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Full convergence of sequential local regularization methods for Volterra inverse problems

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Abstract

Local regularization methods for ill-posed linear Volterra equations have been shown to be efficient regularization procedures preserving the causal structure of the Volterra problem and allowing for sequential solution methods. However questions posed recently in Ring and Prix (2000 *Inverse Problems* **16** 619–34) raise doubts as to whether such methods are convergent for problems which are more than just mildly ill-posed. In this paper we address these questions by reformulating the local regularization method via the use of signed Borel measures instead of the positive Borel measures used in earlier approaches. The result is a new theory for the local regularization of ν-smoothing Volterra problems for which stability and convergence is assured for *all* $\nu = 1, 2, \ldots$. In this paper we discuss this new local regularization theory for general finitely smoothing Volterra problems and demonstrate convergence and stability of the resulting method. In addition we indicate why using signed Borel measures instead of positive measures makes sense in the context of the Volterra problem and also has connections to the theory of mollification and approximate inverses (Louis A K 1996 *Inverse Problems* **12** 175–90). Finally we include numerical examples which illustrate the improvement which comes from using signed measures instead of positive measures and which facilitates an examination of the role played by the regularization parameter.

1. Introduction

We consider the problem of finding $\bar{u} \in C[0, T]$ solving

$$\mathcal{A}u = f \tag{1}$$

where $\mathcal{A}$ is the Volterra operator of convolution type given by

$$\mathcal{A}u(t) = \int_0^t k(t-s)u(s)\,\mathrm{d}s, \qquad t \in [0, T],$$

and f is in range of $\mathcal{A}$. We assume that the desired $\bar{u}$ satisfies the Hölder condition,

$$|\bar{u}(t) - \bar{u}(s)| \leqslant L_{\bar{u}}|t-s|^{\alpha}, \qquad t, s \in [0, T], \tag{2}$$

for $0 < \alpha \leqslant 1$, $L_{\bar{u}} > 0$ although, as discussed in remark 3.1, this hypothesis may be relaxed to $\bar{u}$ only continuous. We will also assume that the kernel k is 'ν-smoothing' [8, 10], or that

$$k \in C^{\nu}[0, T], \quad \text{and} \quad k^{(j)}(0) = 0, \quad j = 0, 1, \ldots, \nu - 2, \quad \text{with} \quad k^{(\nu-1)}(0) \neq 0.$$

Without loss of generality we will henceforth take $k^{(\nu-1)}(0) = 1$.

It is well known that the operator $\mathcal{A}$ is compact on $C[0, T]$ and (unless k is degenerate) that equation (1) is ill-posed due to lack of continuous dependence on data. Further, the ν-smoothing condition on k serves to characterize the degree of ill-posedness of the problem with the severity of the ill-posedness increasing with ν [10, 12]. For this reason a regularization method must be employed in the solution of the problem since one typically never has access to f but rather a perturbation f^{δ} of f.

Volterra problems have special structure that classical regularization techniques tend to ignore. For example, Volterra problems are causal in the sense that the solution $\bar{u}$ at a given value of $t \in [0, T]$ has no influence on the data f on the interval $[0, t)$. Thus it makes sense to use *future* values of the data only when reconstructing the solution at given values of t. In contrast, classical regularization methods (such as Tikhonov regularization) make use of *all* values of the data on $[0, T]$ when reconstructing the solution. The main reason for this is that classical methods rely on constructions using the operator $\mathcal{A}^{\star}\mathcal{A}$ [5, 7], where $\mathcal{A}^{\star}\mathcal{A}$ is a noncausal operator in the case of Volterra $\mathcal{A}$.

This problem becomes even more evident when classical regularization methods are discretized. Standard numerical methods to solve Volterra problems (i.e., methods which are unregularized beyond the regularization which comes from discretization alone) are typically *sequential* in nature. In contrast, numerical realizations of classical regularization methods are generally not sequential and thus tend to be much more expensive computationally.

The idea for developing a *sequential* method for the regularized solution of Volterra problems dates back to Beck [2] in the early 1960s who developed a very efficient and effective numerical procedure for the solution of the inverse heat conduction problem, a severely ill-posed Volterra problem. Although his algorithm was successfully used for many years by heat transfer engineers, the convergence and stability of the method was not established until the mid-1990s (see [8, 10]) when the ideas were extended to a large class of Volterra problems. The methods were generalized further in [4, 9, 11, 15–17]) and even extended to Fredholm (non-Volterra) problems in [13]. In the general Fredholm case we have 'local' methods (leading to the iterative solution of many small localized problems) instead of 'sequential' methods so it is for this reason that the general method is referred to as 'local regularization'; we use the terminology 'sequential local regularization' in the case of sequential methods for Volterra problems.

For the Volterra theory there still remains a number of open questions regarding the convergence of sequential local regularization methods for ν-smoothing problems in the case of general $\nu = 1, 2, \ldots,$ as will be discussed further in the following section. It is the purpose of this paper to address some of these questions.

1.1. Sequential local regularization methods

To motivate the sequential local regularization method for Volterra problems we let $r \in (0, R]$ be a small fixed constant and assume that equation (1) holds on the extended domain $[0, T+R]$, for sufficiently small $R > 0$ fixed; we note that this can always be accomplished by simply decreasing the size of T slightly. Then $\bar{u}$ satisfies

$$\int_0^{t+\rho} k(t + \rho - s)u(s)\, \mathrm{d}s = f(t + \rho), \qquad t \in [0, T], \quad \rho \in [0, r],$$

or, splitting the integral at t and making a change of integration variable,

$$\int_0^t k(t+\rho-s)u(s)\,\mathrm{d}s + \int_0^\rho k(\rho-s)u(t+s)\,\mathrm{d}s = f(t+\rho), \qquad t \in [0,T], \quad \rho \in [0,r].$$

For each $t \in [0, T]$, the ρ variable serves to advance the equation slightly into the future. In order to consolidate this future information, we integrate both sides of the equation with respect to a suitable Borel measure $\eta_r = \eta_r(\rho)$ on $[0, r]$ (the definition of η_r will be made more precise below), i.e.,

$$\int_0^t \int_0^r k(t+\rho-s)\,\mathrm{d}\eta_r(\rho)\,u(s)\,\mathrm{d}s + \int_0^r \int_0^\rho k(\rho-s)u(t+s)\,\mathrm{d}s\,\mathrm{d}\eta_r(\rho)$$
$$= \int_0^r f(t+\rho)\,\mathrm{d}\eta_r(\rho), \qquad t \in [0,T], \tag{3}$$

where we have made a change of order of integration in the first integral above.

We still have an equation that $\bar{u}$ satisfies exactly; however, when the true data f is replaced by a perturbation f^δ,

$$f^\delta \in C[0, T+R], \qquad \|f - f^\delta\|_\infty < \delta \tag{4}$$

(see remark 2.1 regarding the relaxation of the smoothness condition on the noisy data f^δ), a regularized form of this equation is needed in order to find a suitable approximation for $\bar{u}$ in this case. We accomplish this objective by replacing $u(t+s)$ by $u(t)$ in the second term in equation (3). The rationale behind this substitution is that we think of regularizing u by holding it constant (temporarily) on the small *local* interval of length $[t, t+r]$; the length r of this local interval becomes the regularization parameter.

The resulting equation then becomes, in the case of noisy data f^δ,

$$\int_0^t \tilde{k}_r(t-s)u(s)\,\mathrm{d}s + \alpha_r u(t) = \tilde{f}_r^\delta(t), \qquad t \in [0,T], \tag{5}$$

where

$$\tilde{k}_r(t) = \int_0^r k(t+\rho)\,\mathrm{d}\eta_r(\rho), \qquad \tilde{f}_r^\delta(t) = \int_0^r f^\delta(t+\rho)\,\mathrm{d}\eta_r(\rho), \tag{6}$$

$$\alpha_r = \int_0^r \int_0^\rho k(\rho-s)\,\mathrm{d}s\,\mathrm{d}\eta_r(\rho). \tag{7}$$

Under the conditions on η_r given in the following section we will see that $\alpha_r \neq 0$ for all r sufficiently small so that there is a unique solution $u_r^\delta \in L^2(0, T)$ of (5) which depends continuously on data f^δ in the $C[0, T]$ or $L_2(0, T)$ topologies, for example [6]. From assumption (4) on f^δ it is clear that $u_r^\delta \in C[0, T]$.

