Solution of Ill-Posed Volterra Equations via Variable-Smoothing Tikhonov Regularization

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Abstract

We consider a "local" Tikhonov regularization method for ill-posed Volterra problems. In addition to leading to efficient numerical schemes for inverse problems of this type, a feature of the method is that one may impose varying amounts of local smoothness on the solution, i.e., more regularization may be applied in some regions of the solution's domain, and less in others. Here we present proofs of convergence for the infinite-dimensional local regularization problem and discuss the resulting numerical algorithm.

1. Introduction.

Inverse problems of Volterra type arise in many scientific areas, including applications in heat transfer, population dynamics, and geophysical problems (e.g., groundwater and porous media applications). An inverse problem common to these applications is the determination, via a data-matching process, of unknown time-varying boundary conditions in an underlying diffusion-type partial differential equation model. Rather than dealing directly with the parabolic model, it is often preferable to rewrite the inverse problem as a (typically first-kind) Volterra integral equation. Examples of such models, and examples of other applications in which inverse problems of Volterra type arise, may be found in [3, 4, 6, 7, 8, 9].

We shall consider here a problem of this type, namely the problem of finding $u \in L_2(0,1)$ solving

$$Au(t) = f(t), \quad \text{a.a. } t \in [0,1], \tag{1.1}$$

where A is a Volterra operator given by

$$Au(t) = \int_0^t k(t-\tau)u(\tau)\,d\tau, \tag{1.2}$$

and where the kernel k is assumed to be uniformly Hölder continuous on the interval $[0,1]$ (i.e., $|k(t)-k(\tau)| \le L_k|t-\tau|^{\mu_k}$ for some positive constants L_k and μ_k). We also assume that f is Hölder continuous on $[0,1]$ and is such that u uniquely solves (1.1) (see, for example, [6]).

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It is well-known that (1.1) is an ill-posed problem due to lack of continuous dependence of the solution u on data f. Thus, in the usual case where only a measured or computed approximation f^δ to f is available, some kind of regularization method is required in order to obtain a reasonable approximation u^δ to u.

In this paper we describe a "local" regularization method which is based on the splitting of the operator A into local and non-local parts, with the regularization being applied to the inversion of the local part of A only. The advantages of such an approach are twofold:

- An efficient numerical method results, one which reduces to a rapid sequential method in many circumstances.
- The method allows for *variable* regularization, meaning that one may apply different amounts of regularization over different parts of the domain of the solution. The end effect is a method which may allow for more flexibility and control of the regularization process.

Standard (zero$^{\text{th}}$ order) Tikhonov regularization for the Volterra problem requires solving, for some $\alpha > 0$,

$$\min_{u \in L_2(0,1)} \left\{ \int_0^1 \left| \int_0^t k(t-\tau)u(\tau)\,d\tau - f^\delta(t) \right|^2 dt + \alpha \int_0^1 |u(t)|^2 dt \right\}$$

for $\overline{u}_\alpha \in L_2(0,1)$. In contrast the idea of "local Tikhonov regularization" is as follows, where, for the purposes of clarity, we first consider a discrete realization of this method. Given an integer $M \geq 1$, $\Delta t \equiv 1/M$, and $t_i \equiv i\Delta t$, for $i = 0, \ldots, M-1$, one first solves a Tikhonov problem on a smaller interval $[0, r]$, where $r > \Delta t$ is fixed (typically $r = R\,\Delta t$, for some integer $R > 1$). That is, solve

$$\min_{u_1 \in L_2(0,r)} \left\{ \int_0^r \left| \int_0^t k(t-\tau)u_1(\tau)\,d\tau - f^\delta(t) \right|^2 dt + \alpha \int_0^r |u_1(t)|^2 dt \right\}$$

for $\hat{u}_{1,\alpha} \in L_2(0,r)$. Even though $\hat{u}_{1,\alpha}$ is defined on all of $[0,r]$, we retain $\hat{u}_{1,\alpha}$ only on the smaller interval $[0, \Delta t]$, and make the definition $\hat{u}_\alpha(t) = \hat{u}_{1,\alpha}(t)$, for $t \in [0, t_1]$.

Next, we solve a Tikhonov problem on the subsequent interval $[t_1, t_1 + r]$; i.e., we find $\hat{u}_{2,\alpha} \in L_2(0,r)$ solving

$$\min_{u_2 \in L_2(0,r)} \left\{ \int_{t_1}^{t_1+r} \left| \int_0^{t_1} k(t-\tau)\hat{u}_{1,\alpha}(\tau)\,d\tau + \int_{t_1}^t k(t-\tau)u_2(\tau - t_1)\,d\tau \;-\; f^\delta(t) \right|^2 dt \right.$$
$$\left. + \alpha \int_0^r |u_2(t)|^2 dt \right\}$$

and then retain $\hat{u}_{2,\alpha}$ only on the interval $[0, \Delta t]$. Set $\hat{u}_\alpha(t) = \hat{u}_{2,\alpha}(t - t_1)$, for $t \in [t_1, t_2]$. And we continue in this fashion until $\hat{u}_\alpha$ is constructed on all of $[0,1]$.

In this particular realization of local regularization, the smoothing is performed locally, in fact sequentially, on intervals of length r. Increasing or decreasing r leads to more or less (local) smoothing of the regularized local solution, for a fixed value of α. And because only a small part of each local solution is subsequently retained in the construction of the resulting final solution $\hat{u}_\alpha$, this final solution need not be overly smooth.

The above regularization method allowed for a fixed length of local regularization interval. In contrast, variable regularization for this discrete problem occurs when r is allowed to change on each interval, so that more or less smoothing may be performed on the ith interval via a larger or smaller choice of $r = r_i$, for $i = 0, 1, \dots, M-1$.

A convergence theory for a numerical scheme based on the above-described local regularization method may be found in [9] for the special case of constant r and the kernel k satisfying $k(0) = 0$. For more general k, proofs of convergence become more difficult. In this paper, we address the problem of general k and variable r by examining a slightly different local regularization scheme, one which is, however, closely related to the above method when realized numerically. Indeed, a numerical implementation of the method we discuss in this paper leads to an *iterative* version of the above *sequential* numerical method; under assumptions that we indicate later, the new iterative numerical method reduces to the sequential numerical method described above.

Since the local regularization is performed on "future" intervals, we must either accept a final regularized solution which is defined on a slightly *smaller* interval than $[0, 1]$, or require that data be available on a slightly *longer* interval than $[0, 1]$. We take the latter approach here, a restriction which is not unreasonable for many Volterra applications. Thus we assume that u satisfies (1.1) on an extended interval $[0, 1 + \overline{\Delta}]$ where $\overline{\Delta} > 0$ is fixed and typically small. We assume that f, k are each uniformly Hölder continuous on this extended interval, and that $f^\delta \in L_2(0, 1 + \overline{\Delta})$ satisfies $|f - f^\delta|_{\text{ext}} \le \delta$, where $|\cdot|_{\text{ext}}$ denotes the usual $L_2(0, 1 + \overline{\Delta})$ norm.

