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Research Statement

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Overview

My principal interest lies in studies strongly stimulated by practical applications. I am particularly attracted to interdisciplinary projects which require knowledge of many fields in order to produce results that can be applied in real life. I have endeavored to obtain meaningful information from mathematical models motivated by the increased importance of interdisciplinary research. My current research can be summarized in three themes as follows:

- **Identification of the unknown coefficients in partial differential equations:**
  The main portion of my research involves in the inverse problems to identify unknown coefficients in partial differential equations via boundary measurements. Collaborating with Professor Gang Bao, I recently have obtained an improved stability of an inverse wave problem in [1] and continue my efforts to attack another crucial inverse wave problem to identify the unknown wave speed coefficient. During my Ph.D. period, several remarkable results on various inverse problems could also be obtained under the supervision of Professor Hyeonbae Kang in [2, 3, 4].
- **High stress concentration occurring in stiff fiber reinforced materials:**
  The electric field increases toward infinity in the narrow region between closely adjacent perfect conductors as they approach each other. Much attention has been devoted to the blow-up estimate, especially in two dimensions, for the practical relevance to high stress concentration in fiber-reinforced elastic composites. In my recent papers [5, 6], I have established optimal estimates for the electric field in terms of the distance between a pair of conductors in two dimensions. More recently, in cooperation with Professor Mikyoung Lim [7, 8], two advanced estimates have been obtained to explicitly describe what role the geometric parameters of conductors play in blow-up.
- **Evolution of the grain boundaries in three dimensions:**
  In work with Professor David Kinderlehrer, the primary investigations concern mathematical issues of interfaces in polycrystals, a subject of great historical importance. The main objective is to show the existence of the solution to the mathematical system to describe the evolution of granular network in three dimensions. We have established a short time existence of the solution in three dimensions under the assumption that the system is initially close to the equilibrium configuration in [9].

In what follows, I will describe each of these areas and also my future plans in details.

1 Identification of the unknown coefficients in partial differential equations via boundary measurements

The inverse problems to identify unknown objects by boundary measurements have several potential applications such as reconstruction of the interior of the human body from exterior electrical and magnetic measurements, mine and rock detection, the search for underground water from exterior measurements in geophysics and the locating of flying objects from their acoustic or electromagnetic fields.

These works related to inverse problems are the longest continuous subjects that I have been working since my graduate study at Seoul National University. These research topics are not only interesting to me but have also provided me with valuable background that I need as a mathematician.

In studying these topics for a long time, I have been exposed to a wide range of inverse problems. This section focuses on the recent results and the ongoing research projects among my research findings on inverse problems.

1.1 Stabilities of inverse problems for wave equations

In my recent work with Professor Gang Bao [1], I have considered the following hyperbolic problem

$$\begin{cases} u_{tt} - \Delta u + qu = 0 & (x,t) \in \Omega \times (0,T) \\ u = u_t = 0 & x \in \Omega, t = 0 \\ \frac{\partial u}{\partial \nu} = f & (x,t) \in \partial\Omega \times (0,T) \end{cases} \tag{1.1}$$

Here, the potential function $q = q(x)$ above is unknown. The Neumann to Dirichlet map Λ_q is associated with (1.1) as $\Lambda_q : f \to u\big|_{\Gamma}$.

This work is mainly concerned with a stability of an inverse problem inverting the map $q \mapsto \Lambda_q$, which was initiated by Rakesh and Symes. In their paper [10], the technique inspired by Sylverster and Uhlmann [11] was used to prove the uniqueness theorem that q can be uniquely determined by Λ_q. Furthermore, Sun [12] modified their method to get a Hölder continuity with exponent $1/3 - \epsilon$ for the inverse problem as follows: For any small $\epsilon > 0$, there is $\beta_0 > 0$ such that

$$\|q_1 - q_2\|_{L^2(\Omega)} \leq C\|\Lambda_{q_1} - \Lambda_{q_2}\|_*^{\frac{1}{3}-\epsilon}$$

when $\|q_1 - q_2\|_{H^\beta(\mathbb{R}^n)} \leq M$ for some $\beta > \beta_0$. Here, the norm $\|\cdot\|_*$ represents the operator norm $\|\cdot\|_{B(L^2(\Gamma),H^1(\Gamma))}$ and $H^\beta(\mathbb{R}^n)$ is the Sobolev Space of order β.

