LOCAL REGULARIZATION METHODS FOR THE STABILIZATION OF LINEAR ILL-POSED EQUATIONS OF VOLTERRA TYPE

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Inverse problems based on first-kind Volterra integral equations appear naturally in the study of many applications, from geophysical problems to the inverse heat conduction problem. The ill-posedness of such problems means that a regularization technique is required, but classical regularization schemes like Tikhonov regularization destroy the causal nature of the underlying Volterra problem and, in general, can produce oversmoothed results.

In this paper we investigate a class of local regularization methods in which the original (unstable) problem is approximated by a parameterized family of well-posed, second–kind Volterra equations. Being Volterra, these approximating second–kind equations retain the causality of the original problem and allow for quick sequential solution techniques. In addition, the regularizing method we develop is based on the use of a regularization parameter which is a *function* (rather than a single constant), allowing for more or less smoothing at localized points in the domain. We take this approach even further by adopting the flexibility of an additional penalty term (with variable penalty function) and illustrate the sequential selection of the penalty function in a numerical example.

Key Words: Volterra first-kind problem, inverse problem, regularization, fast sequential algorithm

1. INTRODUCTION

We consider the following scalar Volterra first-kind integral problem. Given a suitable function $f(\cdot)$ defined on $[0,1]$, find $\bar{u}(\cdot)$ satisfying, for a.e. $t \in [0,1]$,

$$\mathcal{A}u(t) = f(t), \tag{1}$$

where $\mathcal{A}$ is the bounded linear operator on $L^2(0,1)$ given by

$$\mathcal{A}u(t) := \int_0^t k(t,s)u(s)\,ds, \qquad \text{a.e. } t \in [0,1].$$

Problem (1) is an important one, having many applications. One is the inverse heat conduction problem (IHCP) or sideways heat equation [4] with convolution kernel k. In an application from capillary viscometry, the integral equation takes the form of (1) with non-convolution kernel k. In this case, f (the known function) is the apparent wall shear rate (itself a measured quantity rather than the true wall shear rate) and the desired quantity, the reciprocal of $u(\cdot)$, is the viscosity [29].

In the typical case that the range of $\mathcal{A}$ is not closed, it is well-known that problem (1) is ill-posed, lacking continuous dependence on data $f \in L^2(0,1)$. Thus, when using measured or numerically-approximated data, one must resort to a regularization method to ensure stability. There is a well-developed theory for classical methods such as Tikhonov regularization (see, for example, [13]), but such methods are less than optimal for Volterra problems of the form (1). For example, Tikhonov regularization replaces the original "causal" problem with a full-domain one. By the causal nature of the original problem we mean that problem (1) has the property that, for any $t \in (0,1]$, the solution u on the interval $[0,t]$ is determined only from values of f on that same interval; for this reason, sequential solution techniques are optimal for causal problems. In contrast, to determine a solution via Tikhonov regularization one must use data values from the interval $[t,1]$ (i.e., future data values), thus destroying the causal nature of the original problem and leading to non-sequential solution techniques.

Another difficulty arising in classical regularization techniques involves the use of a single regularization parameter when *a priori* information indicates that a solution is rough in some areas of the domain and smooth in others. In recent years a number of approaches have been developed to handle this difficulty, among them the technique of bounded variation regularization [1, 7, 11, 16, 17, 33], as well as the method of "regularization for curve representations" [31]. Although effective, these approaches do not preserve the causal nature of the original Volterra problem and, in addition, can require a reformulation of the linear problem (or linear least-squares problem) into either a nondifferentiable or nonquadratic optimization problem. In [30], a unified approach to regularization with nondifferentiable functionals (including functionals of bounded variation type) is considered, with theoretical treatment based on the concept of distributional approximation. The approach in [30] may be adapted so that a localized type of regularization is possible, however the application of this approach to Volterra equations has evidently not been studied.

Local regularization techniques form a different class of methods which have been the focus of study in recent years. These methods retain both the linear and causal structure of the original Volterra problem, allowing for solution via fast sequential methods, and rely on differentiable optimization techniques for solution. And, because regularization occurs in local regions only, sharp/fine structures of true solutions can often be recovered. The development in [21, 22, 23, 25, 26] of such methods grew out of a desire to construct a theoretical framework for understanding a popular numerical method developed by J. V. Beck in the 1960's for the IHCP. In this sequential method, Beck held solutions rigid for a short time into the future (forming a locally regularized "prediction"), and then truncated the prediction in order to improve accuracy ("correction") before moving to the next step in the sequence. Generalizations of Beck's ideas also retain this "predictor-corrector" characteristic when discretized. In general, the methods are easy to implement numerically and provide fast results in almost real-time. (We note that mollifier methods for regularization can also be considered local regularization methods; however such methods do not easily apply to *general* equations of the form (1). See [24] and the references therein for a more complete discussion.)

In what follows, we lay the groundwork for a general class of local regularization methods. Our emphasis here is on a *continuous method* in which one may employ *variable* regularization parameters to allow for local control of the stabilization process. This is an extension of the continuous regularization method studied in [22] where a single (constant-valued) local regularization parameter was employed. The change to functional regularization parameters requires a nontrivial redevelopment of the theory of [22]. In addition, it is worth noting that convergence of a discrete version of the method presented below is studied in [27], where it is shown that the resulting sequential regularization algorithm leads to considerable savings in computational costs. Indeed the discrete local regularization algorithm requires only $\mathcal{O}(N^2)$ arithmetic operations while standard Tikhonov regularization requires $\mathcal{O}(N^3)$ operations (to highest order). In [27] we also propose and test a sequential algorithm for the adaptive selection of one of the variable regularization parameters.

2. PRACTICAL IMPLEMENTATION OF A LOCAL REGULARIZATION METHOD

Before discussing the continuous regularization method of interest in this paper, we motivate the work that follows by describing a practical implementation of our ideas. The discussion in this section will focus on a discretization of the original equation (1), as well as a discretization of the continuous regularization method to be analyzed in subsequent sections.