In [8] it was shown that convergence of regularized solutions u_r^δ to $\bar{u}$ occurs in the case of one-smoothing kernels as the level δ of noise in the data goes to zero, with a resulting convergence rate of order $\mathcal{O}(\delta^{1/2})$. For ν-smoothing kernels with $\nu > 1$, a different approach was required and indeed in [10] it was shown that for η_r a *positive* Borel measure satisfying conditions (h1), (h2) below, then convergence of u_r^δ to $\bar{u}$ also obtains

Theorem 1.1 [10]**.** *Let k be a ν-smoothing kernel and let $\bar{u} \in C^\nu[0, T+R]$ denote the solution of (1) for given f. Let η_r be a positive Borel measure satisfying the following conditions:*

(h1) For $i = 0, 1, \ldots, \nu$, there is some $\sigma \in \mathbb{R}$ and $c_i = c_i(\nu) > 0$ independent of r such that

$$\int_0^r \rho^i\,\mathrm{d}\eta_r(\rho) = r^{i+\sigma}(c_i + \mathcal{O}(r)), \qquad \textit{as} \quad r \to 0. \tag{8}$$

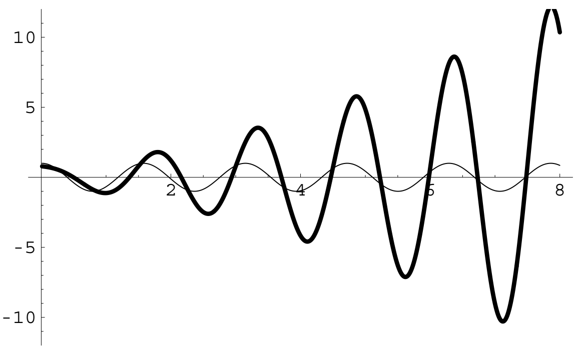

Figure 1. Four-smoothing example from [19], no added noise.

(h2) All ν roots of the polynomial

$$p_\nu(\lambda) = \frac{c_\nu}{\nu!}\lambda^\nu + \frac{c_{\nu-1}}{(\nu-1)!}\lambda^{\nu-1} + \cdots \frac{c_1}{1!}\lambda + \frac{c_0}{0!}$$

have negative real part, where the c_i are defined in (h1).

Then there is a $\hat{C} > 0$ such that if $\|k^{(\nu)}\|_\infty \leqslant \hat{C}$ we have the solution u_r of (5) with exact *data f satisfies $u_r(t) \to \bar{u}(t)$ as $r \to 0$, uniformly in $t \in [0, T]$. If in addition f^δ satisfies (4) for $\delta > 0$, then there is a choice of $r = r(\delta)$ such that $r(\delta) \to 0$ and $u^\delta_{r(\delta)} \to \bar{u}$* in *$L^2(0, T)$ as $\delta \to 0$.*

It was also shown in [10] that some basic positive Borel measures η_r (of the very general types defined in (13) and (17) below) satisfy hypotheses (h1), (h2) in the case of ν-smoothing k for $\nu = 1, 2, 3$, and that conditions (h1), (h2) are also satisfied for some specially constructed measures in the case of $\nu = 4$.

In 2000, Ring and Prix [19] improved upon the above result by weakening the smoothness condition on $\bar{u}$ and by giving a convergence rate. Additionally, they proved the following theorem which did not seem to bode well for sequential regularization methods when applied to problems with ν-smoothing kernels beyond the case of $\nu = 4$.

Theorem 1.2 [19]**.** *There is no family of positive Borel measures η_r for which hypotheses (h1)–(h2) hold in the case of $\nu \geqslant 5$.*

Of course, hypotheses (h1), (h2) are only sufficient conditions for the convergence of sequential regularization methods, so the failure of (h1), (h2) need not mean that convergence cannot occur. Nevertheless, an example in [19] seems to indicate otherwise.

Example 1.1. In this numerical example from [19], we let $k(t) = t^3/6$ (i.e., $\nu = 4$) and take η_r to be defined via $\int_0^r g(\rho)\,\mathrm{d}\eta_r(\rho) = \int_0^r g(\rho)\,\mathrm{d}\rho$, for $g \in C[0, r]$. We note that this measure does not satisfy hypotheses (h1), (h2) in the case of $\nu = 4$. A collocation-based piecewise-constant approximation is used with an approximation level of $N = 600$ on the interval [0, 8] and the regularization parameter is taken to be $r = 0.8$. In this example the true solution is given by $\bar{u}(t) = \cos(4t)$ and the data given by $f(t) = \mathcal{A}u(t)$. No noise is added to the data although of course there is error due to discretization. The results are shown in figure 1, with the approximate solution given in bold.

Thus for a very simple example it seems clear that convergence is not even obtained in the case of $\nu = 4$.

The goal of this paper is to develop *new* conditions on the measure η_r which will allow for the convergence of sequential local approximation methods for *all* $\nu = 1, 2, \ldots$. In developing the old theory, it was felt that the use of *positive* η_r in the definition of $\tilde{f}_t^\delta$ in (6), for example, corresponded to a type of weighted average of future data values over the interval $[t, t+r]$; thus it was somewhat surprising when the right condition on η_r turned out to be that the measure η_r need no longer be positive.

2. Signed Borel measures η_r

Henceforth we will let η_r denote a *signed* Borel measure on $[0, r]$ and require of η_r three conditions, the first two of which are similar to (h1), (h2) (except that now the c_i need no longer be positive) while a third hypothesis partially restores a property lost when dropping the positivity condition on η_r. The hypotheses are the following:

(H1) For $i = 0, 1, \ldots, \nu$, there is some $\sigma \in \mathbb{R}$ and $c_i = c_i(\nu) \in \mathbb{R}$ independent of r such that

$$\int_0^r \rho^i \, \mathrm{d}\eta_r(\rho) = r^{i+\sigma}(c_i + \mathcal{O}(r)), \qquad \text{as} \quad r \to 0, \tag{9}$$

with $c_\nu \neq 0$. Without loss of generality, we will assume that η_r has been scaled so that $c_\nu > 0$.

(H2) The parameters c_i, $i = 0, 1, \ldots, \nu$, satisfy the condition that all roots of the polynomial $p_\nu(\lambda)$ defined by

$$p_\nu(\lambda) = \frac{c_\nu}{\nu!}\lambda^\nu + \frac{c_{\nu-1}}{(\nu-1)!}\lambda^{\nu-1} + \cdots \frac{c_1}{1!}\lambda + \frac{c_0}{0!}$$

have negative real part.

(H3) There exists a $\tilde{C} \geqslant 0$ independent of r such that

$$\left| \int_0^r g(\rho) \, \mathrm{d}\eta_r(\rho) \right| \leqslant \tilde{C} \|g\|_\infty r^\sigma,$$

for all $g \in C[0, r]$ and all $r > 0$ sufficiently small.

We note that the required condition in hypothesis (H2) is a stability condition.

Under the conditions on η_r the following lemma shows that we have $\alpha_r \neq 0$ for all $r > 0$ sufficiently small and all ν-smoothing k; thus equation (5) is always well posed in these cases, with solutions depending continuously on data f^δ.

Lemma 2.1. *Assume η_r satisfies (H1) and (H3). Then*

$$\alpha_r = \frac{c_\nu}{\nu!} r^{\sigma+\nu}(1 + \mathcal{O}(r)), \tag{10}$$

so that $\alpha_r > 0$ for all $r > 0$ sufficiently small.

