u_1\|_{L^\infty(\partial D_1)}$, because the tangential derivative of u_1 is dominated by the given $-H + h_\epsilon$ on ∂D_1 without any blow-up phenomenon. To do so, without loss of generality, we may assume that

$$B_{r_2}(-1 - \epsilon - r_2) \subset \varphi_1^{-1}(D_2) \subset B_{r_1}(-1 - \epsilon - r_1),$$

and by the bound (11), we have

$$0 \leqslant M \cdot \Re\big(\varphi_1(z_1, z_2)\big) + u_1\big(\varphi_1(z_1, z_2)\big) \quad \text{on } \partial B_1(0)$$

where r_1, r_2 and M are independent of sufficiently small ϵ, and $\Re(x_1, x_2) = x_1$. Let μ be the bounded harmonic function in $\mathbb{C} \setminus (B_1(0) \cup \varphi_1^{-1}(D_2))$ with the boundary data

$$\begin{cases} \mu = M \cdot \Re(\varphi_1) & \text{on } \partial B_1(0), \\ \mu = 0 & \text{on } \partial\varphi_1^{-1}(D_2). \end{cases}$$

This implies that $\mu(\varphi^{-1})$ is a bounded harmonic function as follows:

$$\begin{cases} \triangle\mu(\varphi^{-1}) = 0 & \text{in } \mathbb{R}^2 \setminus \overline{D_1 \cup D_2}, \\ \mu(\varphi^{-1}) = Mx_1 & \text{on } \partial D_1, \\ \mu(\varphi^{-1}) = 0 & \text{on } \partial D_2. \end{cases}$$

In [13], a gradient estimate for the solution under applied $H(x_1, x_2) = x_1$ has been established. In the derivation, the bound (3.7) of [13] means

$$\left\|\nabla\big(\mu(\varphi_1^{-1})\big)\right\|_{L^\infty(\mathbb{R}^2/(D_1 \cup D_2))} \leqslant C\left(\frac{1}{\sqrt{\epsilon}}\right), \tag{16}$$

where $\mu(\varphi_1^{-1})$ is the harmonic function u_1 defined in [13] which is different from u_1 in our paper.

Meanwhile, we consider the bounded harmonic functions V_1 and V_2 as follows

$$\begin{cases} \triangle V_i = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\big(B_1(0) \cup B_{r_i}(-1-\epsilon-r_i)\big)}, \\ V_i(z) = \mu(z) + u_1\big(\varphi_1(z)\big) = M \cdot \Re\big(\varphi_1(z)\big) - (H + h_\epsilon)\big(\varphi_1(z)\big) \quad \text{for } z \in \partial B_1(0), \\ V_i(z) = 0 \quad \text{for } z \in \partial B_{r_i}(-1-\epsilon-r_i) \end{cases}$$

for $i = 1, 2$. Since $\mu(z) + u_1(\varphi_1(z)) \geqslant 0$ for $z \in \partial B_1(0)$, it follows from Hopf's Lemma that

$$\partial_\nu V_2(z) \leqslant \big(\partial_{\nu(z)}\mu + \partial_{\nu(z)}u_1\big(\varphi_1(z)\big)\big) \leqslant \partial_\nu V_1(z) \quad \text{for } z \in \partial B_1(0). \tag{17}$$

We now estimate $\partial_\nu V_1(z)$ and $\partial_\nu V_2(z)$. Let Y be the bounded harmonic function in $\mathbb{R}^2 \setminus \overline{B_{r_1}(-1-\epsilon-r_1)}$ with the boundary data

$$Y(z) = \big(M \cdot \Re\big(\varphi_1(z)\big) - H\big(\varphi_1(z)\big) - h_\epsilon\big(\varphi_1(z)\big)\big) \quad \text{on } \partial B_{r_1}(-1-\epsilon-r_1). \tag{18}$$

We consider the Kelvin transforms $K_1 : C^\infty(\mathbb{R}^2 \setminus \overline{B_{r_1}(-1-\epsilon-r_1)}) \to C^\infty(\mathbb{R}^2 \setminus \overline{B_1(0)})$ and $K_2 : C^\infty(\mathbb{R}^2 \setminus \overline{B_1(0)}) \to C^\infty(\mathbb{R}^2 \setminus \overline{B_{r_1}(-1-\epsilon-r_1)})$ as follows

$$K_1(v)(z) = v\left(\frac{z}{|z|^2}\right),$$

$$K_2(v)(z) = v\left(\frac{z+1+\epsilon+r_1}{|z+1+\epsilon+r_1|^2} - 1 - \epsilon - r_1\right).$$