2. Variable Local Regularization.

In the last section we described a discrete version of local Tikhonov regularization. For the infinite-dimensional version of this problem we use $r = r(t)$, and require that $r(t) \in (0, \overline{\Delta})$ for all $t \in [0, 1]$. As was true for the discrete version of the problem, the basic idea of local Tikhonov regularization in an infinite-dimensional setting is that, for each t, we consider a Tikhonov problem on the interval $[t, t + r(t)]$ only. That is, for each t, we split the operator A into "local" and "nonlocal" parts and perform regularization on the domain of the local part of A only. This splitting of A occurs as follows, for fixed $t \in [0, 1]$. That is, for a.a. $\rho \in [0, r(t)]$, we have

$$\begin{aligned} f(t+\rho) &= Au(t+\rho) \\ &= \int_0^\rho k(\rho-\tau)u(t+\tau)\,d\tau + \int_0^t k(t+\rho-\tau)u(\tau)\,d\tau \\ &= \int_0^\rho k(\rho-\tau)\varphi(t)(\tau)\,d\tau + \int_0^t k(t+\rho-\tau)T\varphi(\tau)\,d\tau \qquad (2.1) \end{aligned}$$

where

$$\varphi(t) = u_t,$$

and $u_t \in L_2(0, r(t))$ is defined by

$$u_t(\rho) \equiv u(t+\rho), \quad \text{a.a. } \rho \in [0, r(t)];$$

in addition, $T\varphi(t) = \varphi(t)(0) = u(t)$ (for smooth $\varphi(t)$). We will be more precise in the definitions of operators, variables, and spaces in the subsections which follow. In particular, we will reformulate T in order to accommodate $\varphi(t)$ only square-integrable; for now, however, we will postpone such issues and focus for the moment on simply illustrating the basic ideas.

For a given value of t, the first term in (2.1) represents the action of the original operator A on the *local* part $\varphi(t) = u_t$ of the solution u (i.e., u on the local interval $[t, t+r(t)]$), while the second term in (2.1) represents the action of A on the *non-local* part $T\varphi = u$ of u, (i.e., u on the interval $[0, t]$). We note that the "local" interval (associated with the value t) used here is $[t, t+r(t)]$ rather than $[t-r(t), t+r(t)]$, because we are considering a Volterra, i.e., causal, problem.

As motivation for an infinite-dimensional "local Tikhonov regularization" method in the presence of noisy data f^δ, we assume for the moment that $u(\tau) = T\varphi(\tau)$ is known already for a.a. $\tau \in [0, t]$, for where t is fixed in $(0, 1)$. In this situation, we (approximately) determine the local part $\varphi(t) = u_t \in L_2(0, r(t))$ of the solution u (i.e., u on $[t, t+r(t)]$) by applying Tikhonov regularization to the split problem as motivated by (2.1); that is, for constant $\alpha > 0$ and fixed $t \in (0, 1)$, we find the solution of the problem

$$\min_{\varphi(t)\in L_2(0,r(t))} \left\{ \frac{1}{r(t)} \int_0^{r(t)} \left| \int_0^\rho k(\rho-\tau)\varphi(t)(\tau)\, d\tau + \int_0^t k(t+\rho-\tau) T\varphi(\tau)\, d\tau - f^\delta(t+\rho) \right|^2 d\rho \right.$$
$$\left. + \alpha \int_0^{r(t)} |\varphi(t)(\rho)|^2 d\rho \right\},$$

where the division by $r(t)$ in the first integral above is a type of normalization, ensuring that the fit-to-data criterion in the first integral does not go to zero as we let the regularization parameter $r(t)$ go to zero.

But in actual fact we do not *a priori* know $u = T\varphi$ on *any* interval, so the local regularization problem of interest here is to find φ solving a "uniform-in-t" regularization problem of the form

$$\min_\varphi \int_0^1 \left\{ \frac{1}{r(t)} \int_0^{r(t)} \left| \int_0^\rho k(\rho-\tau)\varphi(t)(\tau)\, d\tau + \int_0^t k(t+\rho-\tau) T\varphi(\tau)\, d\tau - f^\delta(t+\rho) \right|^2 d\rho \right.$$
$$\left. + \alpha \int_0^{r(t)} |\varphi(t)(\rho)|^2 d\rho \right\} dt. \tag{2.2}$$

In addition, we may also wish to decouple the length $r(t)$ of the regularization interval for $\varphi(t)$ from the length of the interval on which the (local) least squares matching of data to model is performed. To this end, we define a second function $s: [0, 1] \to \mathbb{R}$ such that $s(t) \in (0, \overline{\Delta})$ for all $t \in [0, 1]$. Using variable r and s, and constant $\alpha > 0$, the problem in

(2.2) then becomes

$$\min_{\varphi} \int_0^1 \left\{ \frac{1}{s(t)} \int_0^{s(t)} \left| \int_0^{\rho} k(\rho-\tau)\varphi(t)(\tau)\, d\tau + \int_0^t k(t+\rho-\tau)T\varphi(\tau)\, d\tau - f^{\delta}(t+\rho) \right|^2 d\rho \right.$$
$$\left. + \alpha \int_0^{r(t)} |\varphi(t)(\rho)|^2 d\rho \right\} dt, \tag{2.3}$$

where the two interval lengths $r(t)$ (the interval of local regularization) and $s(t)$ (the interval of local data fitting) need not be the same.

We turn now to a more precise statement of the local regularization problem given in (2.3) and to a careful description of the underlying spaces and operators involved.

2.1. Parameter-Dependent Spaces

We assume throughout this section that r and s are fixed functions in $C[0,1]$ satisfying

$$0 \ < \ r_{\min} \equiv \min_{t\in[0,1]} r(t) \ \leq \ r_{\max} \equiv \max_{t\in[0,1]} r(t) \ < \ \overline{\Delta} \tag{2.4}$$

$$0 \ < \ s_{\min} \equiv \min_{t\in[0,1]} s(t) \ \leq \ s_{\max} \equiv \max_{t\in[0,1]} s(t) \ < \ \overline{\Delta}. \tag{2.5}$$

Let X denote $L_2(0,\overline{\Delta})$ and $|\cdot|$ the usual $L_2(0,\overline{\Delta})$ norm. For each $t \in [0,1]$, define $j_{r(t)} : X \mapsto X$ via

$$j_{r(t)}x(\rho) \equiv \begin{cases} x(\rho), & \text{a.a. } 0 \leq \rho \leq r(t), \\ 0, & \text{a.a. } r(t) < \rho \leq 1, \end{cases}$$

and $X_{r(t)} \subseteq X$ by

$$X_{r(t)} \equiv j_{r(t)} X,$$

and we note that $X_{r(t)}$ is a Hilbert space with the norm $|\cdot|_{r(t)}$, where

$$|x|^2_{r(t)} \equiv \int_0^{r(t)} |x(\rho)|^2\, d\rho,$$

for $x \in X_{r(t)}$.