2.1. Collocation-based discretization of the original Volterra problem

We first describe a simple collocation-based discretization of (1). We divide $[0,1]$ into N subintervals $[t_{i-1}, t_i]$, $i = 1, \dots N$, each of width $\Delta t = 1/N$, and seek constants c_i, $i = 1, \dots, N$, so that the step function

$$u(t) = \sum_{i=1}^{N} c_i \chi_i(t), \quad t \in [0,1], \tag{2}$$

satisfies (1) at the collocation points $t = t_i$, $i = 1, \ldots, N$. That is,

$$\mathcal{A}\left(\sum_{j=1}^{i} c_j \chi_j\right)(t_i) = f(t_i), \quad i = 1, \ldots, N. \tag{3}$$

Here χ_i is the usual characteristic function on the interval $(t_{i-1}, t_i]$ for $i = 2, \ldots, N$, while χ_1 is the characteristic function on the interval $[0, t_1]$. Since the operator $\mathcal{A}$ is of Volterra type, the discrete equations (3) reduce to a lower-triangular matrix system in the unknown vector $(c_1, \ldots, c_N)^\top$ and the discretization leads to a sequential solution procedure. Alternatively, we may determine c_i (assuming $c_1, \ldots, c_{i-1}$ have already been found) such that

$$c_i = \arg\min_{c \in \mathbb{R}} J_i(c) \tag{4}$$

where

$$J_i(c) := \left(\mathcal{A}\left(\sum_{j=1}^{i-1} c_j \chi_j + c\chi_i\right)(t_i) - f(t_i)\right)^2. \tag{5}$$

This sequential process is illustrated in Figure 1. We note that discretization of (1) implicitly defines a regularization method (because an ill-posed infinite-dimensional problem has been replaced by a well-posed finite-dimensional problem), however, the stabilization provided by discretization alone is generally not sufficient to provide good results unless N is small.

2.2. Discrete local regularization of the original Volterra problem

We now describe a similar discrete algorithm in which ideas of local regularization are used to further stabilize solutions. To this end we define a functional regularization parameter,

$$r(t) = \sum_{i=1}^{N} r_i \, \chi_i(t),$$

where $r_i = \gamma_i \Delta t$, with integer $\gamma_i := \gamma(t_i)$, $i = 1, \ldots, N$, where γ is a positive integer-valued (piecewise constant) function on $[0, 1]$. At the i^{th} step in the sequential process, r may be used to define a "local regularization interval" of length $r(t_i) = r_i$. We seek a solution of the form

$$u_r(t) = \sum_{i=1}^{N} c_{i,r} \chi_i(t), \quad t \in [0, 1], \tag{6}$$

where the constants $c_{i,r}$ are determined as follows. Assuming $c_{1,r}, \ldots, c_{i-1,r}$ have already been found, we determine $c_{i,r}$ such that

$$c_{i,r} = \arg\min_{c \in \mathbb{R}} J_{i,r}(c) \tag{7}$$

where

$$J_{i,r}(c) := \sum_{k=0}^{\gamma_i} \left(\mathcal{A}\left(\sum_{j=1}^{i-1} c_j \chi_j + c \sum_{\ell=i}^{i+k} \chi_\ell \right)(t_{i+k}) - f(t_{i+k}) \right)^2 . \tag{8}$$

That is, $c_{i,r}$ is determined by a least-squares fitting of a constant-valued solution over the local interval $(t_{i-1}, t_i + r_i] = (t_{i-1}, t_i + \gamma_i \Delta t]$. Because the solution is temporarily held rigid over the small future interval, this is a regularized "prediction" of the optimal solution $u_r(t)$ on the interval $(t_{i-1}, t_i + r_i]$. However, this process is only used to determine $u_r(t)$ on $(t_{i-1}, t_i]$ via the determination of $c_{i,r}$ at the i^{th} step of the process (i.e., the prediction of u_r on $(t_i, t_i + r_i]$ is not retained), and it is this "correction" which avoids over-regularization and improves the overall approximation. See Figure 2.

We note that in the case of $\gamma_i = 0$, $i = 1, \ldots, N$, we have $r(t) = 0$, $t \in [0, 1]$, and the algorithm (7)–(8) reduces to that for the discretized original (unregularized) problem, i.e., to (4)–(5). For $\gamma_i > 0$, the process described above can be viewed as a discrete "predictor-corrector" method of regularization.

2.3. A discrete penalized local regularization method

Implementation of a *penalized* predictor-correction method is one step beyond the sequential regularization algorithm in (7)–(8). Here we define an additional (functional) regularization parameter μ,

$$\mu(t) = \sum_{i=1}^{N} \mu_i \chi_i(t),$$

for $\mu_i \geq 0$, and we modify the above process in order to find $\tilde{u}_r(t) = \sum_{i=1}^N \tilde{c}_{i,r} \chi_i(t)$, $t \in [0, 1]$. In this case, assuming that $\tilde{c}_{1,r}, \ldots, \tilde{c}_{i-1,r}$ have already been found, we determine $\tilde{c}_{i,r}$ such that

$$\tilde{c}_{i,r} = \arg\min_{c \in \mathbb{R}} \tilde{J}_{i,r}(c) \tag{9}$$

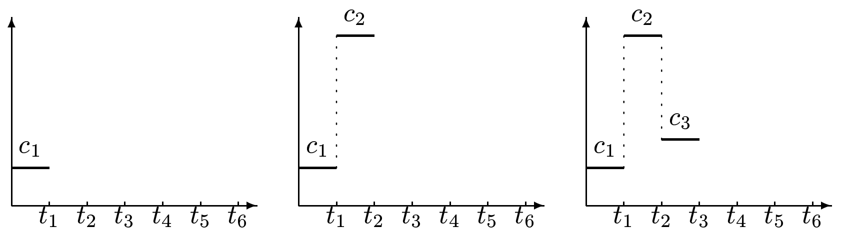
where

$$\tilde{J}_{i,r}(c) := J_{i,r}(c) + \mu_i |c|^2, \tag{10}$$

with $J_{i,r}$ given in (8). Thus for $\mu_i > 0$ the effect is to penalize large absolute values of the constant sought in the i^{th} step of the predictor-corrector scheme.