Then we consider the harmonic function U on $\mathbb{R}^2 \setminus (B_{r_1}(-1-\epsilon-r_1) \cup B_1(0))$ as follows

$$U(z) = \sum_{n=0}^{\infty}(I - K_2)(K_1K_2)^n\big(Y(z)\big).$$

By a standard calculation or referring to [2,3,13], we have

$$\begin{cases} \triangle U = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\big(B_{r_1}(-1-\epsilon-r_1) \cup B_1(0)\big)}, \\ U(z) = O(1) \quad \text{as } |z| \to \infty, \\ U = 0 \quad \text{for } z \in \partial B_{r_1}(-1-\epsilon-r_1), \\ \big(U - \big(M \cdot \Re\big(\varphi_1(z)\big) - H\big(\varphi_1(z)\big) - h_\epsilon\big(\varphi_1(z)\big)\big)\big)\big|_{B_1(0)} = \text{a constant with order } O(\sqrt{\epsilon}) \quad \text{as } \epsilon \to 0. \end{cases}$$

We observe that U is a solution to a conductivity problem with circular inclusions. It follows from the result of Ammari, Kang et al. in [2] that

$$\left\|\nabla U(z)\right\|_{L^\infty(\mathbb{R}^2 \setminus (B_{r_1}(-1-\epsilon-r_1) \cup B_1(0)))} \leqslant C\left(\frac{1}{\sqrt{\epsilon}}\right)\|\nabla Y\|_{L^\infty(\mathbb{R}^2 \setminus B_1(0))}. \tag{19}$$

To give a brief explanation of the proof of (19) in [2], the bound (19) can be derived by estimating $\sum_{n=0}^{N^*}(I - K_2)(K_1K_2)^n(Y(z))$ and the remainders separately, where $N^* = O(\epsilon^{-1/2})$ as $\epsilon \to 0$. For complete details, refer to their paper [2].

The change of small distance ϵ between D_1 and D_2 makes small variation of Y and ∇Y by the small change of $H(\varphi_1)$ in (18), due to the continuity result of the Dirichlet to Neumann map from $C^{1.\alpha}$ onto $C^{0.\alpha}$. Thus, $\|\nabla Y\|_{L^\infty(\mathbb{R}^2\setminus B_1(0))}$ can be bounded by a constant independent of small ϵ. In this respect, the bound (19) is reduced to

$$\|\nabla U\|_{L^\infty(\mathbb{R}^2\setminus(B_{r_1}(-1-\epsilon-r_1)\cup B_1(0)))} \leqslant C\left(\frac{1}{\sqrt{\epsilon}}\right). \tag{20}$$

Here, the remainder can be derived by the same argument as Yun [13]. For self-containedness, this paper includes the derivation. We have

$$\begin{cases} \triangle(V_1-U)=0 \quad \text{in } \mathbb{R}^2\setminus\overline{\big(B_{r_1}(-1-\epsilon-r_1)\cup B_1(0)\big)}, \\ (V_1-U)(z)=O(1) \quad \text{as } |z|\to\infty, \\ (V_1-U)(z)=0 \quad \text{for } z\in\partial B_{r_1}(-1-\epsilon-r_1), \\ (V_1-U)|_{\partial B_1(0)} = \text{a constant with order } O(\sqrt{\epsilon}) \quad \text{as } \epsilon\to 0. \end{cases}$$

Thus, V_1-U is a bounded harmonic function with constant boundary data of order $O(\sqrt{\epsilon})$. Comparing V_1-U and W in Lemma 3.2, we have

$$\big\|\nabla(V_1-U)\big\|_{L^\infty(\mathbb{R}^2\setminus(B_{r_1}(-1-\epsilon-r_1)\cup B_1(0)))} \leqslant C\left(\frac{1}{\sqrt{\epsilon}}\right).$$

By the bound (20) of U, this yields

$$\|\nabla V_1\|_{L^\infty(\mathbb{R}^2\setminus(B_{r_1}(-1-\epsilon-r_1)\cup B_1(0)))} \leqslant C\left(\frac{1}{\sqrt{\epsilon}}\right)$$

and similarly, we also have

$$\|\nabla V_2\|_{L^\infty(\mathbb{R}^2\setminus(B_{r_2}(-1-\epsilon-r_2)\cup B_1(0)))} \leqslant C\left(\frac{1}{\sqrt{\epsilon}}\right).$$

Applying the bounds above of V_i $(i=1,2)$ and (16) to (17), the bound (15) means that

$$\|\nabla u_1\|_{L^\infty(\mathbb{R}^2\setminus(D_1\cup D_2))} \leqslant C_\beta\left(\frac{1}{\sqrt{\epsilon}}\right)$$

where C_β is a constant independent of ϵ. The estimate for u_2 can be derived by the same way as u_1. Therefore, applying (14), (20) and (12) to the decomposition (13), we can obtain the desirable inequality (3).

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