Proof. Under the ν-smoothing condition on k and hypothesis (H1) we have for $t \in [0, T]$,

$$k^{(p)}(t) = \frac{t^{\nu-(p+1)}}{(\nu-(p+1))!} + g_p(t), \qquad p = 0, 1, \ldots, \nu - 1, \tag{11}$$

where $g_p \in C^{\nu-p}[0, T]$,

$$|g_p(t)| \leqslant \frac{\|k^{(\nu)}\|_\infty t^{\nu-p}}{(\nu-p)!}, \qquad t \in [0, T]. \tag{12}$$

Thus,

$$\alpha_r = \int_0^r \int_0^\rho \left[\frac{(\rho - s)^{\nu-1}}{(\nu-1)!} + g_0(\rho - s) \right] \mathrm{d}s \, \mathrm{d}\eta_r(\rho)$$
$$= \frac{c_\nu}{\nu!} r^{\sigma+\nu} \left[1 + \int_0^r G_0(\rho) \, \mathrm{d}\eta_r(\rho) \right] (1 + \mathcal{O}(r)),$$

where $G_0(\rho) = \nu! \int_0^\rho g_0(\rho - s) \, \mathrm{d}s / (c_\nu r^{\sigma+\nu})$ so $\|G_0\|_\infty \leqslant C r^{1-\sigma}$ for some $C \geqslant 0$. Thus from (H3),

$$\left| \int_0^r G_0(\rho) \, \mathrm{d}\eta_r(\rho) \right| \leqslant \tilde{C} r^\sigma C r^{1-\sigma} = \mathcal{O}(r)$$

from which the estimate in (10) follows. □

In what follows we construct two classes of measures satisfying (H1)–(H3) and for which the roots of the polynomial p_ν in (H2) may be arbitrarily placed in $(-\infty, 0)$.

Lemma 2.2. *Let $\nu = 1, 2, \ldots,$ be arbitrary and let $\psi \in L^1(0, 1)$ be given such that*

$$\int_0^1 \rho^\nu \psi(\rho) \, \mathrm{d}\rho > 0.$$

Then the 'density' η_r for $r \in (0, R]$, $0 < R \leqslant 1$, defined via

$$\int_0^r g(\rho) \, \mathrm{d}\eta_r(\rho) = \int_0^r g(\rho) \psi_r(\rho) \, \mathrm{d}\rho, \qquad g \in C[0, r], \tag{13}$$

where $\psi_r \in L^1(0, r)$ is given by

$$\psi_r(\rho) = \psi(\rho/r), \qquad \text{a.a.} \quad \rho \in [0, r] \tag{14}$$

satisfies condition (H1) (with $c_\nu = \int_0^1 \rho^\nu \psi(\rho) \, \mathrm{d}\rho$ and $\sigma = 1$) and condition (H3).

Further, for all $\nu = 1, 2, \ldots$ and given arbitrary positive $\bar{c}, m_1, m_2, \ldots,$ and m_ν, there is a unique polynomial ψ of degree ν so that the resulting family $\{\eta_r\}$ satisfies (H1) with $c_\nu = \bar{c}$ and $\sigma = 1$, (H2) with the roots of the polynomial p_ν in (H2) given by $(-m_i)$, $i = 1, \ldots, \nu$ and (H3).

Proof. Hypothesis (H1) follows from the fact that

$$\int_0^r \rho^i \psi_r(\rho) \, \mathrm{d}\rho = r^i \int_0^1 \rho^i \psi_r(r\rho) r \, \mathrm{d}\rho = r^{i+1} c_i$$

where

$$c_i \equiv \int_0^1 \rho^i \psi(\rho) \, \mathrm{d}\rho, \qquad i = 0, 1, \ldots, \nu,$$

and $c_\nu > 0$. In addition, for $g \in C[0, r]$,

$$\left| \int_0^r g(\rho) \psi_r(\rho) \, \mathrm{d}\rho \right| = r \left| \int_0^1 g(r\rho) \psi(\rho) \, \mathrm{d}\rho \right| \leqslant \tilde{C} \|g\|_\infty r$$

where $\tilde{C} = \int_0^1 |\psi(\rho)| \, \mathrm{d}\rho$.

Let d_i denote the coefficients of the monic polynomial $P_\nu(\lambda) = \prod_{i=1}^\nu (\lambda + m_i)$, i.e.,

$$\lambda^\nu + d_{\nu-1} \lambda^{\nu-1} + \cdots d_1 \lambda + d_0 = \prod_{i=1}^\nu (\lambda + m_i)$$

$(d_\nu = 1)$. We will find a polynomial ψ such that the resulting p_ν defined in (H2) satisfies $p_\nu(\lambda) = \bar{c} P_\nu(\lambda)/\nu!$, $\lambda \in \mathbb{C}$. That is, we seek $a_0, a_1, \ldots, a_\nu$ so that $\psi(\rho) \equiv \sum_{j=0}^\nu a_j \rho^j$

satisfies $c_i \equiv \int_0^1 \rho^i \psi(\rho)\,\mathrm{d}\rho = i!\frac{\bar{c}}{\nu!}d_i$, for $i = 0, 1, \ldots \nu$; i.e., the vector $\mathbf{a} = (a_0, a_1, \ldots, a_\nu)^\top$ satisfies $\mathbf{Ha} = \mathbf{d}$, where $\mathbf{d} = \left(0!\frac{\bar{c}}{\nu!}d_0, 1!\frac{\bar{c}}{\nu!}d_1, \ldots, (\nu-1)!\frac{\bar{c}}{\nu!}d_{\nu-1}, \bar{c}\right)^\top$, and $\mathbf{H}$ is the nonsingular $(\nu+1)$-square Hilbert matrix with entries $\mathbf{H}_{i,j} = 1/(i+j+1)$. There is thus a unique vector $\mathbf{a}$ and a unique ν-degree polynomial ψ generating the measure η_r which satisfies (H2). □

Lemma 2.3. *Let $\nu = 1, 2, \ldots,$ be arbitrary and let $\beta_\ell, \tau_\ell \in \mathbb{R}$, $\ell = 0, 1, \ldots, L$, be fixed so that*

$$0 \leqslant \tau_0 < \tau_1 < \cdots < \tau_{L-1} < \tau_L \leqslant 1, \tag{15}$$

and

$$\sum_{\ell=0}^{L} \beta_\ell \tau_\ell^\nu > 0. \tag{16}$$

Then the discrete measure η_r defined via

$$\int_0^r g(\rho)\,\mathrm{d}\eta_r(\rho) = \sum_{\ell=0}^{L} \beta_\ell g(\tau_\ell r), \qquad g \in C[0, r], \tag{17}$$

satisfies condition (H1) (with $c_\nu = \sum_{\ell=0}^{L} \beta_\ell \tau_\ell^\nu$ and $\sigma = 0$) and condition (H3).

Further, for all $\nu = 1, 2, \ldots$ and given arbitrary positive $\bar{c}, m_1, m_2, \ldots,$ and m_ν and for $L = \nu$, there is a unique choice of $\beta_0, \beta_1, \ldots, \beta_\nu$ satisfying (16) (for each given collection of $\{\tau_\ell\}$ satisfying (15)) and such that the resulting discrete measure η_r satisfies (H1) with $c_\nu = \bar{c}$ and $\sigma = 0$, (H2) with the roots of the polynomial p_ν in (H2) given by $(-m_i)$, $i = 1, 2, \ldots, \nu$ and (H3).

Proof. It is clear that $\int_0^r \rho^i\,\mathrm{d}\eta_r(\rho) = r^i c_i$ with

$$c_0 = \sum_{\ell=0}^{L} \beta_\ell, \qquad c_i = \sum_{\ell=0}^{L} \beta_\ell \tau_\ell^i, \qquad i = 1, \ldots, \nu,$$

with $c_\nu > 0$, so that (H1) is satisfied with $\sigma = 0$. In addition, for $g \in C[0, r]$,

$$\left| \int_0^r g(\rho)\,\mathrm{d}\eta_r(\rho) \right| \leqslant \tilde{C}\|g\|_\infty$$

with $\tilde{C} = \sum_{\ell=0}^{L} |\beta_\ell|$ so that (H3) holds. In addition, using the same d_i and $\mathbf{d}$ as defined in the proof of lemma 2.2, hypothesis (H2) requires that the c_i in (9) satisfy $c_i = i!\frac{\bar{c}}{\nu!}d_i$, for $i = 0, \ldots, \nu$; indeed, for $\mathbf{b} = (\beta_0, \beta_1, \ldots, \beta_\nu)^\top$, one requires $\mathbf{Vb} = \mathbf{d}$, where the Vandermonde matrix

$$\mathbf{V} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \tau_0^1 & \tau_1^1 & \cdots & \tau_\nu^1 \\ \vdots & \vdots & \ddots & \vdots \\ \tau_0^\nu & \tau_1^\nu & \cdots & \tau_\nu^\nu \end{pmatrix}$$

is nonsingular because the τ_i are distinct. Thus hypothesis (H2) holds. □

Remark 2.1. We note that the use of a Borel measure facilitates the type of discrete measure constructed in lemma 2.3, but at the same time necessitates the use of smooth (or piecewise smooth) perturbed data f^δ (which can be accomplished *a priori* by a smoothing procedure). If measures of the type defined by (13) in lemma 2.2 are all that is needed, then one may obviously relax the smoothness conditions on f^δ. However, as was shown in [8] the original sequential method created by Beck can be thought of as a discretization of (5) with a special discrete measure; thus it seems appropriate to include the possibility of discrete measures in our theory here.