In the penalized regularization case, there are two functional regularization parameters and it is unlikely that both will be independently selected. Indeed the theory in subsequent sections reinforces this idea.

We note that an alternate way to implement variable local regularization is to vary the degree of the polynomial used to determine the predicted solution on the future interval instead of varying the values of r and/or μ. That is, instead of the approach taken at

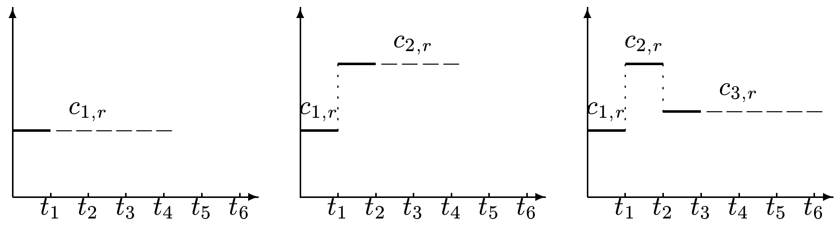


Let u be given by (2). *Step 1:* Pick c_1 which matches $\mathcal{A}u(t_1)$ to f_1.

Step 2: Pick c_2 which matches $\mathcal{A}u(t_2)$ to f_2.

Step 3: Pick c_3 which matches $\mathcal{A}u(t_3)$ to f_3. And so on.

FIG. 1. Unregularized collocation method: highly oscillatory solutions are possible.



Let u_r be given by (6). *Step 1:* Suppose $r_1 = 3\Delta t$. Then we pick $c_{1,r}$ to match $\mathcal{A}u_r(t_i)$ to f_i, (in a least squares sense), for $i = 1, 2, 3$, and 4.

Step 2: Suppose $r_2 = 2\Delta t$. Then we pick $c_{2,r}$ to match $\mathcal{A}u_r(t_i)$ to f_i, (in a least squares sense), for $i = 2, 3$, and 4.

Step 3: Suppose $r_3 = 3\Delta t$. Then we pick $c_{3,r}$ to match $\mathcal{A}u_r(t_i)$ to f_i, (in a least squares sense), for $i = 3, 4, 5, 6$. And so on.

FIG. 2. Predictor-corrector implementation of local regularization method

the i^{th} step in (7)–(8) of finding an optimal constant-valued solution over a future interval of length r_i, we may instead find the optimal d_i-degree polynomial solution over a future interval of fixed length, where the integer d_i is allowed to change with each i. Smaller d_i values lead to stronger regularization on the i^{th} subinterval, while larger d_i values lead to less localized regularization. An analysis of regularization methods for Volterra problems based on localized polynomial fitting may be found in [8].

2.4. Numerical Examples and the Case for Variable r and/or μ

We present an example where the kernel $k(t, s) = t - s$ is of convolution type, examining the stability of solutions of algorithm (9)–(10) in the usual situation where the data is in error. The discretization parameter is $N = 40$ ($\Delta t = 1/40$), and the relative error in data (randomly generated) is of order 3%. In the figures shown below, the true solution is represented by a dashed curve, and approximate solutions are shown using a solid curve (joining midpoints of stepfunctions).

EXAMPLE 2.1. In Figure 3 we first illustrate the application of standard Tikhonov regularization for a true solution which has both steep and flat features. We show the results for $\alpha = 0$ (no regularization) and for various other choices of the Tikhonov regularization parameter α. As can be seen from this figure, both peaks in $\bar{u}$ are adequately resolved with the choice of $\alpha = 2.5 \times 10^{-7}$, but the flat region on $[.6, 1.0]$ is not recovered well in this case. When α is increased in the later graphs to better handle the flat area at the end of the interval, the first peak in $\overline{u}$ is lost due to oversmoothing. (We note that with Volterra problems it is well-known that one cannot accurately recover values of solutions at the very end of the interval – on say, $[.8, 1.]$, for example – using any regularization algorithm.)

EXAMPLE 2.2. In Figure 4 we illustrate an application of local regularization (algorithm (9)–(10)) to the same example. The length r_i of the local interval is held constant at $r_i = 2\Delta t = .05$, for $i = 1, \ldots, 40$, while the values of μ_i are changed sequentially through the regularization process. The determination of μ_i is made using a sequential Morozov discrepancy principle and knowledge of the magnitude of the error (error in the data, and error in the propagated solution) at the i^{th} step. Even though having good information about the magnitude of the exact error is an unrealistic expectation, it nevertheless shows that the sequential selection of μ_i has much potential for applications in which a good model of error is available. We plot the (rescaled) μ_i determined by this process in the second graph in Figure 4, connecting with line segments between discrete μ_i values. The general increase in μ_i over the interval $[0, 1]$ is likely to be due to the growth in propagated error of the sequential method as the number of steps increases. It is interesting to also note the way in which the computed μ_i decreases sharply near .35 and .55, roughly corresponding to sharp increases in the true solution. The issue of sequential selection of μ_i is discussed further in [27], where a simple analytic formula for μ_i (based on a sequential Morozov discrepancy principle) is derived. In addition, numerical examples using non-exact error magnitudes are also given in [27].