When $(Z, \|\cdot\|)$ is a Hilbert space, we shall use the notation $L_2((0,1),Z)$ to denote the Hilbert space of Z-valued "functions" φ which are strongly Lebesgue-measurable on $[0,1]$ and which satisfy $\|\varphi\|^2_{L_2((0,1),Z)} \equiv \int_0^1 \|\varphi(t)\|^2_Z\, dt < \infty$ (see, for example, [11]). Accordingly, we shall define $\mathcal{X}$ by

$$\mathcal{X} \equiv L_2((0,1),X),$$

and use the notation $\|\cdot\|_{\mathcal{X}}$ to designate the associated $\|\cdot\|_{L_2((0,1),X)}$ norm. We also define an r-dependent space $\mathcal{X}_r \subseteq \mathcal{X}$ via

$$\mathcal{X}_r \equiv j_r \mathcal{X},$$

where $j_r : \mathcal{X} \mapsto \mathcal{X}$ is defined for $\varphi \in \mathcal{X}$ by

$$j_r\varphi(t) \equiv j_{r(t)}(\varphi(t)), \quad \text{a.a. } t \in [0,1].$$

Then $\mathcal{X}_r$ is also a Hilbert space with the norm $\|\cdot\|_{\mathcal{X}_r}$,

$$\begin{aligned}\|\varphi\|^2_{\mathcal{X}_r} &\equiv \int_0^1 |\varphi(t)|^2_{r(t)}\,dt, \quad \varphi \in \mathcal{X}_r,\\ &= \int_0^1 \int_0^{r(t)} |\varphi(t)(\rho)|^2\,d\rho\,dt.\end{aligned}$$

For "data spaces", we shall let $Y = X = L_2(0, \overline{\Delta})$ and, for each $t \in [0,1]$, define $Y_{s(t)}$ as the Hilbert space

$$Y_{s(t)} \equiv j_{s(t)} Y$$

equipped with the (weighted) norm $|\cdot|_{\frac{1}{s(t)}}$, where for $y \in Y_{s(t)}$,

$$|y|^2_{\frac{1}{s(t)}} \equiv \frac{1}{s(t)} \int_0^{s(t)} |y(\rho)|^2\,d\rho.$$

Defining also $\mathcal{Y} \equiv L_2((0,1), Y)$ with norm $\|\cdot\|_{\mathcal{Y}}$, and $\mathcal{Y}_s \equiv j_s\mathcal{Y}$, we have that $\mathcal{Y}_s$ is a Hilbert space with (weighted) norm $\|\cdot\|_{\mathcal{Y}_s}$,

$$\begin{aligned}\|\psi\|^2_{\mathcal{Y}_s} &\equiv \int_0^1 |\psi(t)|^2_{\frac{1}{s(t)}}\,dt, \quad \psi \in \mathcal{Y}_s,\\ &= \int_0^1 \frac{1}{s(t)} \int_0^{s(t)} |\psi(t)(\rho)|^2\,d\rho\,dt.\end{aligned}$$

We will also formulate f in the setting of the data space $\mathcal{Y}_s$. To this end we define $F_s \in \mathcal{Y}_s$ via

$$\begin{aligned}F_s(t)(\rho) &\equiv j_{s(t)} f_t(\rho)\\ &= \begin{cases} f(t+\rho), & \text{a.a. } \rho \in [0, s(t)],\ t \in [0,1],\\ 0, & \text{a.a. } \rho \in (s(t), 1],\ t \in [0,1],\end{cases}\end{aligned}$$

and note that $\|F_s\|_{\mathcal{Y}_s} \le \sqrt{s_{\max}/s_{\min}}\,|f|_{\text{ext}}$. A similar definition is made for F_s^δ, using f^δ. In addition, we define $\overline{F}_s \in \mathcal{Y}_s$ by

$$\overline{F}_s(t) \equiv f(t)\, j_{s(t)}\mathbf{1}, \quad \text{a.a. } t \in [0,1], \tag{2.6}$$

where $\mathbf{1} \in X$ is defined by $\mathbf{1}(\rho) = 1$, a.a. $\rho \in [0,1]$.

It will also be useful to define analogous functions for u. That is, $U_r \in \mathcal{X}_r$ is defined by

$$U_r(t) \equiv j_{r(t)} u_t, \quad \text{a.a. } t \in [0,1], \tag{2.7}$$

where $\|U_r\|_{\mathcal{X}_r} \le \sqrt{r_{\max}}\,|u|_{\text{ext}}$, and $\overline{U}_r \in \mathcal{X}_r$ may be defined by $\overline{U}_r(t) = u(t)\, j_{r(t)}\mathbf{1}$, a.a. $t \in [0,1]$, where $\|\overline{U}_r\|_{\mathcal{X}_r} \le \sqrt{r_{\max}}|u|$.

2.2. Parameter-Dependent Operators

For simplicity in some of the computations that follow (and without loss of generality), we will henceforth take $\overline{\Delta} = 1$.

For each $t \in [0,1]$, we define the dilation operator $I_{r(t)} : X_{r(t)} \mapsto X$ by

$$I_{r(t)}x(\rho) \equiv x(\rho \cdot r(t)), \quad \text{a.a. } \rho \in [0, \overline{\Delta}] \equiv [0,1], \;\; x \in X_{r(t)},$$

and note that $I_{r(t)}$ is an isomorphism for all $t \in [0,1]$, with

$$|I_{r(t)}\, x|^2 = \frac{1}{r(t)} |x|^2_{r(t)}, \quad x \in X_{r(t)}.$$

Similarly we define $\mathcal{I}_r : \mathcal{X}_r \mapsto \mathcal{X}$ via

$$\mathcal{I}_r\varphi\,(t) \equiv I_{r(t)}(\varphi(t)), \quad \text{a.a. } t \in [0,1], \;\; \varphi \in \mathcal{X}_r,$$

and note that

$$\frac{1}{r_{\max}}\, \|\varphi\|^2_{\mathcal{X}_r} \;\le\; \|\mathcal{I}_r\varphi\|^2_{\mathcal{X}} \;\le\; \frac{1}{r_{\min}}\, \|\varphi\|^2_{\mathcal{X}_r}, \quad \varphi \in \mathcal{X}_r.$$

In earlier motivating the idea of local regularization in an infinite-dimensional setting, we used a definition of T involving point evaluations of $\varphi(t)$, under the assumption that $\varphi(t)$ was smooth on $[0, r(t)]$; in actuality, we only have $\varphi(t) \in X_{r(t)} \subseteq L_2(0,1)$, so point evaluation of $\varphi(t)$ is not a continuous operation. We will depart here somewhat from the motivating comments in the preceding section and instead reformulate T (and a parameter-dependent T_r) as a continuous operator. To this end, let $\ell \in X^\star$ (the continuous dual of X) be given satisfying $\ell(\mathbf{1}) = 1$. Then $\ell(x) = (x,\gamma)$ for all $x \in X$ and some $\gamma \in X$, where $(\cdot,\cdot)$ denotes the usual inner product on X. Using ℓ, we define $T \in \mathcal{L}(\mathcal{X}, X)$ by

$$T\varphi(t) \equiv \ell(\varphi(t)), \quad \text{a.a. } t \in [0,1], \;\; \varphi \in \mathcal{X},$$

where here we are using $\mathcal{L}(Z,W)$ to denote the space of bounded linear operators defined on Z with range in W, for given Hilbert spaces Z and W.

For each $t \in [0,1]$, let $\ell_{r(t)} \in X^\star_{r(t)}$ be defined via

$$\ell_{r(t)}(x) \equiv \ell(I_{r(t)}x), \quad x \in X_{r(t)},$$

and let $T_r \in \mathcal{L}(\mathcal{X}_r, X)$ be defined by

$$T_r\varphi(t) \equiv \ell_{r(t)}(\varphi(t)), \quad \text{a.a. } t \in [0,1], \;\; \varphi \in \mathcal{X}_r.$$

We note that $T_r\varphi = T(\mathcal{I}_r\varphi)$ for all $\varphi \in \mathcal{X}_r$.