3. Main results

We turn now to the convergence of regularized approximations u_r^δ to $\bar{u}$. To this end we define $y_r(t) = u_r^\delta(t) - \bar{u}(t)$ and $d(t) = f^\delta(t) - f(t)$, where $\|d\|_\infty \leqslant \delta$, and note that y_r satisfies for $t \in [0, T]$,

$$\int_0^t \tilde{k}_r(t-s) y_r(s)\,\mathrm{d}s + \alpha_r y_r(t)$$
$$= \int_0^r \mathrm{d}(t+\rho)\,\mathrm{d}\eta_r(\rho) + \int_0^r \int_0^\rho k(\rho - s)[\bar{u}(t+s) - \bar{u}(t)]\,\mathrm{d}s\,\mathrm{d}\eta_r(\rho),$$

or

$$K_r \star y_r(t) + y_r(t) = F_r^\delta(t), \qquad t \in [0, T], \tag{18}$$

where for $t \in [0, T]$,

$$K_r(t) = \frac{1}{\alpha_r}\tilde{k}_r(t), \tag{19}$$

$$F_r^\delta(t) = \frac{1}{\alpha_r}\left[\int_0^r \mathrm{d}(t+\rho)\,\mathrm{d}\eta_r(\rho) + \int_0^r \int_0^\rho k(\rho - s)[\bar{u}(t+s) - \bar{u}(t)]\,\mathrm{d}s\,\mathrm{d}\eta_r(\rho)\right]. \tag{20}$$

We will first need some estimates on the size of F_r^δ and of derivatives of K_r:

Lemma 3.1. *For K_r and F_r^δ given by (19) and (20), respectively, and for $\bar{u}$ satisfying (2) on $[0, T+R]$ for $R > 0$ small, we have*

$$K_r^{(p)}(t) = r^{-(\sigma+\nu)}\frac{\nu!}{c_\nu}\int_0^r k^{(p)}(t+\rho)\,\mathrm{d}\eta_r(\rho)(1 + \mathcal{O}(r)), \qquad p = 0, \ldots, \nu, \tag{21}$$

$$K_r^{(p)}(0) = \frac{\nu!}{(\nu - (p+1))!}\frac{r^{-(p+1)}}{c_\nu}[c_{\nu-(p+1)} + \mathcal{O}(r)], \qquad p = 0, \ldots, \nu - 1, \tag{22}$$

$$\left|F_r^\delta(t)\right| \leqslant C_1\frac{\delta}{r^\nu} + C_2 r^\alpha, \qquad t \in [0, T], \tag{23}$$

for non-negative constants C_1 and C_2 independent of r and where $\alpha > 0$ is the Hölder exponent for $\bar{u}$ and δ is given in (4).

Proof. Equation (21) follows from lemma 2.1, while for (22) and $p = 0, 1, \ldots, \nu - 1$, we use (11) to obtain

$$\int_0^r k^{(p)}(\rho)\,\mathrm{d}\eta_r(\rho) = \int_0^r \left[\frac{\rho^{\nu-(p+1)}}{(\nu - (p+1))!} + g_p(\rho)\right]\mathrm{d}\eta_r(\rho)$$
$$= \frac{r^{\nu-(p+1)+\sigma}}{(\nu - (p+1))!}c_{\nu-(p+1)}(1 + \mathcal{O}(r)) + \int_0^r g_p(\rho)\,\mathrm{d}\eta_r(\rho),$$

where from (12) and (H3) we obtain

$$\left|\int_0^r g_p(\rho)\,\mathrm{d}\eta_r(\rho)\right| \leqslant C r^{\nu-p+\sigma},$$

for some $C > 0$ independent of r. It follows that

$$\int_0^r k^{(p)}(\rho)\,\mathrm{d}\eta_r(\rho) = \frac{r^{\nu-(p+1)+\sigma}}{(\nu - (p+1))!}(c_{\nu-(p+1)} + \mathcal{O}(r)).$$

Thus since $K_r^{(p)}(0) = \int_0^r k^{(p)}(\rho)\,\mathrm{d}\eta_r(\rho)/\alpha_r$, the estimate in (22) follows from (10).

For (23), we first note that from the Hölder condition on $\bar{u}$ we have

$$\left|\int_0^\rho k(\rho - s)[\bar{u}(t+s) - \bar{u}(t)]\,\mathrm{d}s\right| \leqslant L_{\bar{u}}\rho^\alpha \left[\int_0^\rho \frac{(\rho - s)^{\nu-1}}{(\nu-1)!}\,\mathrm{d}s + \int_0^\rho |g_0(\rho - s)|\,\mathrm{d}s\right]$$
$$\leqslant L_{\bar{u}}\frac{r^{\nu+\alpha}}{\nu!}\left[1 + \frac{r}{(\nu+1)}\|k^{(\nu)}\|_\infty\right],$$

where we have used (11), (12). Thus from (H3) and (10)

$$\left|F_r^\delta(t)\right| \leqslant \tilde{C}\frac{\nu!(1+\mathcal{O}(r))}{c_\nu r^{\sigma+\nu}} r^\sigma \left[\|d\|_\infty + L_{\bar{u}}\frac{r^{\nu+\alpha}}{\nu!}(1+\mathcal{O}(r))\right],$$

so that we obtain the results of the lemma. □

Our central convergence result is given in theorem 3.1. As lemmas 2.2, 2.3 show, there are an infinite number of choices of families η_r which satisfy (H1)–(H3) for *all* $\nu = 1, 2, \ldots$. Thus we are assured from theorem 3.1 that we have at hand a large choice of sequential local regularization methods which converge for *all* finitely smoothing Volterra problems.

Theorem 3.1. *Let $\bar{u}$ denote the solution of (1) given 'true' data $f \in C[0, T+R]$ and assume $\bar{u}$ satisfies (2) on $[0, T+R]$ with some $\alpha \in (0, 1]$ and $R > 0$ small. Assume k is ν-smoothing and that $\{\eta_r\}$ is a family of signed Borel measures satisfying hypotheses (H1)–(H3) for all $r \in (0, R]$. Then there is a constant $\hat{C} > 0$ (depending only on the c_i defined in (H1) and independent of r) such that if*

$$\|k^{(\nu)}\|_\infty < \hat{C},$$

then for all $t \in [0, T]$ and in the case of exact data f we have

$$|u_r(t) - \bar{u}(t)| = \mathcal{O}(r^\alpha) \to 0 \qquad \text{as} \quad r \to 0.$$

If in addition $f^\delta \in C[0, T+R]$ satisfies (4), then

$$\left|u_r^\delta(t) - \bar{u}(t)\right| \leqslant C_1\frac{\delta}{r^\nu} + C_2 r^\alpha, \qquad t \in [0, T],$$

for some $C_1, C_2 \geqslant 0$, so that the choice

$$r = r(\delta) \sim \delta^{\frac{1}{\alpha+\nu}}$$

gives

$$\left|u_r^\delta(t) - \bar{u}(t)\right| = \mathcal{O}\left(\delta^{\frac{\alpha}{\alpha+\nu}}\right) \qquad \text{as} \quad \delta \to 0,$$

uniform in $t \in [0, T]$.