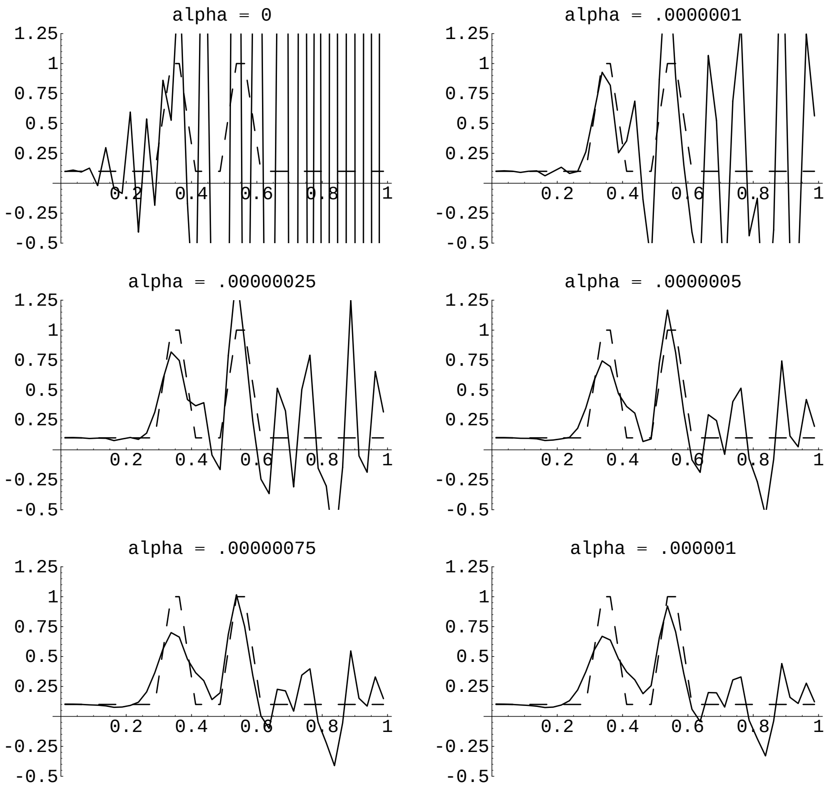


FIG. 3. Tikhonov regularization for several α values (Example 2.1)

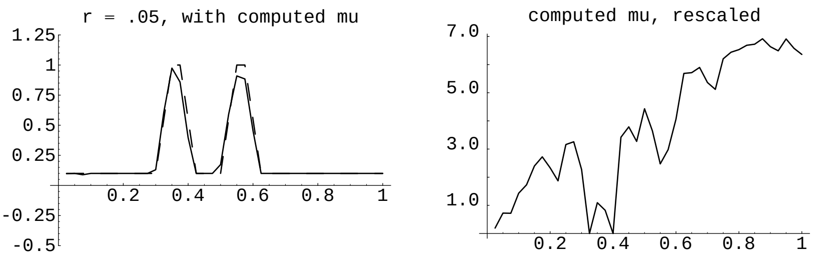


FIG. 4. Local regularization with μ computed using a sequential discrepancy principle (Example 2.2)

3. A CONTINUOUS LOCAL REGULARIZATION METHOD

For suitable k and f, it can be shown [27] that the penalized predictor-corrector numerical algorithm given by (9)–(10) above is simply a collocation-based discretization of the following second-kind Volterra equation:

$$\int_0^t \tilde{k}(t,s;r)u(s)\,ds + (\alpha(t;r)+\mu(t))u(t) = \tilde{f}(t;r), \text{ a.e. } t\in[0,1], \tag{11}$$

where

$$\tilde{k}(t,s;r) := \int_0^{r(t)} k(t+\rho,s)\,d\eta_r(\rho;t), \tag{12}$$

$$\alpha(t;r) := \int_0^{r(t)}\int_0^{\rho} k(t+\rho,s+t)\,ds\,d\eta_r(\rho;t), \tag{13}$$

$$\tilde{f}(t;r) := \int_0^{r(t)} f(t+\rho)\,d\eta_r(\rho;t), \tag{14}$$

for a.e. $t\in[0,1]$. In the above, $\eta_r(\cdot;t)$ is the r-dependent discrete measure defined for a bounded Borel function g and a.e. t $\in[0,1]$ by

$$\int_0^{r(t)} g(\rho)d\eta_r(\rho;t) = \int_0^{\|r\|_\infty} g(\rho)\chi_{[0,r(t)]}(\rho)\,d\eta_r(\rho;t),$$

where $\chi_{[0,r(t)]}$ denotes the usual characteristic function on $[0,r(t)]$, and where

$$\int_0^{\|r\|_\infty} g(\rho)d\eta_r(\rho;t) := \sum_{\ell=1}^{L} s_\ell(t)g(\tau_\ell), \quad \text{a.e. t} \in [0,1], \tag{15}$$

with

$$\begin{aligned} L &:= 1+\max_{1\le i\le N}\gamma_i \\ s_\ell(t) &:= \begin{cases} \Delta t - t + \int_{-t}^{t_{\ell-1}} k(t+t_{\ell-1},t+s)\,ds, & 0\le t\le \Delta t, \\ \int_{-\Delta t}^{t_{\ell-1}} k(t+t_{\ell-1},t+s)\,ds, & t>\Delta t, \end{cases} \\ \tau_\ell &:= (\ell-1)\Delta t, \end{aligned} \tag{16}$$

for $\ell = 1 \ldots, L$.

For a.e. $t \in [0,1]$ the measure $\eta_r(\cdot;t)$ serves to "sample" a small amount of future data, which is then used in the regularization process. In order for $\tilde{k}$ and $\tilde{f}$ in (13)–(14) to be well-defined, we will require that both k and f be defined on an extended interval $[0,T]$, for some $T > 1$, and that the original Volterra equation (1) holds on this extended interval as well. (The alternative is to determine a reconstructed regularized solution of (1) on the interval $[0, 1-\varepsilon]$, for some $\varepsilon > 0$ small.)

We thus make the following standing hypothesis:

Hypothesis 3.1. *For $T > 1$ fixed, let $k \in \mathcal{C}^1([0,T] \times [0,T])$ and let $f : [0,T] \to \mathbb{R}$ be bounded, and assume that equation (1) has a unique solution $\bar{u}$ on $[0,T]$. Further, assume $k(t,t) \neq 0$ for $0 \leq t \leq T$; without loss of generality, we assume that $k(t,t) = 1$ for $0 \leq t \leq T$.*

We note that in the above we assume that the kernel k satisfies $k(t,t) \neq 0$ for $t \in [0,T]$. Because this assumption limits the convergence theory to those operators $\mathcal{A}$ for which equation (1) is only mildly ill-posed, a couple of remarks are in order.