Example: An example of T_r which matches the spirit of the earlier discussion is given via

$$\ell(x) = \frac{1}{c}\int_0^c x(\rho)\, d\rho, \quad x \in X, \tag{2.8}$$

where $0 < c \ll 1$. In this case then,

$$T_r\varphi(t) = \frac{1}{c\,r(t)} \int_0^{c\,r(t)} \varphi(t)(\rho)\,d\rho, \quad \text{a.a. } t \in [0,1],$$

for arbitrary $\varphi \in \mathcal{X}_r$, while for the case of $\varphi \equiv U_r$, we obtain

$$T_rU_r(t) = \frac{1}{c\,r(t)} \int_t^{t+c\,r(t)} u(\rho)\,d\rho, \quad \text{a.a. } t \in [0,1],$$

so that, for a.a. $t \in [0,1]$, $T_rU_r(t)$ represents the integral average of u over $[t, t + c\,r(t)]$, a small subinterval of the usual regularization interval $[t, t+r(t)]$. In numerical discretizations with piecewise-constant approximations (on a uniform mesh of size Δt), one obtains the desired "pointwise evaluation" used in the last section, namely $T_r\,(\varphi(t)) = \varphi(t)(0)$, when ℓ is defined as in (2.8) with $0 < c \leq \Delta t$.

We turn now to the "operator splitting" of the original operator A. Reflecting the operators associated with the two terms appearing in (2.1), we define $\mathcal{A} \in \mathcal{L}(X, Y)$ and $\mathcal{B} \in \mathcal{L}(X, \mathcal{Y})$ by

$$\begin{aligned}
\mathcal{A}\eta(\rho) &\equiv \int_0^\rho k(\rho - \tau)\eta(\tau)\,d\tau, \qquad \text{a.a. } \rho \in [0,1], \\
\mathcal{B}\eta(t)(\rho) &= \int_0^t k(t + \rho - \tau)\eta(\tau)\,d\tau, \quad \text{a.a. } t, \rho \in [0,1],
\end{aligned}$$

for any $\eta \in X$. Then in place of the original operator A we shall use $\mathcal{C}_{s,r} \in \mathcal{L}(\mathcal{X}_r, \mathcal{Y}_s)$, where

$$\mathcal{C}_{s,r} = \mathcal{A}_{s,r} + \mathcal{B}_sT_r.$$

Here $\mathcal{A}_{s,r} \in \mathcal{L}(\mathcal{X}_r, \mathcal{Y}_s)$ is defined by

$$\mathcal{A}_{s,r}\varphi(t) = j_{s(t)}\mathcal{A}(\varphi(t)), \quad \text{a.a. } t \in [0,1], \;\; \varphi \in \mathcal{X}_r,$$

and $\mathcal{B}_s \in \mathcal{L}(X, \mathcal{Y}_s)$ is given by

$$\mathcal{B}_s\eta(t) = j_{s(t)}(B\eta(t)), \quad \text{a.a. } t \in [0,1], \;\; \eta \in X.$$

It is not difficult to show that $\|\mathcal{A}_{s,r}\| \leq \sqrt{3\,\Delta_{\max}/2}\;\|k\|_\infty$ and $\|\mathcal{B}_s\| \leq \|k\|_\infty$, where $\|k\|_\infty \equiv \sup_{0\leq t\leq 1} |k(t)|$ and

$$\Delta_{\max} \equiv \min\{r_{\max}, s_{\max}\}. \tag{2.9}$$

2.3. The Local Tikhonov Regularization Problem

The local regularization problem is given as follows, for continuous functions r, s satisfying (2.4), (2.5), and for a given constant $\alpha > 0$:

Problem $\mathcal{P}^{\delta}_{\alpha,s,r}$:

Find $\varphi^{\delta}_{\alpha,s,r} \in \mathcal{X}_r$ satisfying

$$\varphi^{\delta}_{\alpha,s,r} = \arg\min_{\varphi \in \mathcal{X}_r} \left\{ \|\mathcal{C}_{s,r}\,\varphi - F^{\delta}_s\|^2_{\mathcal{Y}_s} + \alpha\,\|\varphi\|^2_{\mathcal{X}_r} \right\}.$$

It follows from standard theory [5] that there exists a unique solution $\varphi^{\delta}_{\alpha,s,r} \in \mathcal{X}_r$ of Problem $\mathcal{P}^{\delta}_{\alpha,s,r}$ for every function r, s, and $\alpha > 0$ so prescribed.

In this paper we treat the convergence of $T_r\varphi^{\delta}_{\alpha,s,r}$ to u as $\delta \to 0$, and also discuss an efficient numerical implementation of this scheme. Questions about the actual selection of regularization parameters will be addressed elsewhere.

3. Convergence Theorems.

We shall state the main convergence results for solutions $\varphi^{\delta}_{\alpha,s,r}$ of $\mathcal{P}^{\delta}_{\alpha,s,r}$, and give conditions under which the quantity $T_r\varphi^{\delta}_{\alpha,s,r}$ converges to the solution u of (1.1). Because three parameters α, s, and r, are involved, the results tend to include fairly lengthy technical statements of conditions on these parameters, as related to the size of δ. Therefore, to more clearly illustrate the roles of the various parameters, we will follow the general convergence result given below (Theorem 3.1) by statements of two special cases of this theorem. The proof of the general result is sketched in Section 5..

We shall simplify notation by letting $\varphi^{\delta_n}_n \equiv \varphi^{\delta_n}_{\alpha_n,s_n,r_n}$ and $\mathcal{P}^{\delta_n}_n \equiv \mathcal{P}^{\delta_n}_{\alpha_n,s_n,r_n}$, where α_n, s_n, and r_n are given sequences of parameters. In addition, we shall need the definition, for $n = 1, 2, \ldots$, of

$$\hat{s}_{n,\max} \equiv \begin{cases} 0, & \text{if } s_n(t) \le r_n(t), \text{ all } t \in [0,1], \\ s_{n,\max}, & \text{otherwise.} \end{cases}.$$

Theorem 3.1 *Let $\delta_n > 0$ be given with $\delta_n \to 0$ as $n \to \infty$. Let $\{r_n\} \subseteq C[0,1]$ and $\alpha_n > 0$ satisfy*

(i) $0 < r_{n,\min},\ r_{n,\max} < \overline{\Delta}$*, for* $n = 1, 2, \ldots$,

(ii) $\frac{r_{n,\max}}{r_{n,\min}} \to 1$ *as* $n \to \infty$,

(iii) $\alpha_n\, r_{n,\max} \to 0$ *as* $n \to \infty$,

(iv) $\frac{\delta_n^2}{\alpha_n\, r_{n,\min}} \to 0$ *as* $n \to \infty$.