Proof. We may rewrite the error equation (18) as

$$y_r(rt) + \int_0^{rt} K_r(rt - s)y_r(s)\,\mathrm{d}s = F_r^\delta(rt), \qquad t \in [0, T/r],$$

or, making a change of integration variable and defining $\tilde{y}_r(t) = y_r(rt)$, $\tilde{f}_r^\delta(t) = F_r^\delta(rt)$, for $t \in [0, T/r]$,

$$\tilde{y}_r(t) + r\int_0^t K_r(r(t-s))\tilde{y}_r(s)\,\mathrm{d}s = \tilde{f}_r^\delta(t), \quad t \in [0, T/r],$$

so that

$$\tilde{y}_r(t) = \tilde{f}_r^\delta(t) - \tilde{R}_r \star \tilde{f}_r^\delta(t), \qquad t \in [0, T/r], \tag{24}$$

where $\tilde{R}_r$ solves the resolvent equation

$$\tilde{R}_r(t) + r\int_0^t K_r(r(t-s))\tilde{R}_r(s)\,\mathrm{d}s = rK_r(rt), \qquad t \in [0, T/r] \tag{25}$$

(see, e.g., [6]). Clearly $\tilde{R}_r$ is sufficiently smooth to allow ℓ-times differentiation of equation (25) for $\ell = 1, 2, \ldots, \nu$,

$$\begin{aligned}\tilde{R}_r^{(\ell)}(t) &+ rK_r(0)\tilde{R}_r^{(\ell-1)}(t) + r^2K_r'(0)\tilde{R}_r^{(\ell-2)}(t) + \cdots + r^\ell K_r^{(\ell-1)}(0)\tilde{R}_r(t)\\ &+ \int_0^t r^{\ell+1}K_r^{(\ell)}(r(t-s))\tilde{R}_r(s)\,\mathrm{d}s = r^{\ell+1}K_r^{(\ell)}(rt), \qquad t \in [0, T/r],\end{aligned} \tag{26}$$

or for $\ell = \nu$ and using lemma 3.1,

$$\begin{aligned}\tilde{R}_r^{(\nu)}(t) = &-\sum_{i=1}^{\nu} \frac{c_{\nu-i} + \mathcal{O}(r)}{c_\nu}\frac{\nu!}{(\nu-i)!}\tilde{R}_r^{(\nu-i)}(t)\\ &- \frac{\nu!(1+\mathcal{O}(r))}{c_\nu r^{\sigma+\nu}}\int_0^t r^{\nu+1}\int_0^r k^{(\nu)}(r(t-s)+\rho)\,\mathrm{d}\eta_r(\rho)\tilde{R}_r(s)\,\mathrm{d}s\\ &+ \frac{\nu!(1+\mathcal{O}(r))}{c_\nu r^{\sigma+\nu}} r^{\nu+1}\int_0^r k^{(\nu)}(rt+\rho)\,\mathrm{d}\eta_r(\rho),\end{aligned}$$

for $t \in [0, T/r]$. We note that the introduction of the resolvent equation allows us to use the inherent smoothness of $\tilde{R}_r$ to perform ν differentiations of equation (25) instead of equation (24), thus avoiding the requirement that the true solution $\bar{u}$ have ν derivatives [19].

Defining $\mathbf{r}_r(t) = \left(\tilde{R}_r(t), \tilde{R}_r'(t), \ldots, \tilde{R}_r^{(\nu-1)}(t)\right)^\top$, we have

$$\mathbf{r}_r'(t) = \mathbf{A}_r\mathbf{r}_r(t) + \mathbf{M}_r\mathbf{r}_r(t) + \int_0^t \mathbf{D}_r(t-s)\mathbf{r}_r(s)\,\mathrm{d}s + \mathbf{g}_r(t), \qquad t \in [0, T/r],$$

where

$$\mathbf{A}_r = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \ddots & 0 & 1\\ -\gamma_0 & -\gamma_1 & -\gamma_2 & \cdots & -\gamma_{\nu-1}\end{pmatrix},$$

$$\gamma_i = \frac{c_i}{c_\nu}\frac{\nu!}{i!}, \qquad i = 0, \ldots, \nu-1,$$

$$\|\mathbf{M}_r\| = \mathcal{O}(r) \qquad \text{as} \quad r \to 0,$$

while for $t \in [0, T/r]$,

$$\mathbf{D}_r(t) = \begin{pmatrix} 0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ -H_r(t) & 0 & \cdots & 0\end{pmatrix},$$

$$\mathbf{g}_r(t) = (0, 0, \ldots, 0, H_r(t))^\top,$$

$$H_r(t) = \frac{\nu! r^{1-\sigma}(1+\mathcal{O}(r))}{c_\nu}\int_0^r k^{(\nu)}(tr+\rho)\,\mathrm{d}\eta_r(\rho).$$

Since $k^{(\nu)}(tr+\rho) \leqslant \|k^{(\nu)}\|_\infty$ for $t \in [0, T/r]$ and $\rho \in [0, r]$, we have from (H3) that

$$|H_r(t)| \leqslant \frac{2\tilde{C}\|k^{(\nu)}\|_\infty \nu! r}{c_\nu}, \qquad t \in [0, T/r],$$

for all r sufficiently small. From (H2), the eigenvalues λ_i of $\mathbf{A}_r$ are the roots of the polynomial $p_\nu(\lambda)$ and as such all have negative real part.

We note in addition that from (26)

$$\tilde{R}^{(\ell)}(0) = -\sum_{i=1}^{\ell} r^i K_r^{(i-1)}(0)\tilde{R}_r^{(\ell-i)}(0) + r^{\ell+1}K_r^{(\ell)}(0), \tag{27}$$

for $\ell = 1, \ldots, \nu - 1$. But from lemma 3.1 we have

$$\begin{aligned} r^i K_r^{(i-1)}(0) &= \frac{\nu!}{(\nu - i)!}\frac{c_{\nu-i}}{c_\nu}(1 + \mathcal{O}(r)) \\ &= \kappa_{i-1}(1 + \mathcal{O}(r)), \qquad i = 1, \ldots, \nu, \end{aligned}$$

where κ_{i-1} is independent of r. We then have $\tilde{R}_r(0) = rK_r(0) = \kappa_0(1 + \mathcal{O}(r))$ and an inductive argument applied to (27) yields

$$\tilde{R}_r^{(\ell)}(0) = \hat{\kappa}_\ell(1 + \mathcal{O}(r)), \qquad \ell = 1, \ldots, \nu - 1,$$

where $\hat{\kappa}_\ell$ is independent of r. Thus we may bound $\|\mathbf{r}_r(0)\| \leqslant C < \infty$ for all r sufficiently small, where $C > 0$ is independent of r.

We may use arguments similar to those in the proof of lemma 1 from [19] to claim that there exist constants $\hat{C} > 0$ (depending on $\nu, c_0, \ldots c_\nu$) and $M > 0$ (depending on $k, \nu, c_0, \ldots c_\nu$) so that $\|\tilde{R}_r\|_{L^1(0,T/r)} \leqslant M$, provided $\|k^{(\nu)}\|_\infty < \hat{C}$. The only estimates which change in [19] are those in (2.29) of that reference and those which rely on (2.29). For the estimate in (2.29) in [19] we note that

$$\int_t^{T/r} \|\mathbf{D}_r(s - t)\|\,\mathrm{d}s = \int_t^{T/r} |H_r(s - t)|\,\mathrm{d}s \leqslant \frac{2\tilde{C}T\|k^{(\nu)}\|_\infty \nu!}{c_\nu}.$$

The rest of the proof of lemma 1 in [19] follows if one replaces $\|k^{(\nu)}\|_{L^1(0,1+r)}$ in that reference by $\|k^{(\nu)}\|_\infty$ (which is needed in the case of η_r a signed measure). We note that the notation in [19] differs somewhat from the notation used here. In [19], the parameter ρ replaces the r used here; the interval [0, 1] is $[0, T]$ here; the resolvent r in [19] becomes $\tilde{R}_r$ here, while the parameter k_ν in [19] satisfies $k_\nu = 1$ under the assumptions in this paper. Briefly the work remaining in lemma 1 of [19] is to construct a Lyapunov functional for use in analysing the stability of the resolvent equation. The details, while technical, are largely unchanged from [19].

Returning to equation (24), we thus have

$$|\tilde{y}_r(t)| \leqslant \left\|\tilde{f}_r^\delta\right\|_{C[0,T/r]}(1 + M), \qquad t \in [0, T/r],$$

or, for $t \in [0, T]$,

$$\begin{aligned} |y_r(t)| &\leqslant (1 + M)\left\|F_r^\delta\right\|_{C[0,T]} \\ &\leqslant (1 + M)\left(C_1\frac{\delta}{r^\nu} + C_2 r^\alpha\right), \end{aligned}$$

where we have used lemma 3.1. □

Remark 3.1. We note that one also obtains a stability estimate for the continuous dependence of approximate solutions u_r^δ on data, and for convergence in the case of $\bar{u} \in C[0, T]$ not satisfying (2). These results are similar to those in [19] and will not be repeated here.