In our work [1], an effective way to use the same condition as Sun has been obtained, which has been motivated by the optimal mass transport. Based on this, a Hölder stability with exponent $1-\epsilon$ is established under essentially the same assumption as Sun [12]: For any small $\epsilon > 0$, there is $\beta_0 > 0$ such that

$$\|q_1 - q_2\|_{L^2(\Omega)} \leq C\|\Lambda_{q_1} - \Lambda_{q_2}\|_*^{1-\epsilon}$$

when $\|q_1 - q_2\|_{H^\beta(\mathbb{R}^n)} \leq M$ for some $\beta > \beta_0$.

After a successful completion of the work above, I have started to investigate another inverse problem to determine an unknown coefficient $\rho(x)$ in the following hyperbolic problem

$$\rho u_{tt} - \Delta u = 0 \quad (x,t) \in \Omega \times (0,T).$$

In this work, the main objective is to establish the stability estimate for the unknown coefficient $\rho(x)$ by the Dirichlet to Neumann map. Some initial progress has recently been made in stability by Stefanov and Uhlmann.

In both of the inverse problems above, the unknown Fourier transformed coefficients $\widehat{p}(\eta)$ and $\widehat{\rho}(\eta)$ have been considered. The previous studies provided the good estimates for $\widehat{p}(\eta)$ and $\widehat{\rho}(\eta)$ only in a bounded area $|\eta| \ll \infty$. This is the reason why we could not have the Lipschitz type stability. Therefore, my current investigation concerns the behavior of $\widehat{p}(\eta)$ and $\widehat{\rho}(\eta)$ near infinity to make improvement.

During the visiting period at Michigan State University, I have developed a strong interest in the inverse scattering problems. There are many interesting questions that remain open about these subjects with significant applications. Another current research direction is to study the dependency of the solution with respect to high wave number in a Helmholtz type problem with applications to EM cavities. I intend to continue attacking those crucial questions in inverse problems and the related subjects.

1.2 Early works on inverse problems for elliptic equations

This subsection provides brief descriptions of my early works I have done under the supervision of Professor Hyeonbae Kang during my Ph.D. period.

Identification of sound hard obstacles in scattering problems

In this work, I have solved a challenging problem in the inverse scattering problem. The interest of this subject lies in the detection of an object (an obstacle D) from the pattern (or the scattering amplitude) of the field generated by (plane) incident waves of frequency k. The field in the simplest situation of scattering by an obstacle D is a solution to a Helmholtz equation with the homogeneous Dirichlet (or Neumann) boundary data a sound-soft obstacle (or a sound-hard obstacle) respectively. For details, refer to [14].

In the case of sound-soft obstacles, owing to Colton's theorem in [15], C. Liu proved the uniqueness within the class of *sound-soft balls* by a scattering amplitude corresponding to a single incident direction and one fixed wave number k in [16].

In the case of sound-hard obstacles, the uniqueness within the class of *sound-hard balls* had been a challenging problem for a long time. However, after twenty four years from Colton [15] and ten years from Liu [16], I have established the analogue of Colton for Neumann data. This immediately implies the desirable uniqueness within the class of *sound-hard balls* by a scattering amplitude corresponding to a single incident direction and one fixed wave number k in [2]. This remarkable result was cited by R. Kress in [17] et al.

Identification of a source term in elliptic equations

Collaborating with Professor Hyeonbae Kang and Doctor Kiwoon Kwon, I have studied identification of a source term in elliptic equations appearing in the determination of the metal-to-semiconductor contact and its resistivity of electric devices. Current density g is applied to a side of semiconductor layer Ω (called the diffusion layer) and the corresponding electric potential u satisfies an elliptic equation

$$\triangle u - p\chi(D)u = 0 \quad \text{in } \Omega$$

and $\frac{\partial u}{\partial \nu} = g$ on $\partial\Omega$ where $p = \frac{R_s}{\rho_c}$ (R_s : the sheet resistance of the semiconductor layer and ρ_c : the contact resistivity), D is the metal-to-semiconductor contact, ν is the unit outward normal to $\partial\Omega$. There have been several studies of this inverse problem (e.g. [18, 19]).