(1) We note that this assumption (or an assumption of even milder ill-posedness than assumed here) is standardly found in theoretical convergence arguments for methods which preserve the Volterra nature of the original problem. Indeed, the theory for classical regularization methods is typically based on the special spectral properties of the operator $\mathcal{A}^\star\mathcal{A}$, where $\mathcal{A}^\star$ is the (Hilbert) adjoint of $\mathcal{A}$. However, $\mathcal{A}^\star\mathcal{A}$ is a non-causal operator, and is thus not of use in the theoretical treatment of methods which preserve the Volterra nature of the original problem. For this reason, theoretical findings for Volterra-preserving methods typically require much more stringent conditions than do classical regularization methods. The hypotheses of several well-known Volterra-preserving methods are discussed in [24, 27].

(2) Our second comment is that the theoretical assumption $k(t,t) \neq 0$ does *not* appear to be needed in practice for the method we present here. Indeed, as mentioned in Section 1, our method is a generalization of one which has been used for over thirty years in the practical solution of the inverse heat conduction problem [4] (a severely ill-posed problem for which $k(t,s)$ and all its derivatives are zero on the line $s = t$). Other numerical examples for k not satisfying the assumption $k(t,t) \neq 0$ may be found in [21, 27]. We note that Ring and Prix have shown that a certain sufficient condition for convergence fails to hold for a class of kernels k satisfying $k(t,t) = 0$. In [32], numerical results are given to illustrate lack of stability for a particular example of this type; nonetheless, a change in the measure and numerical parameters seems to restore stability. Clearly there is much work still needed to complete our understanding of stability and convergence in the case of $k(t,t) = 0$. In particular, the convergence of the method with a kernel of the class in [32] using constant-valued r and $\mu = 0$ is still an open problem.

3.1. A general framework for the continuous local regularization problem

For the discrete regularization algorithm described in Section 2, it was useful to prescribe the functional regularization parameters r and μ as piecewise-constant functions. This restriction is not needed for the continuous regularization problem so instead we generalize the definitions of r and μ used earlier.

DEFINITION 3.1. We define a functional regularization parameter $r \in \mathcal{P}$, where

$$\mathcal{P} := \{r : [0,1] \to \mathbb{R} : r \text{ piecewise continuous on } [0,1], \text{with} \\ \min_{t\in[0,1]} r(t) > 0, \ \max_{t\in[0,1]} t + r(t) \le T \ \}.$$

For $r \in \mathcal{P}$, we make the definition $r_{min} := \min_{t\in[0,1]} r(t) > 0$ and note also that $\|r\|_\infty := \max_{t\in[0,1]} r(t) \le T$.

DEFINITION 3.2. We take the penalty regularization parameter μ to be a Lebesgue-measurable function on $[0,1]$ satisfying $\mu(t) \ge 0$ a.e. $t \in [0,1]$.

Corresponding to the generalizations of r and μ put forth in the above definitions, we consider a larger class of measures $\eta_r = \eta_r(\cdot;t)$ than the one defined earlier in this section (the one that led to the algorithm described in Section 2):

DEFINITION 3.3. Given $r \in \mathcal{P}$, we assume that $\eta_r = \eta_r(\cdot;t)$ is a Borel measure which satisfies (15) for a.e. $t \in [0,1]$, where the r-dependent parameters L, s_ℓ, τ_ℓ satisfy

(1) $L > 0$ integer;
(2) $s_\ell \in L^\infty(0,1)$ with $0 \le s_\ell(t)$, a.e. $t \in [0,1]$, $\ell = 1, \ldots, L$; and,
(3) $0 \le \tau_1 < \tau_2 < \cdots < \tau_L \le \|r\|_\infty$, with $\tau_\ell \in (0, r_{min}]$ and $s_\ell(t) \ge \underline{s}_\ell > 0$, for some $\ell = 1, \ldots, L$.

Given r, μ, and η_r, the continuous regularization problem is to find $u = u(t; r, \mu)$ satisfying the second-kind Volterra equation (11).

We note that the continuous regularization equation (11) could also arise by viewing regularization of the original Volterra problem (1) from a different perspective. Indeed we note that, given $r \in \mathcal{P}$, the true solution $\bar{u}$ satisfies

$$\int_0^{t+\rho} k(t+\rho, s)u(s)\, ds = f(t+\rho), \quad \text{a.e. } \rho \in [0, \mathrm{r(t)}], \ \mathrm{t} \in [0,1].$$

Splitting the left-hand side of this equation into the sum of two integrals, we have (after a change in variables)

$$\int_0^t k(t+\rho, s)u(s)\, ds + \int_0^\rho k(t+\rho, s+t)u(s+t)\, ds = f(t+\rho), \tag{17}$$

for a.e. $\rho \in [0, r(t)]$, $t \in [0,1]$. Now integrating both sides of equation (17) with respect to the measure η_r, we get that $\bar{u}$ satisfies

$$\int_0^{r(t)} \int_0^t k(t+\rho, s) u(s)\, ds\, d\eta_r(\rho; t) + \int_0^{r(t)} \int_0^\rho k(t+\rho, s+t) u(s+t)\, ds\, d\eta_r(\rho; t)$$
$$= \int_0^{r(t)} f(t+\rho)\, d\eta_r(\rho; t), \quad \text{a.e. } t \in [0,1], \tag{18}$$

or, after a change in order of integration in the first term of (18),

$$\int_0^t \tilde{k}(t, s; r) u(s)\, ds + \int_0^{r(t)} \int_0^\rho k(t+\rho, s+t) u(s+t)\, ds\, d\eta_r(\rho; t)$$
$$= \int_0^{r(t)} f(t+\rho)\, d\eta_r(\rho; t), \quad \text{a.e. } t \in [0,1]. \tag{19}$$

Stabilization occurs when we consider a regularized variation of (19) (which was suggested in [22]) where the idea is to replace the second integral term in (19) by a term which serves to restrict the variation (in some sense) of u on the local interval $[t, t+r(t)]$ for each t. The result is equation (11) with $\mu = 0$. The case of nonzero μ simply allows for a penalized variation of this regularization problem.