Then a sequence of $s_n \in C[0,1]$ may be selected such that the following conditions hold:

(v) $0 < s_{n,\min},\ s_{n,\max} < \overline{\Delta}$*, for* $n = 1, 2, \ldots$,

(vi) $\frac{s_{n,\max}}{s_{n,\min}}$ *is bounded, for* $n = 1, 2, \ldots$,

(vii) $s_{n,\max} \to 0$ *as* $n \to \infty$,

(viii) $\frac{\hat{s}_{n,\max}}{\alpha_n\, r_{n,\min}} \to 0$ *as* $n \to \infty$.

Then, with α_n, r_n, s_n, *satisfying (i)-(viii), it follows that the solution* $\varphi_n^{\delta_n}$ *of Problem* $\mathcal{P}_n^{\delta_n}$ *satisfies*

$$\eta_n^{\delta_n} \equiv T_{r_n}\varphi_n^{\delta_n} \to u \quad \text{as } n \to \infty.$$

We now state two corollaries of special, but not untypical, cases of Theorem 3.1. The first is the case of $s(t) = r(t)$.

Corollary 3.2 *Let* $\delta_n > 0$ *satisfy* $\delta_n \to 0$ *as* $n \to \infty$. *For any sequence* $\{r_n\}$ *of positive functions in* $C[0,1]$ *satisfying* $r_{n,\max} \to 0$ *as* $n \to \infty$, *we may select* $\alpha_n > 0$ *such that*

- $\alpha_n\, r_{n,\max} \to 0$ *as* $n \to \infty$.

Then for such $\alpha_n = \alpha_n(r_n)$ *and for positive* $r_n \in C[0,1]$ *satisfying*

- $r_{n,\max} \to 0$ *and* $\frac{r_{n,\max}}{r_{n,\min}} \to 1$ *as* $n \to \infty$, *and*

- $\frac{\delta_n^2}{\alpha_n\, r_{n,\min}} \to 0$ *as* $n \to \infty$,

we have that the solution $\varphi_n^{\delta_n}$ *of* $\mathcal{P}_n^{\delta_n}$, *for* $s_n = r_n$, *satisfies*

$$\eta_n^{\delta_n} \equiv T_{r_n}\varphi_n^{\delta_n} \to u$$

as $n \to \infty$.

We note that the above corollary allows for the selection of $\alpha_n \to 0$, which is not unexpected since this result is also true for standard Tikhonov regularization. What is interesting in this new setting is that the possibility exists for selecting α_n constant or even increasing, at a rate determined by r_n. For example, one may select

$$\alpha_n = C\, r_{n,\max}^p, \quad p > -1,\ C > 0.$$

The second special case that we consider covers the situation where $s(t) = r(t)$, as before, but now we assume that α remains constant (e.g., $\alpha = 1$) as the level δ of noise goes to zero.

Corollary 3.3 *Let* $\alpha > 0$ *be fixed and let* $\delta_n > 0$ *satisfy* $\delta_n \to 0$ *as* $n \to \infty$. *For each* n, *let the positive function* r_n *in* $C[0,1]$ *satisfy the following conditions:*

- $r_{n,\max} \to 0$ *and* $\frac{r_{n,\max}}{r_{n,\min}} \to 1$ *as* $n \to \infty$, *and*

- $\frac{\delta_n^2}{r_{n,\min}} \to 0$ *as* $n \to \infty$.

Then if $s_n = r_n$ *for all* n, *the solution* $\varphi_n^{\delta_n}$ *of* $\mathcal{P}_n^{\delta_n}$ *satisfies*

$$\eta_n^{\delta_n} \equiv T_{r_n}\varphi_n^{\delta_n} \to u$$

as $n \to \infty$.

4. Numerical Implementation.

We discuss here the numerical implementation of the infinite-dimensional local regularization scheme discussed in the last two sections. In particular, we describe an *iterative* numerical method which reduces to a sequential method under some circumstances.

For simplicity we shall let $s(t) = r(t) > 0$, for all $t \in [0, 1]$. Let $M \geq 1$ be a fixed integer and define $t_i = 1/\Delta t$, $i = 0, 1, \ldots$, where $\Delta t = 1/M$. For $i = 0, \ldots, M-1$, define R_i by

$$R_i = \left[\frac{r(t_i)}{\Delta t}\right] + 1,$$

where $[\,\cdot\,]$ denotes the greatest integer function. We note that $1 \leq R_i \leq M$ for all i.

In the discrete formulation, we seek $\varphi \in U_r$ of the special form

$$\varphi(t)(\rho) = \sum_{i=0}^{M-1} \sum_{j=0}^{R_i-1} c_{ij} \chi_i(t) \chi_j(\rho), \quad t \in [0, 1],\ \rho \in [0, R_i\, \Delta t], \tag{4.1}$$

where c_{ij}, $j = 0, \ldots, R_i$, $i = 0, \ldots, M-1$, are unknown constants to be determined, and where

$$\chi_i(t) = \begin{cases} 1, & t \in [t_i, t_{i+1}), \\ 0, & \text{otherwise.} \end{cases}$$

Using ℓ as defined by (2.8) with $c \equiv \Delta t$, the special form of φ leads to a discrete version of $T_r\varphi$, i.e.,

$$T_r\varphi(t) = \sum_{i=0}^{M-1} c_{i0} \chi_i(t), \quad t \in [0, 1].$$

A discrete analog of the problem $\mathcal{P}^{\delta}_{\alpha,s,r}$ is then to seek the array $\mathbf{c} = (c_{ij})$ of unknowns which minimizes the functional $J_M(\mathbf{c})$,

$$\begin{aligned} J_M(\mathbf{c}) \;=\; & \sum_{m=0}^{M-1} \sum_{r=0}^{R_m-1} \frac{1}{R_m \Delta t} \left| \int_0^{\rho_{r+1}} k(\rho_{r+1} - \tau) \sum_{i=0}^{M-1} \sum_{j=0}^{R_i-1} c_{ij} \chi_i(t_m) \chi_j(\tau)\, d\tau \right. \\ & \left. + \int_0^{t_m} k(t_m + \rho_{r+1} - \tau) \sum_{i=0}^{M-1} c_{i0} \chi_i(\tau)\, d\tau - f^{\delta}(t_m + \rho_{r+1}) \right|^2 \\ & + \alpha \sum_{m=0}^{M-1} \sum_{r=0}^{R_m-1} |c_{mr}|^2, \end{aligned}$$

which corresponds to a collocation-based discretization (collocating to points (t_m, ρ_{r+1}), $\rho_{r+1} \equiv t_{r+1}$, $r = 0, 1, \ldots, R_m - 1$, $m = 0, 1, \ldots, M-1$) of the objective functional appearing in problem $\mathcal{P}^{\delta}_{\alpha,s,r}$. The functional J_M may be simplified considerably, due to the Volterra/convolution nature of the problem; that is,

$$\begin{aligned} J_M(\mathbf{c}) \;=\; & \sum_{m=0}^{M-1} \sum_{r=0}^{R_m-1} \frac{1}{R_m \Delta t} \left| \sum_{j=0}^{r} c_{mj} \Delta_{r+1-j} + \sum_{i=0}^{m-1} c_{i0} \Delta_{m-i+1+r} - f^{\delta}(t_{m+r+1}) \right|^2 \\ & + \alpha \sum_{m=0}^{M-1} \sum_{r=0}^{R_m-1} |c_{mr}|^2, \end{aligned}$$

