Lemma 4.1-revised: Let y satisfy

$$y(t) = -\int_0^t \left(\frac{k(t,s)}{\varepsilon} + K(t,s)\right) y(s)\, ds + E(t) + F(t), \tag{30}$$

a.e. $t \in [0,1]$, where $k \in C^1([0,1]\times[0,1])$ satisfies $k(t,t) = 1$ for $t \in [0,1]$; $K(\cdot,\cdot)$ is bounded, measurable on $[0,1]\times[0,1]$; $F(\cdot) \in C^1[0,1]$ with $F(0) = 0$; $E(\cdot)$ is bounded, measurable on $[0,1]$; and where ε is a positive real number (independent of t). Then $y \in L^\infty(0,1)$ and

$$\|y\|_\infty \le (2\|E\|_\infty + \varepsilon\|F'\|_\infty) \exp\left(\|k\|_{1,\infty} + 2\|K\|_\infty\right).$$

Proof. The proof extends the ideas found in [22] (using a variation of an argument in [9]), differing here in the presence of the K and E terms. We first note that the assumptions of the lemma give $y \in L^2(0,1)$ [14] so that from the form of (34) it follows that $y \in L^\infty(0,1)$.

Given $\varepsilon > 0$, define

$$\psi(t,\varepsilon) := \begin{cases} 0, & t < 0, \\ \frac{1}{\varepsilon} e^{-t/\varepsilon}, & t \ge 0. \end{cases} \tag{31}$$

Convolving both sides of (34) with $\psi(t,\varepsilon)$ we obtain

$$\begin{aligned} &\int_0^t \psi(t-s,\varepsilon) y(s) ds \\ &= -\int_0^t \psi(t-\tau,\varepsilon) \int_0^\tau \left(\frac{k(\tau,s)}{\varepsilon} + K(\tau,s)\right) y(s)\, ds\, d\tau + \psi(t,\varepsilon) * (E(t) + F(t)) \\ &= -\int_0^t \int_s^t \psi(t-\tau,\varepsilon) \left(\frac{k(\tau,s)}{\varepsilon} + K(\tau,s)\right) d\tau\, y(s)\, ds + \psi(t,\varepsilon) * (E(t) + F(t)), \end{aligned}$$

where we use an integration by parts on the first term on the right-hand side above to obtain

$$\begin{aligned} &\int_0^t \psi(t-s,\varepsilon) y(s) ds \\ &= -\int_0^t \left(\frac{k(t,s)}{\varepsilon} - e^{-(t-s)/\varepsilon}\frac{k(s,s)}{\varepsilon}\right) y(s)\, ds + \frac{1}{\varepsilon}\int_0^t \int_s^t e^{-(t-\tau)/\varepsilon} D_1 k(\tau,s)\, d\tau\, y(s)\, ds \\ &\quad - \int_0^t \int_s^t \psi(t-\tau,\varepsilon) K(\tau,s)\, d\tau\, y(s)\, ds + \psi(t,\varepsilon) * (E(t) + F(t)) \\ &= -\int_0^t \left(\frac{k(t,s)}{\varepsilon} - \psi(t-s,\varepsilon)\right) y(s)\, ds + \int_0^t \int_s^t \psi(t-\tau,\varepsilon) D_1 k(\tau,s)\, d\tau\, y(s)\, ds \\ &\quad - \int_0^t \int_s^t \psi(t-\tau,\varepsilon) K(\tau,s)\, d\tau\, y(s)\, ds + \psi(t,\varepsilon) * (E(t) + F(t)), \end{aligned}$$

for $t \in [0,1]$. Subtracting the last equation from equation (34), we have for a.e. $t \in [0,1]$,

$$\begin{aligned} y(t) &= -\int_0^t \int_s^t \psi(t-\tau,\varepsilon) D_1 k(\tau,s)\, d\tau\, y(s)\, ds \\ &\quad + \int_0^t \int_s^t \psi(t-\tau,\varepsilon) K(\tau,s)\, d\tau\, y(s)\, ds - \int_0^t K(t,s) y(s)\, ds \\ &\quad + [E(t) - \psi(t,\varepsilon) * E(t)] + [F(t) - \psi(t,\varepsilon) * F(t)], \end{aligned}$$

or

$$
\begin{aligned}
y(t) &= \int_0^t G(t,s)y(s)\,ds + [E(t) - \psi(t,\varepsilon) * E(t)] \\
&\quad + [F(t) - \psi(t,\varepsilon) * F(t)], \quad \text{a.e. } t \in [0,1],
\end{aligned} \tag{32}
$$

where

$$
G(t,s) := \int_s^t \psi(t-\tau,\varepsilon)\left(K(\tau,s) - D_1 k(\tau,s)\right)\,d\tau - K(t,s)
$$

for $0 \le s \le t \le 1$. But

$$
\begin{aligned}
|G(t,s)| &\le \int_s^t \frac{e^{-(t-\tau)/\varepsilon}}{\varepsilon}\left(|K(\tau,s)| + |D_1 k(\tau,s)|\right)\,d\tau + |K(t,s)| \\
&\le \left(\|k\|_{1,\infty} + \|K\|_\infty\right)\left(1 - e^{-(t-s)/\varepsilon}\right) + \|K\|_\infty \\
&\le \|k\|_{1,\infty} + 2\|K\|_\infty,
\end{aligned}
$$

for a.e. $0 \le s \le t \le 1$. Further, for a.e. $t \in [0,1]$,

$$
\begin{aligned}
E(t) - \psi(t,\varepsilon) * E(t) &\le \|E(\cdot)\|_\infty \left[1 + \int_0^t \psi(t-\tau,\varepsilon)\,d\tau\right] \\
&\le 2\|E(\cdot)\|_\infty.
\end{aligned}
$$