In [22], consideration of equation (11) was restricted to the case of a convolution kernel, a constant regularization parameter $r(t) \equiv \bar{r}$, $t \in [0,1]$, and zero penalty parameter μ; the fact that the kernel was of convolution type meant that the associated measure η_r was necessarily t-independent. A central goal of this paper is to extend the regularized convergence results in [22] to the case of nonconvolution kernels (which leads to the need in practical applications for measures $\eta_r(\cdot; t)$ which depend on t), and, more importantly, to *variable* regularization parameters $r(\cdot)$ and $\mu(\cdot)$. The use of a variable regularization parameter appears to allow for better resolution of solutions having sharp features, even discontinuities, than does a constant regularization parameter. (See the numerical examples in Section 2 and in [27].)

REMARK 3.1. We note that the idea of regularizing first-kind Volterra problems via the formulation of a related second-kind equation is not a new one. Indeed, the "small parameter method" (see, for example, [3, 10, 18, 19, 20, 28, 36]) is based on adding a term of the form $\varepsilon(u(t) - \bar{u}(0))$ to the right-hand side of the *original* Volterra equation (1). However, for this particular approach one requires accurate knowledge of the initial value $\bar{u}(0)$ of the true solution, otherwise a boundary layer effect can lead to an unsatisfactory approximation of $\bar{u}$ in the neighborhood of $t = 0$. See, for example, [2, 24]. In contrast, knowledge of $\bar{u}(0)$ is not required for the local regularization method described in this paper (and in the other local regularization references given above). The only exception to this statement is the fairly-specific result in Theorem 4.2 below (and Theorem 5.3, which generalizes this result).

Well-posedness of equation (11) follows from the next theorem.

THEOREM 3.1. *Let r, μ, and η_r be given as in Definitions 3.1–3.3, and let k and f satisfy Hypothesis 3.1. Then for $\|r\|_\infty$ sufficiently small, there is a unique solution $u(\cdot;r,\mu) \in L^2(0,1)$ of equation (11) which depends continuously on $f \in L^\infty(0,T)$.*

Proof. We may use a Taylor expansion on the integrand of α in (13) to write

$$\alpha(t;r) = \int_0^{r(t)} \int_0^{\rho} [k(t,t) + \rho D_1 k(t+\xi, t+\zeta) + s D_2 k(t+\xi, t+\zeta)]\, ds\, d\eta_r(\rho;t) \qquad (20)$$

$$= \left(\int_0^{r(t)} \rho\, d\eta_r(\rho;t) \right) (1 + h(t;r)) \qquad (21)$$

for suitable $\xi = \xi_{t,s}(\rho)$, $\zeta = \zeta_{t,s}(\rho)$, and a.e. $t \in [0,1]$, where it follows from the definition of η_r that $\int_0^{r(t)} \rho\, d\eta_r(\rho;t) > 0$, a.e. $t \in [0,1]$. Here

$$|h(t;r)| = \frac{|\int_0^{r(t)} \int_0^{\rho} [\rho D_1 k(t+\xi, t+\zeta) + s D_2 k(t+\xi, t+\zeta)]\, ds\, d\eta_r(\rho;t)|}{\int_0^{r(t)} \rho\, d\eta_r(\rho;t)}$$

$$\leq \frac{3}{2} \|k\|_{1,\infty} \|r\|_\infty, \qquad (22)$$

for a.e. $t \in [0,1]$. Thus, for $\|r\|_\infty$ sufficiently small,

$$\alpha(t;r) + \mu(t;r) \geq \frac{1}{2} \int_0^{r(t)} \rho\, d\eta_r(\rho;t) > 0$$

for a.e. $t \in [0,1]$. Using this fact and the assumptions on k, we have that the quantities $\tilde{k}(t,s;r)/(\alpha(t;r)+\mu(t;r))$, $0 \leq s \leq t \leq 1$, and $\tilde{f}(t;r)/(\alpha(t;r)+\mu(t;r))$, $0 \leq t \leq 1$, are well-defined and prescribe Lebesgue square-integrable "functions" on their respective domains.

Dividing equation (11) by $\alpha(t;r) + \mu(t;r)$, we obtain

$$u(t) = -\int_0^t \frac{\tilde{k}(t,s;r)}{\alpha(t;r) + \mu(t;r)}\, u(s)\, ds + \frac{\tilde{f}(t;r)}{\alpha(t;r) + \mu(t;r)}, \quad \text{a.e. } t \in [0,1], \qquad (23)$$

for which there is a unique solution $u(\cdot;r,\mu)$ in $L^2(0,1)$ depending continuously on the quantity $\tilde{f}(\cdot;r)/(\alpha(\cdot;r)+\mu(\cdot;r)) \in L^2(0,1)$ (see, for example, [14]). It easily follows that u depends continuously on $f \in L^\infty(0,T)$.

∎

In the next section we give proofs of convergence of continuous regularization schemes based on equation (11). Convergence results for discretizations of (11) (with formulations generalizing the algorithms in (7)–(8) and (9)–(10)) may be found in [27]. The theoretical findings in [27] extend the results in [21] and provide an alternative to the sequential Tikhonov regularization algorithm considered in [26].

4. CONVERGENCE RESULTS

We now turn to the issue of convergence of the regularized solution $u(\cdot; r, \mu)$ of (11) to $\bar{u}$ as the regularization parameters r and μ go to zero. Specifically we will look at conditions guaranteeing $\|u(\cdot; r_n, \mu_n) - \bar{u}\|_\infty \to 0$ as $n \to \infty$, for appropriate sequences $\{r_n\}$, $\{\mu_n\}$ satisfying $\|r_n\|_\infty$, $\|\mu_n\|_\infty \to 0$ as $n \to \infty$.

Once a sequence $\{r_n\} \subset \mathcal{P}$ has been specified, we will also require a corresponding sequence $\{\eta_{r_n}\}$ of measures. This is done in a natural way following Definition 3.3.