4. Discussion concerning the use of signed versus positive measures η_r

We will briefly try to motivate why the use of a signed measure is a reasonable choice in the context of sequential local regularization of a Volterra problem.

Suppose for simplicity that the polynomial p_ν in (H2) has a repeated real root $(-m)$, $m > 0$ so that

$$p_\nu(\lambda) = \bar{C}(\lambda + m)^\nu,$$

with $\bar{C} = \frac{c_\nu}{\nu!} > 0$ from the assumptions on c_ν in (H1), (H2). For sufficiently smooth $g : [0, r] \mapsto \mathbb{R}$, we may expand g in a Taylor series about the point $\rho = 0$ and obtain

$$\begin{aligned}
\int_0^r g(\rho)\,\mathrm{d}\eta_r(\rho) &\approx \sum_{i=0}^{\nu} \frac{1}{i!} g^{(i)}(0) \int_0^r \rho^i\,\mathrm{d}\eta_r(\rho)\\
&\approx \left(\sum_{i=0}^{\nu} \frac{c_i}{i!} r^{i+\sigma} D^i\right)(g)(0)\\
&= r^\sigma p_\nu(rD)g(0)\\
&= \bar{C}r^\sigma\,(rD + m)^\nu g(0)\\
&= \bar{C}r^{\nu+\sigma}\left(D + \frac{m}{r}\right)^\nu g(0),
\end{aligned}$$

where D^i denotes the ith derivative operator.

Thus for data f sufficiently smooth and for ν-smoothing k, the quantities $\tilde{f}_r$ and $\tilde{k}_r$ defined by (6) with $\delta = 0$ become (approximately)

$$\tilde{f}_r(t) \approx \bar{C}r^{\nu+\sigma}\left(D + \frac{m}{r}\right)^\nu f(t), \qquad \tilde{k}_r(t) \approx \bar{C}r^{\nu+\sigma}\left(D + \frac{m}{r}\right)^\nu k(t). \tag{28}$$

Further, from (10) we have $\alpha_r \approx \bar{C}r^{\nu+\sigma}$, so equation (5) becomes (approximately, dropping highest order terms),

$$\int_0^t \left(D + \frac{m}{r}\right)^\nu k(t-s)u(s)\,\mathrm{d}s + u(t) = \left(D + \frac{m}{r}\right)^\nu f(t), \qquad t \in [0, T]. \tag{29}$$

There are several points of interest regarding the above discussion:

- One is the appearance of ν derivatives of k, f in (29). It is well known that for ν-smoothing k, the original equation (1) may be differentiated ν times to obtain a second-kind equation
$$\int_0^t D^\nu k(t-s)u(s)\,\mathrm{d}s + u(t) = D^\nu f(t), \qquad t \in [0, T], \tag{30}$$
which is well posed with solutions depending continuously on the *differentiated* data $f^{(\nu)}$. (Indeed, if $k(t) = t^{\nu-1}/(\nu-1)!$, then $\mathcal{A}\bar{u} = f$ is equivalent to $\bar{u} = D^\nu f$.) Thus when $\frac{m}{r} = 0$, equation (29) reduces to (30) and solutions depend continuously on $f^{(\nu)}$. On the other hand, for $\frac{m}{r} > 0$ we have from theorem 3.1 and remark 3.1 that the regularized equation (5) (for which (29) is an approximation) has solutions depending continuously on the *undifferentiated* data f.
- The second point concerns the presence of the ratio $\frac{m}{r}$ in (29). Obviously as $r \to 0$ equation (29) degenerates, as one would expect when the amount of stability supplied by regularization decreases to zero. However, for given $r > 0$ (fixed, perhaps, for a numerical procedure), it is clear that the ratio $\frac{m}{r}$ is important to the stability of the numerical result. Recall that for *any* $m > 0$ we can find classes of η_r for which the hypotheses (H1)–(H3) hold for all $\nu = 1, 2, \ldots$. The appropriate regularization parameter r for a given level δ of error in the data is thus linked to m (i.e., to the choice of family η_r). This will be illustrated further in example 5.3 in section 5.

- Additionally, as we indicated earlier, with a signed measure η_r we no longer have the interpretation that the quantity

$$\tilde{f}_r(t) = \int_0^r f(t+\rho)\,\mathrm{d}\eta_r(\rho)$$

is a type of weighted average (with *positive* weights) of data f on the future interval $[t, t+r]$. However from (28),

$$\begin{aligned}\tilde{f}_r(t) &\approx \bar{C} r^{\nu+\sigma}\left(D+\frac{m}{r}\right)^{\nu} f(t)\\ &= \bar{C} r^{\nu+\sigma}\sum_{i=0}^{\nu}\binom{\nu}{i}\left(\frac{m}{r}\right)^{\nu-i} D^i f(t)\end{aligned} \tag{31}$$

so that $\tilde{f}_r(t)$ *does* approximate a weighted sum (with *positive* weights) of *derivatives* of f. This construction has some of the flavour of the ideas in [1] where the problem of ν-numerical differentiation of noisy data is considered (i.e., a problem equivalent to a discretization of (1) when $k(t) = t^{\nu-1}/(\nu-1)!$). In [1] it was found that the proper way to approximate $f^{(\nu)}$ when one only has access to a perturbation f^δ of f is via a weighted running average of centred differences applied to f^δ, where the number of differences in the weighted average corresponds to the regularization parameter r in this paper. We have a similar construction here when D^i in (31) is approximated by a central difference operator.
- Finally we mention an analogy with the method of mollifiers and approximate inverses found, for example, in [18]. There the idea is that one is not necessarily interested in recovering $\bar{u}(t)$ exactly but may be happy to instead recover a linear functional such as

$$\int_0^T \bar{u}(t)\mu(t)\,\mathrm{d}t \tag{32}$$

where μ is a suitably defined (positive) mollifier function. The approach typically is to assume that μ is in the range of $\mathcal{A}^\star$, i.e., $\mu = \mathcal{A}^\star\phi$, for some ϕ, so that one may then write

$$\begin{aligned}\int_0^T \bar{u}(t)\mu(t)\,\mathrm{d}t &= \int_0^T \bar{u}(t)(\mathcal{A}^\star\phi(t))\,\mathrm{d}t\\ &= \int_0^T (\mathcal{A}\bar{u}(t))\,\phi(t)\,\mathrm{d}t\\ &= \int_0^T f(t)\phi(t)\,\mathrm{d}t.\end{aligned}$$

That is, one uses the quantity $\tilde{f}_\phi \equiv \int_0^T f(t)\phi(t)\,\mathrm{d}t$, or in the case of perturbed data,

$$\tilde{f}_\phi^\delta \equiv \int_0^T f^\delta(t)\phi(t)\,\mathrm{d}t,$$

(which corresponds to our $\tilde{f}_r^\delta$ in (6)) where ϕ need *not* be a positive function, as part of the process of reconstructing the desired quantity (32) which is defined using a (positive) mollification of $\bar{u}$.

5. Numerical results

In all numerical examples in this paper, we used Mathematica to evaluate collocation-based discretizations over the space of piecewise constant functions (defined on a uniform grid

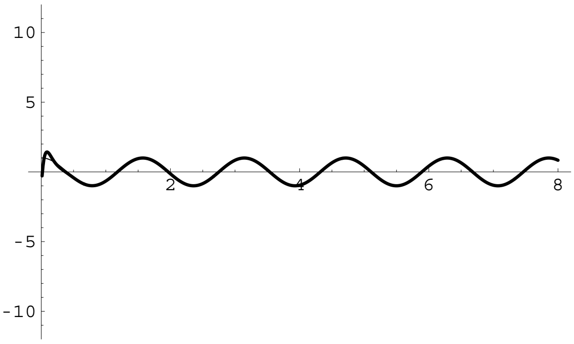

Figure 2. Example 5.1: repeat of example 1.1 (figure 1), but now with a signed Borel measure η_r.