Combining these estimates with equation (32), we see that for a.e. $t \in [0,1]$,

$$
\begin{aligned}
|y(t)| \le \int_0^t \left(\|k\|_{1,\infty} + 2\|K\|_\infty\right)|y(s)|\,ds + 2\|E(\cdot)\|_\infty \\
+ |F(t) - \psi(t,\varepsilon) * F(t)|.
\end{aligned} \tag{33}
$$

Now, if $F \in C^1[0,1]$ satisfies $F(0) = 0$, then we can show that for any $\varepsilon > 0$,

$$
|F(t) - \psi(t,\varepsilon) * F(t)| \le \varepsilon\,\|F'\|_\infty.
$$

Indeed, integrating by parts we have

$$
\begin{aligned}
\psi(t,\varepsilon) * F(t) &= \int_0^t \frac{1}{\varepsilon} e^{-(t-s)/\varepsilon} F(s)\,ds \\
&= e^{-(t-s)/\varepsilon} F(s)\Big|_0^t - \int_0^t e^{-(t-s)/\varepsilon} F'(s)\,ds \\
&= F(t) - \int_0^t e^{-(t-s)/\varepsilon} F'(s)\,ds.
\end{aligned}
$$

Thus

$$
\begin{aligned}
|F(t) - \psi(t,\varepsilon) * F(t)| &= \left|\int_0^t e^{-(t-s)/\varepsilon} F'(s)\,ds\right| \\
&\le \|F'\|_\infty \int_0^t e^{-(t-s)/\varepsilon}\,ds \\
&= \|F'\|_\infty\,\varepsilon\, e^{-(t-s)/\varepsilon}\Big|_0^t \\
&= \varepsilon\,\|F'\|_\infty \left(1 - e^{-t/\varepsilon}\right) \\
&\le \varepsilon\,\|F'\|_\infty.
\end{aligned}
$$

An application of a generalized Gronwall inequality (see, e.g., [34,37]) to the bound in (33) thus gives

$$|y(t)| \leq (2\|E\|_\infty + \varepsilon\|F'\|_\infty) \exp\left(\|k\|_{1,\infty} + 2\|K\|_\infty\right), \quad \text{a.a. } t \in [0,1].$$

Corollary 1 to Lemma 4.1-revised: Under the conditions of Lemma 4.1-revised, with $F(t) = 0$, $t \in [0,1]$, we have

$$\|y\|_\infty \leq 2\|E\|_\infty \exp\left(\|k\|_{1,\infty} + 2\|K\|_\infty\right).$$

Corollary 2 to Lemma 4.1-revised: For each $n = 1, 2, \ldots$, let y_n satisfy

$$y_n(t) = -\int_0^t \left(\frac{k(t,s)}{\varepsilon_n} + K_n(t,s)\right) y_n(s)\, ds + E_n(t) + F(t), \tag{34}$$

a.e. $t \in [0,1]$, where $k \in C^1([0,1] \times [0,1])$ satisfies $k(t,t) = 1$ for $t \in [0,1]$; $K_n(\cdot,\cdot)$ is bounded, measurable on $[0,1] \times [0,1]$; $F(\cdot) \in C^1[0,1]$ with $F(0) = 0$; $E_n(\cdot)$ is bounded, measurable on $[0,1]$; and where ε_n is a positive real number (independent of t) for each $n = 1, 2, \ldots$. Then $y_n \in L^\infty(0,1)$ for each $n = 1, 2, \ldots$. Further, if

- $|K_n(t,s)| \leq M$, a.e. $0 \leq s \leq t \leq 1$,

as $n \to \infty$, for some $M > 0$ independent of n, we have

$$\|y_n\|_\infty \leq (2\|E_n\|_\infty + \varepsilon_n\|F'\|_\infty) \exp\left(\|k\|_{1,\infty} + 2M\right).$$

Corollary 3 to Lemma 4.1-revised: Under the conditions of Corollary 2, with the addition of the two assumptions

- $\varepsilon_n \to 0$, and
- $\|E_n(\cdot)\|_\infty \to 0$,

as $n \to \infty$, we have

$$\|y_n\|_\infty \to 0 \quad \text{as } n \to \infty,$$

with rate determined by the worse of the two rates of convergence of $\varepsilon_n \to 0$ and $\|E_n(\cdot)\|_\infty \to 0$.

Corollary 4 to Lemma 4.1-revised: Under the conditions of Corollary 2, with the addition of the two assumptions

- $F(t) = 0, \;\; t \in [0, 1], \;\;$ and
- $\|E_n(\cdot)\|_\infty \to 0$ as $n \to \infty$,

we have

$$\|y_n\|_\infty \leq 2\|E_n\|_\infty \exp\left(\|k\|_{1,\infty} + 2M\right) \;\to 0 \quad \text{as} \;\; n \to \infty,$$

with rate of convergence the same as the rate for $\|E_n(\cdot)\|_\infty \to 0$.

NOTE: From Corollary 4 we see that when $F(t) = 0$, we *do not* require $\varepsilon_n \to 0$ nor do we even require that $\{\varepsilon_n\}$ remain bounded! All that is needed is that each $\varepsilon_n > 0$, for $n = 1, 2, \ldots$.