Definition 4.1. Given $\{r_n\} \subset \mathcal{P}$, we will define for each $n = 1, 2, \ldots$, $\eta_{r_n} = \eta_{r_n}(\cdot; t)$ satisfying

$$\int_0^{\|r_n\|_\infty} g(\rho) d\eta_{r_n}(\rho; t) := \sum_{\ell=1}^{L_n} s_{\ell,n}(t) g(\tau_{\ell,n}), \quad \text{a.e. t} \in [0,1], \tag{24}$$

where for each $n = 1, 2, \ldots$, the parameters L_n, $s_{\ell,n}$, and $\tau_{\ell,n}$ satisfy the conditions (1)–(3) of the corresponding (n-independent) parameters L, s_ℓ, τ_ℓ, in Definition 3.3.

Remark 4.2. Although we will not require additional conditions on $\{\eta_{r_n}\}$ in general, it will be useful in certain situations to note that the practical implementation in Section 2 was naturally associated with a sequence $\{\eta_{r_n}\}$, $n = N = 1, 2, \ldots$ of measures, where we see from (15)–(16) that, in addition to the conditions given above, certain other properties naturally occur. For example, for all n sufficiently large it follows that for this particular sequence $\{\eta_{r_n}\}$ there is some $L > 0$ for which

$$L_n = L, \tag{25}$$

$$s_{\ell,n}(t) = \sigma_\ell \|r_n\|_\infty \left(1 + \mathcal{O}(\|r_n\|_\infty)\right), \quad \ell = 1, \ldots, L, \tag{26}$$

for a.e. $t \in [0, 1]$, and

$$\tau_{\ell,n} = c_\ell \|r_n\|_\infty, \quad \ell = 1, \ldots, L, \tag{27}$$

where, for at least one $\ell = 1, \ldots, L$,

$$c_\ell \sigma_\ell > 0. \tag{28}$$

(We note that $L = 1 + \|\gamma\|_\infty$, $\sigma_\ell = \ell/\|\gamma\|_\infty$, and $c_\ell = (\ell - 1)/\|\gamma\|_\infty$ for the particular sequence of measures associated with the practical implementation in Section 2.) We will need conditions like these in Theorem 4.1 below.

It will be useful for the analysis in this section to make the following definition. We define for $\nu = 0, 1$,

$$a_\nu(t; r_n) := \int_0^{r_n(t)} \rho^\nu \, d\eta_{r_n}(\rho, t), \quad \text{a.e. t} \in [0,1]. \tag{29}$$

¿From assumptions on r_n and η_{r_n}, we note that $a_0(t; r_n) \geq a_1(t; r_n)/\|r_n\|_\infty > 0$, a.e. $t \in [0,1]$. In addition, even if r_n is a constant, a_ν still depends on t through the t-dependence of η_{r_n}.

The following technical lemma will be used to prove the main results of this section.

LEMMA 4.1. *For each $n = 1, 2, \ldots$, let y_n satisfy*

$$y_n(t) = -\int_0^t \left(\frac{k(t,s)}{\varepsilon_n} + K_n(t,s) \right) y_n(s) \, ds + E_n(t) + F(t), \tag{30}$$

a.e. $t \in [0,1]$, where $k \in C^1([0,1] \times [0,1])$ satisfies $k(t,t) = 1$ for $t \in [0,1]$; $K_n(\cdot,\cdot)$ is bounded, measurable on $[0,1] \times [0,1]$; $F(\cdot) \in C[0,1]$ with $F(0) = 0$; $E_n(\cdot)$ is bounded, measurable on $[0,1]$; and where ε_n is a positive real number (independent of t) for each $n = 1, 2, \ldots$. Then $y_n \in L^\infty(0,1)$ for each $n = 1, 2, \ldots$. Further, if

- $\varepsilon_n \to 0$,
- $\|E_n(\cdot)\|_\infty \to 0$
- $|K_n(t,s)| \leq M$, *a.e.* $0 \leq s \leq t \leq 1$,

as $n \to \infty$, for some $M > 0$ independent of n, it follows that $y_n \to 0$ in $L^\infty(0,1)$ as $n \to \infty$.

Proof. The proof extends the ideas found in [22] (using a variation of an argument in [9]), differing here in the presence of the K_n and E_n terms. We first note that the assumptions of the lemma give $y_n \in L^2(0,1)$ [14] so that, from the form of (30) it follows that $y_n \in L^\infty(0,1)$ for all n.

Given $\varepsilon_n > 0$, define

$$\psi(t, \varepsilon_n) := \begin{cases} 0, & t < 0, \\ \frac{1}{\varepsilon_n} e^{-t/\varepsilon_n}, & t \geq 0. \end{cases} \tag{31}$$

Convolving both sides of (30) with $\psi(t, \varepsilon_n)$ we obtain

$$\begin{aligned} &\int_0^t \psi(t-s, \varepsilon_n) y_n(s) ds \\ &= -\int_0^t \psi(t-\tau, \varepsilon_n) \int_0^\tau \left(\frac{k(\tau,s)}{\varepsilon_n} + K_n(\tau,s) \right) y_n(s) \, ds \, d\tau + \psi(t,\varepsilon_n) * (E_n(t) + F(t)) \\ &= -\int_0^t \int_s^t \psi(t-\tau, \varepsilon_n) \left(\frac{k(\tau,s)}{\varepsilon_n} + K_n(\tau,s) \right) d\tau \, y_n(s) \, ds + \psi(t,\varepsilon_n) * (E_n(t) + F(t)), \end{aligned}$$

where we use an integration by parts on the first term on the right-hand side above to obtain