We have proved a uniqueness of D within the class of two- or three-dimensional balls from a single boundary measurement in [3]. This result have laid the foundation for further research by S. Kim and M. Yamamoto [20, 21].

Boundary determination of anisotropic conductivities

The purpose of inverse conductivity problems is mainly to identify an interior conductivity or to detect inclusions in a conductor from the Dirichlet-to-Neumann map (or the voltage-to-current map). Those problems are strongly motivated by electrical impedance tomography, which are new and promising methods to prospect the interior of the human body by surface electromagnetic measurements.

In cooperation with Professor Hyeonbae Kang, I established an explicit reconstruction of boundary values of less regular anisotropic (possibly non-analytic) conductivities and their derivatives in [4], while the previous results are associated with the analytic regularity. This subject had been brought by J. Lee and G. Uhlmann, who proved the uniqueness in identifying an anisotropic conductivity for three dimension under some restrictions when conductivities are real analytic in a domain (see [23]). Moreover, M. Lassas and G. Uhlmann [22] extended the result to the case when g is real analytic up to a portion of the boundary and removed some restrictions of [23]. Furthermore, we have obtained the improved identification and reconstruction results with the relaxed regularity restrictions by a new method motivated by G. Nakamura and K. Tanuma [24].

2 High stress concentration in fiber reinforced materials

In fiber-reinforced compositions, even though stiff fibers are present for reinforcement, the stress blows up as the fibers approximate to each other. These high shear stress concentrations cause low strengths of fiber-reinforced composites. Much attention has been devoted to the blow-up estimate for stress due to the practical significance.

In my recent work [5, 6], I have established the optimal growth rate on the stresses occurring between a pair of closely spaced stiff fibers with arbitrary shaped cross-sections. Furthermore, in the collaboration with Lim, two significant improvements [7, 8] also have been made to describe explicitly what role the geometric parameters play in the blow-up phenomenon. In particular, the estimate presented in [8] reveals a surprising fact that the high stress concentration can be significantly increased by a small geometrical change of a fiber.

Referring to a model of anti-plane shear, the estimate for stress is reduced to the gradient estimate for the solution u to conductivity problems as follows: for a given entire harmonic function H in $\mathbb{R}^2$, the out-of-displacement u is satisfied with

$$\begin{cases} \nabla \cdot \left\{ \left(1 + \sum_{i=1,2} (a_i - 1)\chi(D_i)\right) \nabla u \right\} = 0 \\ u(\mathbf{x}) - H(\mathbf{x}) = O(|\mathbf{x}|^{-1}) \quad \text{as } |\mathbf{x}| \to \infty \end{cases}$$

where D_1 and D_2 are closely spaced inclusions ϵ apart in $\mathbb{R}^2$, representing the cross-sections of fibers.

For the case that the inclusions have finite conductivities (shear moduli), with the effort of E. Bonnetier and M. Vogelius [25] as the beginning point, Y.Y. Li and M. Vogelius [26] derived the uniform bound of the stress that are independent of the distance ϵ. Moreover, this result has been extended to elliptic systems by Y.Y. Li and L. Nirenberg in [27]. These results point out that the extremely high conductivity (or the stiffness of fibers) is indispensable to the blow-up phenomena. In this respect, much attention has been focused on the model of a pair of perfect conductors which are ϵ apart. H. Ammari, H. Kang, H. Lee, J. Lee and M. Lim [28, 29] have established the optimal blow-up rate $\epsilon^{-1/2}$ as the distance ϵ goes to zero, only when conductors are of circular shape in two dimensions.

In my recent papers [5, 6], the above mentioned result restricted only to circular inclusions was extended to a sufficiently general class of the conductors' shapes in two dimensions. Furthermore, the technique I used is completely different from the previous one restricted to the case of circular shape. My result has been extended by S. Bao, Y.Y. Li and B. Yin [30] to higher dimensions. However, their estimates are only given by the distance between two conductors and geometric information of conductors are not incorporated into the estimates.