where $\Delta_i \equiv \int_0^{\Delta t} k(t_i - \tau)\, d\tau$. If we make the following definitions for $m = 0, 1, \ldots, M-1$,

$$\begin{aligned} \mathbf{c}_m &\equiv \left(c_{m0},\, c_{m1},\, \ldots,\, c_{m,R_m-1}\right)^\top \in \mathbb{R}^{R_m}, \\ \mathbf{f}_m^\delta &\equiv \left(f^\delta(t_{m+1}), \ldots, f^\delta(t_{m+R_m})\right) \in \mathbb{R}^{R_m}, \end{aligned}$$

as well as the definition of the R_m-square matrix K_{R_m},

$$K_{R_m} = \begin{pmatrix} \Delta_1 & 0 & \ldots & 0 \\ \Delta_2 & \Delta_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{R_m} & \Delta_{R_m-1} & \ldots & \Delta_1 \end{pmatrix},$$

the finite-dimensional optimization problem becomes

$$\min_{\mathbf{c}_0,\ldots,\mathbf{c}_M} \sum_{m=0}^{M-1} J_m(\mathbf{c}_0, \ldots, \mathbf{c}_m)$$

where

$$J_m(\mathbf{c}_0, \ldots, \mathbf{c}_m) = \frac{1}{R_m \Delta t} \left\| K_{R_m} \mathbf{c}_m + \sum_{i=0}^{m-1} c_{i0}\, \hat{\Delta}_{m-i+1}^m - \mathbf{f}_m^\delta \right\|_{R_m}^2 + \alpha \|\mathbf{c}_m\|_{R_m}^2;$$

here $\|\cdot\|_{R_m}$ denotes the usual $\mathbb{R}^{R_m}$ norm and $\hat{\Delta}_i^m \equiv (\Delta_i, \ldots, \Delta_{i+R_m-1})^\top \in \mathbb{R}^{R_m}$, for $i = 0, \ldots, M-1$.

Using the theory of [1], a decomposition of this minimization problem is possible, setting up an iterative relaxation-type minimization. The following algorithm finds $\mathbf{c}_I$ for $I = 0, 1, \ldots, M-1$, using at each step $\beta \in \mathbb{R}^{R_m}$ to store the unknown $\mathbf{c}_I$. The vectors $\mathbf{d}_0$, $\mathbf{d}_1$, …, $\mathbf{d}_{M-1}$ used below may be initialized to zero and correspond to initial guesses for $\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{M-1}$.

“Local Tikhonov Regularization” Algorithm:

1. Initialize vectors $\mathbf{d}_0$, $\mathbf{d}_1$, …, $\mathbf{d}_{M-1}$.
2. Let $I = 0$.
3. Holding the previously determined values of $\mathbf{c}_0$, $\mathbf{c}_1$, …, $\mathbf{c}_{I-1}$ fixed, find $\overline{\beta} \in \mathbb{R}^{R_m}$ solving

$$\begin{aligned} \min_\beta \Bigg\{ \sum_{m=0}^{I-1} J_m(\mathbf{c}_0, \ldots, \mathbf{c}_m) \;+\; J_I(\mathbf{c}_0, \ldots, \mathbf{c}_{I-1}, \beta) \\ + \sum_{m=I+1}^{M-1} J_m(\mathbf{c}_0, \ldots, \mathbf{c}_{I-1}, \beta, \mathbf{d}_{I+1}, \ldots \mathbf{d}_m) \Bigg\} \end{aligned} \tag{4.2}$$

(i.e., $\mathbf{c}_{I+1} = \mathbf{d}_{I+1}$, …, $\mathbf{c}_{M-1} = \mathbf{d}_{M-1}$ temporarily).

4. Set $\mathbf{c}_I = \overline{\beta}$.

5. If $I = M-1$, let $\mathbf{d}_i = \mathbf{c}_i$ for $i = 0, \ldots, M-1$, and return to step 2. Otherwise, increment I by 1 and return to step 3 (leaving the $\mathbf{d}_i$ unchanged).

Under reasonable conditions, convergence of the relaxation-type minimization algorithm is guaranteed [1] and the converged values c_{00}, c_{10}, ..., $c_{M-1,0}$ are approximations for $u(t_0)$, $u(t_1)$, ..., $u(t_{M-1})$, respectively.

Remark: For $R_i = R_i(\delta)$ appropriately chosen (given the level δ of noise in the problem), it has been observed in practice that very accurate results are obtained if, in place of solving (4.2) for $\overline{\beta}$ in step 3 above, one drops the last summation in (4.2) and instead finds $\overline{\beta} \in \mathbb{R}^{R_I}$ solving

$$\min_{\beta} \; J_I(\mathbf{c}_0, \ldots, \mathbf{c}_{I-1}, \beta) \tag{4.3}$$

(the first summation in (4.2) may be neglected automatically since it is independent of β). Indeed, in [2] a case (based on physical considerations) is made for the adoption of this approach for a different, but related regularization method, as applied to (1.1) for the special case of the inverse heat conduction problem. See [7, 8] for a mathematical analysis of this related local regularization method.

Since J_I is independent of $\mathbf{d}_{I+1}, \ldots, \mathbf{d}_{M-1}$, the entire iteration (using (4.3) in place of (4.2) in Step 3) for $I = 0, \ldots, M-1$, is performed once only, without any initializing or updating of the values of $\mathbf{d}_i$. The resulting algorithm, given below, is precisely a rescaled version of the sequential algorithm (here in fully discrete form) which was discussed in Section 1.

Sequential "Local Tikhonov Regularization" Algorithm:

1. Let $I = 0$.

2. Holding the previously determined values of c_{00}, c_{10}, ..., $c_{I-1,0}$ fixed, solve for $\overline{\beta} \in \mathbb{R}^{R_I}$:

$$\min_{\beta} \; \frac{1}{R_I \Delta t} \left\| K_{R_I}\beta + \sum_{i=0}^{I-1} c_{i0} \hat{\Delta}^I_{I-i+1} - \mathbf{f}^\delta_I \right\|^2_{R_I} + \alpha \|\beta\|^2_{R_I}$$

3. Set $c_{I0} = \overline{\beta}_0$, the first component of $\overline{\beta}$.

4. If $I = M-1$, **stop**. Otherwise, increment I by 1 and return to step 2.

Because R_I is typically much smaller than M, and because the governing matrix K_{R_I} is similar at every step, the computational cost is quite low with the effectiveness about the same as full Tikhonov regularization. The reader is referred to [9] for numerical examples and for comparisons in terms of operation counts for standard Tikhonov regularization contrasted with an efficient implementation of the second algorithm above (in the case of $R_i = R$ constant). The results in [9] are valid for collocation-type approximations as well as several standard methods based on the numerical quadrature of (1.1).

5. Proofs of Convergence.

In this section we sketch the ideas behind the proof of Theorem 3.1. Details of the proof may be found in [10] where a more general case (i.e., higher-order local Tikhonov regularization) is treated.