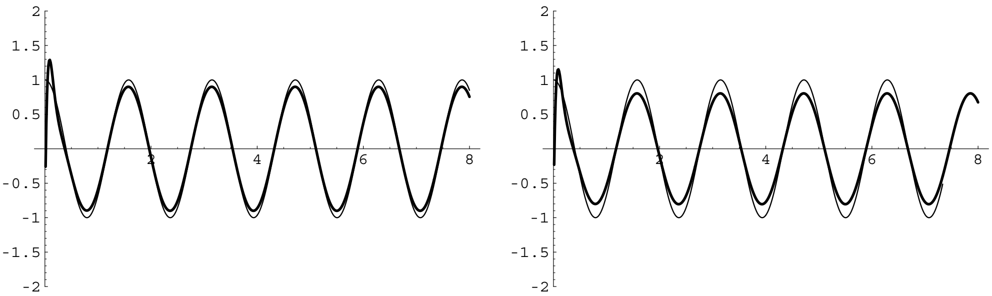

Figure 3. Example 5.1, continued: 0.1% (left) and 0.2% (right) relative error in f.

of $N + 1$ points starting at 0 and ending at T). Unless otherwise indicated, the measure η_r is defined as in lemma 2.2, with c_i constructed so that the polynomial p_ν in (H2) is given by

$$p_\nu(\lambda) = (\lambda + m)^\nu, \tag{33}$$

where $m > 0$ will be specified in each example below. We note that the polynomial ψ generated in lemma 2.2 can be highly oscillatory when ν is somewhat large; thus accurate quadrature routines should be used when evaluating the integrals involving η_r.

In each case below we illustrate the results with a plot of $\bar{u}$ compared to a plot of the approximate curve (in bold). The 'true' solution $\bar{u}$ was selected *a priori* and 'true' data found by integrating to find $f(t) = \mathcal{A}\bar{u}(t)$. Uniformly distributed random error was added to f in some examples to generate f^δ, where the relative error $\|f^\delta - f\|/\|f\|$ will be given in applicable examples below.

Example 5.1. First we return to the problem with a four-smoothing kernel for example 1.1, but now we use the methods developed in sections 2 and 3. As in example 1.1 we use $N = 600$, $T = 8$ and $r = 0.8$; we also let $m = 10$ in (33). The results are shown in figure 2 (compare with figure 1). In figure 3 we show the same example (with a zoomed scale) where 0.1% and 0.2% levels of error have been added to the data.

Example 5.2. We now consider a two-smoothing example, with $k(t)=t^2/2!$. In this case, $T = 1.0$, $N = 100$, $r = 0.15$ and $m = 10$ (in (33)). We used $\bar{u}(t) = 1 + 3t(\sin(10t) - \sin t)$. The results are shown in figure 4 for no noise in the data, and for 1% and 3% relative error in data.

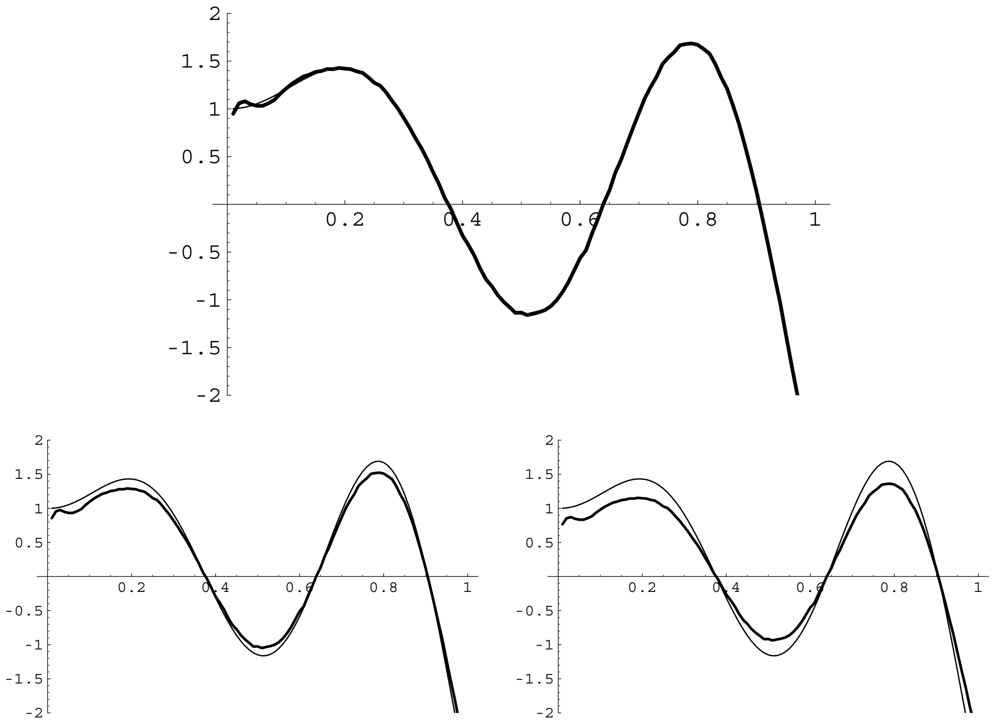

Figure 4. Example 5.2: two-smoothing example with no noise (top), and with 1% (bottom left) and 3% (bottom right) relative error in f.

Example 5.3. Repeating Example 5.2, we look at the effects of varying the root m in (33). Throughout $r = 0.35$ is held fixed (and other parameters are set as in example 5.2); there is no noise in the data. See figures 5 for the results when $m = \frac{1}{2}, 1, 5, 10, 15$ and 20. As was discussed in section 4, it is the ratio m/r that is important and this example illustrates that point. For m too large or r too small, we have under-regularization; for m too small or r too large, we have over-regularization. (Note, the effects of changing r alone in numerical examples using a positive measure η_r have been documented in several references [8–10] including the possibility of variable $r = r(t)$ [11, 16, 17]).

In general there seems to be a rather wide window of 'acceptable' m-values, given a value of r, but the question of the exact relationship of r to m and the selection of appropriate regularization parameters remains an open one. Another interesting question is the relationship between m and the bound on $\|k^{(\nu)}\|_\infty$ in theorem 3.1. As seen in the proof of the theorem, the bound on $\|k^{(\nu)}\|_\infty$ depends only on the parameters c_i and these parameters in term depend on the choice of m (or on $m_1, m_2, \ldots, m_\nu$ in the case of multiple roots of the polynomial p_ν in (H2)). The overall question of the selection of r and m, with regard to these and other issues, is the focus of an ongoing study.

Remark 5.1. It is interesting to note that the approximations using signed Borel measures tend to be in error (slightly) near $t = 0$, a phenomenon not seen for approximations using positive Borel measures. The reason for this behaviour is not clear but is also the subject of current study.