In collaboration with Professor Mikyoung Lim, under the assumption that the perfect conductors have spherical shapes in three and higher dimensions, the gradient estimates have been established in terms of the radii as well as the distance ϵ between inclusions in a way different from Bao et al. [30].

Furthermore, we have recently obtained another crucial result in [8]. This work focuses on the case when a smaller fiber is located in-between area of two fibers, see Figure 1. This paper reveals that, with the addition of a smaller fiber, possibly attached on a fiber, the growth of stress is significantly increased: if the diameter d of the fiber in the middle is sufficiently small and the distance between adjoining fibers is ϵ, then the stress blows up at the rate of $\frac{1}{\sqrt{d\epsilon}}$ in the narrow region, even though the blow-up rate has been known as $\frac{1}{\sqrt{\epsilon}}$ as in the case of a pair of fibers. This means that the defect of fiber as a protrusion causes much lower strengths in composites than had been thought. This subject still has many crucial questions which I shall continue.

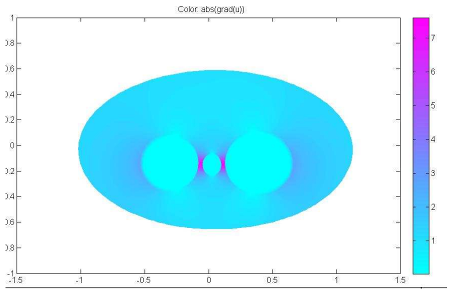


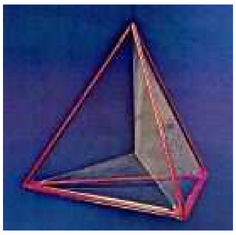
Figure 1: The electric field increased by the small inclusion between a pair of inclusions.

3 Evolution of grain boundaries in three dimensions

In work with Professor David Kinderleher, our investigation concerns the evolution of granular network governed by the Mullins Equations of a curvature driven growth supplemented by the Herring Condition force balanced by the junctions. In particular, the main challenge is to show the existence of the solution to the mathematical systems of the evolution. Including triple junctions in rigorous mathematical models had very limited success [31], even though it has been successfully modeled via simulation [32].

In our recent work [9], we consider the soap bubble with six facets in the tetrahedron as a simplified model in three dimensions. The motion of each surface is determined by the curvature driven growth, so called the Mullins Equation:

$$(X_t \cdot n) - \kappa = 0 \text{ on the facets}$$



with the Herring Condition for the interior angle $\frac{2\pi}{3}$ at the triple junction lines, where κ is the mean curvature. We have obtained a short time existence of the solution X satisfying the growth above under the assumption that the system is initially close to the equilibrium configuration in [9].

In the case of grain boundaries in the plane, L. Bronsard and F. Reitich derived the short time evolution in [33]. From the viewpoint of methodology, the governing nonlinear equation of a motion by curvature can be separated into a linear parabolic equation and a higher order term as a source term as follows: Let $X = I + Z$ where I is the identity function. Then

$$\left(\frac{\partial}{\partial t} - \Delta\right) Z = F(DZ, D^2 Z)$$

with $|F| \leq C|DZ||D^2 Z|$ for small $|DX|$. The result can be derived by an iteration of solving the linear equation and substituting the solution for the higher order term repeatedly. Based on Solonnikov-type estimates, D. Kinderlehrer and C. Liu [34] established the long time evolution by similar argument to [33] under the assumption that the system is initially closed to some equilibrium.

However, in the case of the soap bubble in three dimensions, the previous straightforward iteration by Bronsard *et.al.*

$$\left(\frac{\partial}{\partial t} - \Delta\right) Z_{n+1} = F(DZ_n, D^2 Z_n)$$

is not available due to the cornered shape of six surfaces. According to Grisvard's result on the angular domains [35], a singular term with an unexpectedly low regularity can be often contained in the solution defined in the angled domains. Thus, a solution Z_{n+1} with a low regularity may not be available for the recursive process.

In our work, a new iteration has been established to solve the difficulties occurring in the previous process. Based on this, a short time existence of X above has also been obtained under the assumption that the system is initially close to the equilibrium configuration. Moreover, I would like to investigate the long time existence with the convergence of the surface to the stable bubble.

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