From the definition of $\varphi^\delta_{\alpha,s,r}$, we obtain

$$\|\mathcal{C}_{s,r}\,\varphi^\delta_{\alpha,s,r} - F^\delta_s\|^2_{\mathcal{Y}_s} + \alpha\,\|\varphi^\delta_{\alpha,s,r}\|^2_{\mathcal{X}_r} \le \|\mathcal{C}_{s,r}\,\overline{U}_r - F^\delta_s\|^2_{\mathcal{Y}_s} + \alpha\,\|\overline{U}_r\|^2_{\mathcal{X}_r}$$

where

$$\|\mathcal{C}_{s,r}\,\overline{U}_r - F^\delta_s\|_{\mathcal{Y}_s} \le \|\mathcal{A}_{s,r}\left(\overline{U}_r - U_r\right)\|_{\mathcal{Y}_s} + \|\mathcal{A}_{s,r}U_r + \mathcal{B}_s T_r \overline{U}_r - F_s\|_{\mathcal{Y}_s} + \|F_s - F^\delta_s\|_{\mathcal{Y}_s}.$$

A standard calculation yields that

$$\begin{aligned}&\|\mathcal{C}_{s,r}\,\varphi^\delta_{\alpha,s,r} - F^\delta_s\|^2_{\mathcal{Y}_s} + \alpha\,\|\varphi^\delta_{\alpha,s,r}\|^2_{\mathcal{X}_r}\\ &\quad\le \Big(24\,\|k\|^2_\infty(\Delta_{\max} r_{\max} + \hat{s}_{\max}) + \alpha\, r_{\max}\Big)\,|u|^2_{\text{ext}} + 2\,\frac{s_{\max}}{s_{\min}}\,\delta^2,\end{aligned} \tag{5.1}$$

where $\hat{s}_{\max}$ is defined in the statement of Theorem 3.1 and $\Delta_{\max}$ is given by (2.9). It follows that

$$\begin{aligned}\|\mathcal{I}_r\varphi^\delta_{\alpha,s,r}\|^2_{\mathcal{X}} &\le \frac{1}{\alpha\, r_{\min}}\left\{\|\mathcal{C}_{s,r}\varphi^\delta_{\alpha,s,r} - F^\delta_s\|^2_{\mathcal{Y}_s} + \alpha\,\|\varphi^\delta_{\alpha,s,r}\|^2_{\mathcal{X}_r}\right\},\\ &\le \left(24\|k\|^2_\infty\,\frac{\Delta_{\max} r_{\max} + \hat{s}_{\max}}{\alpha\, r_{\min}} + \frac{r_{\max}}{r_{\min}}\right)|u|^2_{\text{ext}} + 2\frac{s_{\max}}{s_{\min}}\,\frac{\delta^2}{\alpha\, r_{\min}}.\end{aligned}$$

Therefore, if the conditions of Theorem 3.1 hold, we have that $\{\mathcal{I}_r\varphi^\delta_{\alpha,r,s}\}$ is bounded in $\mathcal{X}$.

Let $\delta_n > 0$ be such that $\delta_n \to 0$ as $n \to \infty$. Throughout this section we will simplify notation by writing $\varphi^{\delta_n}_n \equiv \varphi^{\delta_n}_{\alpha_n,s_n,r_n}$, $\mathcal{P}^{\delta_n}_n \equiv \mathcal{P}^{\delta_n}_{\alpha_n,s_n,r_n}$, $\mathcal{X}_n \equiv \mathcal{X}_{r_n}$, $\mathcal{Y}_n \equiv \mathcal{Y}_{s_n}$, $\mathcal{C}_n \equiv \mathcal{C}_{s_n,r_n}$, $\mathcal{A}_n \equiv \mathcal{A}_{s_n,r_n}$, $\mathcal{B}_n \equiv \mathcal{B}_{s_n}$, $T_n \equiv T_{r_n}$, $F^{\delta_n}_n \equiv F^{\delta_n}_{s_n}$, etc..

It follows from the above arguments that there is a weakly convergent subsequence of $\{\mathcal{I}_n\varphi^{\delta_n}_n\}$, relabeled $\{\mathcal{I}_n\varphi^{\delta_n}_n\}$, such that

$$\mathcal{I}_n\varphi^{\delta_n}_n \rightharpoonup \tilde{\varphi} \quad \text{in } \mathcal{X} \text{ as } n \to \infty,$$

for some $\tilde{\varphi} \in \mathcal{X}$. In addition, defining $\eta^{\delta_n}_n \equiv T\left(\mathcal{I}_n\varphi^{\delta_n}_n\right)$ and $\eta \equiv T(\tilde{\varphi})$, we have from the continuity of T that $\eta^{\delta_n}_n \rightharpoonup \eta$ in X as $n \to \infty$.

In fact, $\eta = u$, as can be shown in arguments that we sketch below. Indeed, η satisfies

$$\begin{aligned}|A\eta - f|^2 &= \int_0^1 \frac{1}{s_n(t)}\int_0^{s_n(t)} |A\eta(t) - f(t)|^2\, d\rho\, dt\\ &= \|\overline{A}_{s_n}\eta - \overline{F}_n\|^2_{\mathcal{Y}_n}\end{aligned}$$

where $\overline{A}_{s_n} \in \mathcal{L}(X, \mathcal{Y}_n)$ is defined by

$$\overline{A}_{s_n}\eta(t) = A\eta(t)\cdot j_{s_n(t)}\mathbf{1}, \quad \text{a.a. } t \in [0,1],\ \eta \in X,$$

and $\overline{F}_n$ is defined for s_n by (2.6). Then, for some $C > 0$ independent of n,

$$\begin{aligned}
&|A\eta - f|^2 \qquad\qquad (5.2)\\
&\leq \ C\Big\{\|\mathcal{C}_n\varphi_n^{\delta_n} - F_n^{\delta_n}\|^2_{\mathcal{Y}_n} + \alpha_n\|\varphi_n^{\delta_n}\|^2_{\mathcal{X}_n} + \|\mathcal{A}_n\varphi_n^{\delta_n}\|^2_{\mathcal{Y}_n} + \|\overline{A}_{s_n}\eta - \mathcal{B}_n\eta_n^{\delta_n}\|^2_{\mathcal{Y}_n}\\
&\qquad + \|F_n^{\delta_n} - F_n\|^2_{\mathcal{Y}_n} + \|F_n - \overline{F}_n\|^2_{\mathcal{Y}_n}\Big\}
\end{aligned}$$

where we have used the fact that $T_n\varphi_n^{\delta_n} = T(\mathcal{I}_n\varphi_n^{\delta_n}) = \eta_n^{\delta_n}$. We sketch below the estimates which show that each term in the right-hand side of (5.2) converges to zero as $n \to \infty$; throughout we require the assumptions of Theorem 3.1.