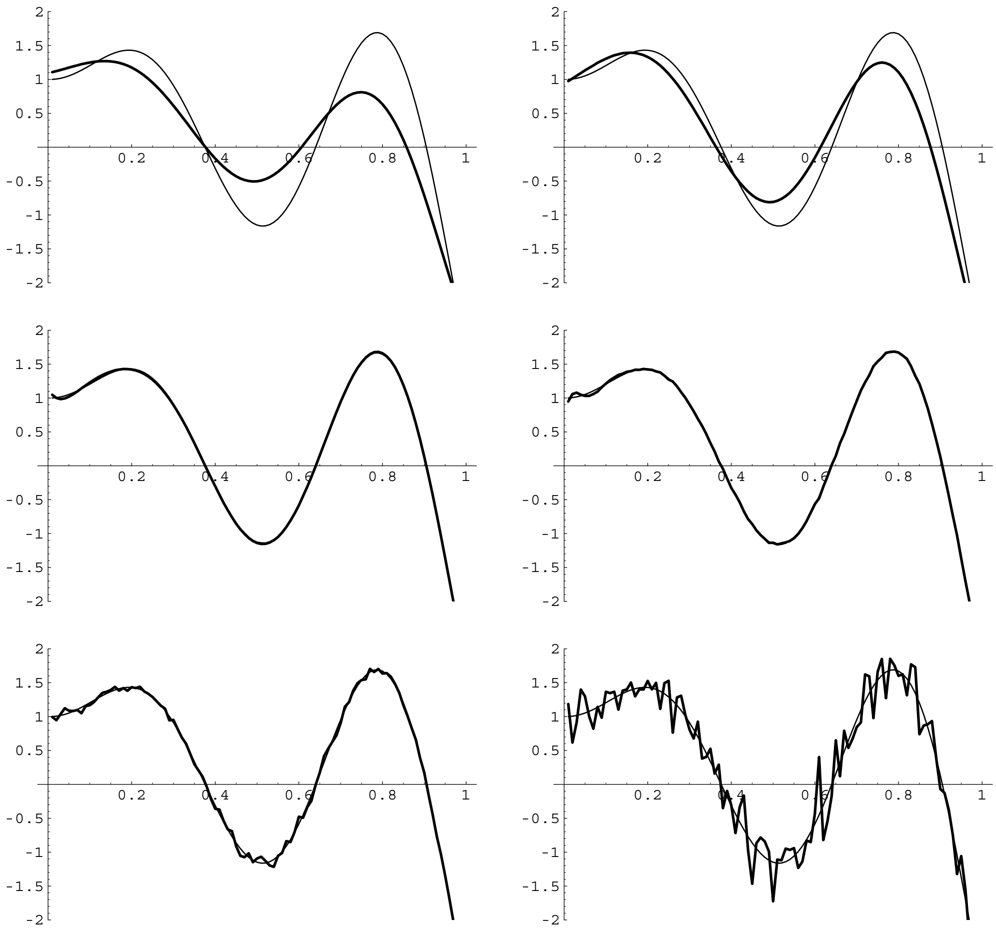

Figure 5. Example 5.3: dependence of solutions on m, for fixed r. First row: $m = \frac{1}{2}, 1$; second row: $m = 5, 10$; third row: $m = 15, 20$.

6. One η_r fits all?

In all applications of the theory thus far we have constructed η_r according to lemma 2.2 and have selected ψ exactly of degree ν so that hypotheses (H1)–(H3) hold. An interesting question is whether a ψ of degree $\bar{\nu}$ can be found for which hypotheses (H1)–(H3) hold for all ν-smoothing problems in the case of $\nu \leqslant \bar{\nu}$. Although our investigations into this question are ongoing, one observation is worth noting.

Let $p_{\bar{\nu}}$ denote the $\bar{\nu}$ degree polynomial

$$p_{\bar{\nu}}(\lambda) = (\lambda + 1)^{\bar{\nu}} = \sum_{\ell=0}^{\bar{\nu}} \binom{\bar{\nu}}{\ell} \lambda^{\ell}$$

for $\bar{\nu} = 1, 2, \ldots, 10$, or 11. Then it is a fact that, for all $\nu = 1, 2, \ldots \bar{\nu}$, the ν-degree polynomial p_ν which is a truncation of the $\bar{\nu} - \nu$ highest order terms in $p_{\bar{\nu}}$, given by

$$p_\nu(\lambda) = \sum_{\ell=0}^{\nu} \binom{\bar{\nu}}{\ell} \lambda^{\ell},$$

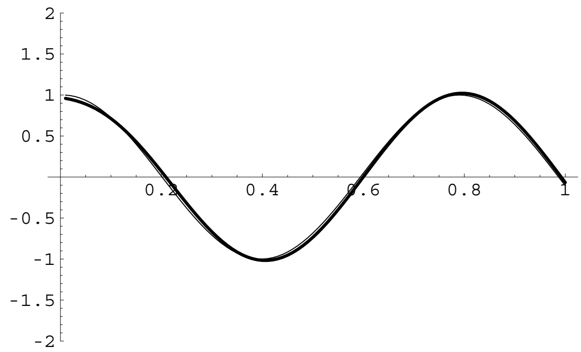

Figure 6. Example 6.1: three-smoothing problem with 11th degree polynomial ψ used to generate η_r.

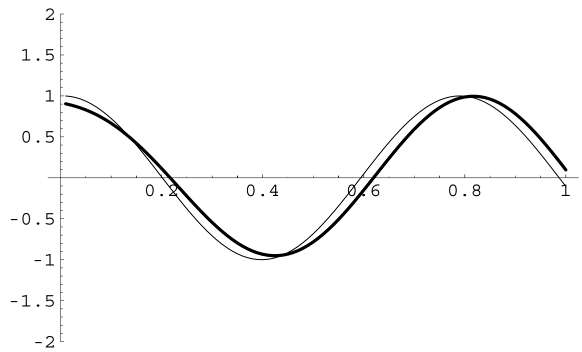

Figure 7. Example 6.1: three-smoothing problem with 20-degree polynomial ψ used to generate η_r.

is such that all ν roots of p_ν have negative real part. This phenomenon, which can be observed numerically, is known to be related to an interesting problem from the area of combinatorics known as the problem of *roots of sections of the binomial*. In fact, once one goes to $\bar{\nu} = 12$, then the truncated polynomials p_ν associated with $\nu = 7, 8$ and 9 have roots with positive real part [20]. Suppose then we construct ψ_{11} via (13) so that (H2) is satisfied using the polynomial $p_{11}(\lambda) = (\lambda + 1)^{11}$. From the above observation, this *same* η_r can then be used to get convergence of approximations in the ν-smoothing case for all $\nu = 1, \ldots, 11$.

We illustrate this point using the following numerical example:

Example 6.1. We consider the three-smoothing problem with kernel given by $k(t) = (3t^2 + t^5)/6$, with true $\bar{u}(t) = \cos 8t$ on the interval $[0, 1]$ and no noise in the data. In this example, $r = 0.2$ and the polynomial $p_{\bar{\nu}}$ used to generate the family of measures η_r is given for $\bar{\nu} = 11$ by

$$p_{11}(\lambda) = (\lambda + 15)^{11}.$$

From the results in figure 6, it is clear that η_r constructed using the 11th degree polynomial ψ (using p_{11} in (H2)) gives a good numerical approximation in the case of the three-smoothing problem.

It is also interesting to note that for this three-smoothing problem we are able to get very good numerical results using a ν-degree ψ generating p_ν for several values of $\nu > 12$ despite the fact that hypothesis (H2) fails in this case. We show in figure 7 the result for $\nu = 20$ where

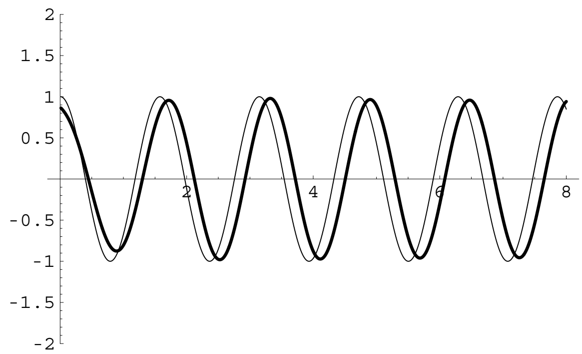

Figure 8. Example 6.1: four-smoothing problem from figures 1 and 2 with 12th degree polynomial ψ used to generate η_r.

we have increased the regularization parameter m to 20 (for $m = 15$ the approximating graph undergoes a slightly larger phase shift).

In fact a more difficult test example is the four-smoothing example from figures 1 and 2. In figure 8 we illustrate the outcome when a 12-degree polynomial ψ generating p_{12} with $m = 12$ is used to regularize the problem; as before an approximation level of $N = 600$ is used. Although the results are better than those found in figure 1, there is a distinct phase shift; in addition it is worth noting that the numerical findings in this case are quite sensitive to the size of m, leading in fact to slow unstable growth of the approximate solution when m is increased to 15.

7. Conclusion

In this paper we have been able to show that the sequential local regularization of ill-posed Volterra problems is indeed an efficient and effective regularization method provided one uses signed measures in the construction of the regularized equation. Our central result is that convergence is obtained for this method in the case of ν-smoothing kernels k for *all* $\nu = 1, 2, \ldots$. If the true solution is assumed to be Hölder continuous we obtain a convergence rate depending on the Hölder parameter, ν, and the level δ of noise present in the data. These convergence results serve to answer an open question raised in [19] about the efficacy of sequential local regularization methods. There are still many remaining issues to be addressed with regard to sequential local regularization, including the important question of selection of the regularization parameter r (as was discussed only briefly in sections 4 and example 5.3 here). Such questions are the subject of an ongoing study.

It is worth noting that the ideas in this paper also extend to the nonlinear Hammerstein Volterra problem [14]. Further, it can be shown that the 'future polynomial regularization method' developed by Cinzori in [3] for linear Volterra problems (a method which is in fact also a sequential regularization method) relies on a particular signed measure which can be shown [14] to satisfy the hypotheses (H1)–(H3) given in section 2 above in the case of a one-smoothing kernel k.

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