From (5.1) it follows that $\|\mathcal{C}_n\varphi_n^{\delta_n} - F_n^{\delta_n}\|^2_{\mathcal{Y}_n} + \alpha_n\|\varphi_n^{\delta_n}\|^2_{\mathcal{X}_n} \to 0$ as $n \to \infty$, while

$$\begin{aligned}
\|\mathcal{A}_n\varphi_n^{\delta_n}\|^2_{\mathcal{Y}_n} &\leq \|\mathcal{A}_n\|^2\|\varphi_n^{\delta_n}\|^2_{\mathcal{X}_n}\\
&\leq \Big(3\Delta_{n,\max}\|k\|^2_\infty/2\Big)\cdot r_{n,\max}\,\|\mathcal{I}_n\varphi_n^{\delta_n}\|^2_{\mathcal{X}}
\end{aligned}$$

so that $\|\mathcal{A}_n\varphi_n^{\delta_n}\|^2_{\mathcal{Y}_n} \to 0$ as $n \to \infty$. In addition,

$$\|\overline{A}_{s_n}\eta - \mathcal{B}_n\eta_n^{\delta_n}\|_{\mathcal{Y}_n} \leq \|\overline{A}_{s_n}(\eta - \eta_n^{\delta_n})\|_{\mathcal{Y}_n} + \|\Big(\overline{A}_{s_n} - \mathcal{B}_n\Big)\eta_n^{\delta_n}\|_{\mathcal{Y}_n}$$

where $\|\overline{A}_{s_n}(\eta - \eta_n^{\delta_n})\|^2_{\mathcal{Y}_n} = |A(\eta - \eta_n^{\delta_n})|^2$ so that the compactness of the operator A may be used to conclude that $A(\eta - \eta_n^{\delta_n}) \to 0$ in X as $n \to \infty$. Further, the Hölder continuity of k may be used to show that

$$\|\Big(\overline{A}_{s_n} - \mathcal{B}_n\Big)\eta_n^{\delta_n}\|^2_{\mathcal{Y}_n} \leq L_k^2\, s_{n,\max}^{2\mu_k}\,|\eta_n^{\delta_n}|^2/(2\mu_k + 1),$$

and standard calculations give

$$\begin{aligned}
\|F_n^{\delta_n} - F_n\|^2_{\mathcal{Y}_n} &\leq \delta_n^2\, s_{n,\max}/s_{n,\min},\\
\|F_n - \overline{F}_n\|^2_{\mathcal{Y}_n} &\leq L_f^2\, s_{n,\max}^{2\mu_f}/(2\mu_f + 1),
\end{aligned}$$

so that all three expressions converge to zero as $n \to \infty$, under the assumptions of Theorem 3.1. Combining these estimates, we have from (5.2) that $A\eta = f$ and, since $u \in L_2(0,1)$ uniquely solves (1.1), the weak (subsequential) convergence of η_n to u.

To complete the proof of Theorem 3.1, we examine the role of $\tilde{\varphi}$, the weak subsequential limit of $\{\mathcal{I}_n\varphi_n^{\delta_n}\}$. Since $T\tilde{\varphi} = \eta = u$, it is easily seen that $\tilde{\varphi}$ solves the operator equation

$$\tilde{A}\varphi = f \qquad (5.3)$$

where $\tilde{A} \in \mathcal{L}(\mathcal{X}, X)$ is defined for $\varphi \in \mathcal{X}$ by $\tilde{A}\varphi(t) = A(T\varphi)(t)$, a.a. $t \in [0,1]$.

It may also be shown that $\tilde{U} \in \mathcal{X}$ is the (unique) minimum-norm solution of (5.3), where $\tilde{U}$ is given by $\tilde{U}(t) \equiv u(t)\,\gamma/|\gamma|^2$, a.a. $t \in [0,1]$. But

$$\|\mathcal{I}_n\varphi_n^{\delta_n}\|^2_{\mathcal{X}} \leq \frac{1}{\alpha_n\, r_{n,\min}}\Big\{\|\mathcal{C}_n\varphi_n^{\delta_n} - F_n^{\delta_n}\|^2_{\mathcal{Y}_n} + \alpha_n\|\varphi_n^{\delta_n}\|^2_{\mathcal{X}_n}\Big\}$$

$$
\begin{aligned}
&\leq \frac{1}{\alpha_n\, r_{n,\min}} \left\{ \|\mathcal{C}_n \left(\mathcal{I}_n^{-1}\tilde{U}\right) - F_n^{\delta_n}\|^2_{\mathcal{Y}_n} + \alpha_n \|\, \mathcal{I}_n^{-1}\tilde{U}\,\|^2_{\mathcal{X}_n} \right\} \\
&\leq \frac{C}{\alpha_n\, r_{n,\min}} \left\{ \|\mathcal{A}_n U_n + \mathcal{B}_n T_n \left(\mathcal{I}_n^{-1}\tilde{U}\right) - F_n\|^2_{\mathcal{Y}_n} + \|\mathcal{A}_n \left(\mathcal{I}_n^{-1}\tilde{U}_n - U_n\right)\|^2_{\mathcal{Y}_n} \right. \\
&\qquad\qquad \left. + \|F_n - F_n^{\delta_n}\|^2_{\mathcal{Y}_n} + \alpha_n\, r_{n,\max} \|\tilde{U}\|^2_{\mathcal{X}} \right\}
\end{aligned}
$$

where U_n was defined for r_n in (2.7) and C is independent of n. It may be argued that

$$
\begin{aligned}
\|\mathcal{A}_n U_n + \mathcal{B}_n T_n \left(\mathcal{I}_n^{-1}\tilde{U}\right) - F_n\|_{\mathcal{Y}_n} &= \|\mathcal{A}_n U_n + \mathcal{B}_n u - F_n\|_{\mathcal{Y}_n} \\
&\leq \sqrt{\hat{s}_{n,\max}}\, \|k\|_\infty \, |u|_{\mathrm{ext}}
\end{aligned}
$$

where we used the fact that $T_n \left(\mathcal{I}_n^{-1}\tilde{U}\right) = T\tilde{U} = u$, and

$$
\|\mathcal{A}_n \left(\mathcal{I}_n^{-1}\tilde{U} - U_n\right)\|_{\mathcal{Y}_n} \leq \left(\sqrt{3\Delta_{n,\max}/2}\, \|k\|_\infty\right) \cdot \sqrt{r_{n,\max}} \left(\|\tilde{U}\|_{\mathcal{X}} + |u|_{\mathrm{ext}}\right).
$$

Under the assumptions of Theorem 3.1, the above estimates yield

$$
\begin{aligned}
\|\tilde{\varphi}\|^2_{\mathcal{X}} &\leq \liminf_{n\to\infty} \|\mathcal{I}_n \varphi_n^{\delta_n}\|^2_{\mathcal{X}} \\
&\leq \limsup_{n\to\infty} \|\mathcal{I}_n \varphi_n^{\delta_n}\|^2_{\mathcal{X}} \\
&\leq \limsup_{n\to\infty} \left\{ h(n) + \frac{r_{n,\max}}{r_{n,\min}} \|\tilde{U}\|^2_{\mathcal{X}} \right\}.
\end{aligned}
$$

where $h(n) \to 0$ as $n \to \infty$. Thus, under the assumption that $r_{n,\max}/r_{n,\min} \to 1$ as $n \to \infty$ it follows that $\|\tilde{\varphi}\|_{\mathcal{X}} \leq \|\tilde{U}\|_{\mathcal{X}}$ and thus, by the definition of $\tilde{U}$, it must be that $\tilde{\varphi} = \tilde{U}$. One also obtains that $\|\mathcal{I}_n \varphi_n^{\delta_n}\| \to \|\tilde{\varphi}\|$ from which the strong (subsequential) convergence of $\mathcal{I}_n \varphi_n^{\delta_n}$ to $\tilde{\varphi}$ obtains, and (again using the continuity of T) $\eta_n^{\delta_n} \to u$ as $n \to \infty$. Standard arguments then may be used to obtain *full* sequential convergence for both sequences. △

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