$$\begin{aligned}
&\int_0^t \psi(t-s,\varepsilon_n) y_n(s) ds \\
&= -\int_0^t \left(\frac{k(t,s)}{\varepsilon_n} - e^{-(t-s)/\varepsilon_n} \frac{k(s,s)}{\varepsilon_n} \right) y_n(s)\, ds + \frac{1}{\varepsilon_n} \int_0^t \int_s^t e^{-(t-\tau)/\varepsilon_n} D_1 k(\tau,s)\, d\tau\, y_n(s)\, ds \\
&\quad - \int_0^t \int_s^t \psi(t-\tau,\varepsilon_n) K_n(\tau,s)\, d\tau\, y_n(s)\, ds + \psi(t,\varepsilon_n) * (E_n(t) + F(t)) \\
&= -\int_0^t \left(\frac{k(t,s)}{\varepsilon_n} - \psi(t-s,\varepsilon_n) \right) y_n(s)\, ds + \int_0^t \int_s^t \psi(t-\tau,\varepsilon_n) D_1 k(\tau,s)\, d\tau\, y_n(s)\, ds \\
&\quad - \int_0^t \int_s^t \psi(t-\tau,\varepsilon_n) K_n(\tau,s)\, d\tau\, y_n(s)\, ds + \psi(t,\varepsilon_n) * (E_n(t) + F(t)),
\end{aligned}$$

for $t \in [0,1]$. Subtracting the last equation from equation (30), we have for a.e. $t \in [0,1]$,

$$\begin{aligned}
y_n(t) \;=\; & -\int_0^t \int_s^t \psi(t-\tau,\varepsilon_n) D_1 k(\tau,s)\, d\tau\, y_n(s)\, ds \\
& + \int_0^t \int_s^t \psi(t-\tau,\varepsilon_n) K_n(\tau,s)\, d\tau\, y_n(s)\, ds - \int_0^t K_n(t,s) y_n(s)\, ds \\
& + [E_n(t) - \psi(t,\varepsilon_n) * E_n(t)] + [F(t) - \psi(t,\varepsilon_n) * F(t)],
\end{aligned}$$

or

$$\begin{aligned}
y_n(t) \;=\; & \int_0^t G_n(t,s) y_n(s)\, ds + [E_n(t) - \psi(t,\varepsilon_n) * E_n(t)] \\
& + [F(t) - \psi(t,\varepsilon_n) * F(t)], \quad \text{a.e. } t \in [0,1],
\end{aligned} \tag{32}$$

where

$$G_n(t,s) := \int_s^t \psi(t-\tau,\varepsilon_n) \left(K_n(\tau,s) - D_1 k(\tau,s) \right) d\tau - K_n(t,s)$$

for $0 \le s \le t \le 1$. But

$$\begin{aligned}
|G_n(t,s)| \;&\le\; \int_s^t \frac{e^{-(t-\tau)/\varepsilon_n}}{\varepsilon_n} \left(|K_n(\tau,s)| + |D_1 k(\tau,s)| \right) d\tau + |K_n(t,s)| \\
&\le\; (\|k\|_{1,\infty} + M) \left(1 - e^{-(t-s)/\varepsilon_n} \right) + M \\
&\le\; \|k\|_{1,\infty} + 2M,
\end{aligned}$$

for a.e. $0 \le s \le t \le 1$. Further, for a.e. $t \in [0,1]$,

$$\begin{aligned}
E_n(t) - \psi(t,\varepsilon_n) * E_n(t) \;&\le\; \|E_n(\cdot)\|_\infty \left[1 + \int_0^t \psi(t-\tau,\varepsilon_n)\, d\tau \right] \\
&\le\; 2\|E_n(\cdot)\|_\infty.
\end{aligned}$$

Combining these estimates with equation (32), we see that for a.e. $t \in [0,1]$,

$$\begin{aligned}
|y_n(t)| \le \int_0^t (\|k\|_{1,\infty} + 2M) |y_n(s)|\, ds + 2\|E_n(\cdot)\|_\infty \\
+ |F(t) - \psi(t,\varepsilon_n) * F(t)|.
\end{aligned} \tag{33}$$

Finally we use the result [9, 12] that if $g : [0,1] \to \mathbb{R}$ is a continuous function satisfying $g(0) = 0$, then $\psi(t, \varepsilon) * g(t) \to g(t)$ as $\varepsilon \to 0^+$, uniformly in $t \in [0,1]$. We thus have that

$$|F(t) - \psi(t, \varepsilon_n) * F(t)| \to 0$$

as $n \to \infty$, uniformly in $t \in [0,1]$. An application of a generalized Gronwall inequality (see, e.g., [34, 37]) to the bound in (33) completes the proof of the lemma. ▌

4.1. Convergence with constant r and variable μ

We obtain convergence results in this section as the regularization parameters go to zero, for functional parameters μ_n and constant parameters r_n; the same results hold for nearly constant r_n, as will be seen in Section 5. For the result that follows we also require the additional conditions (25)–(28) on $\{\eta_{r_n}\}$, conditions which occur naturally in practice.

THEOREM 4.1. *Assume that $\bar{u} \in C^1[0,1]$. Let $\{r_n\} \subset \mathbb{R}$ be a given sequence with $r_n \to 0$ as $n \to \infty$. Assume that the r_n-dependent measures η_{r_n} are given as in Definition 4.1 and such that the additional conditions (25)–(28) hold.*

Then if, for some $C_\mu \geq 0$ and $p \geq 3$, $\{\mu_n\} \subset L^\infty(0,1)$ is selected satisfying

$$0 \leq \mu_n(t) \leq C_\mu r_n^p, \quad \text{a.e. t} \in [0,1], \tag{34}$$

for all n sufficiently large, it follows that $u(\cdot; r_n, \mu_n)$, the solution of

$$\int_0^t \tilde{k}(t,s;r_n)u(s)\,ds + [\alpha(t;r_n) + \mu_n(t)]\,u(t) = \tilde{f}(t;r_n), \quad \text{a.e. t} \in [0,1], \tag{35}$$

belongs to $L^\infty(0,1)$ and converges to $\bar{u}$ in $L^\infty(0,1)$ as $n \to \infty$.

Moreover, if the sequence $\{f^{\delta_n}\} \in L^\infty(0,T)$ is given with $d_n(t) := f^{\delta_n}(t) - f(t)$, a.e. $t \in [0,1]$, $n = 1,2,\ldots$, where for a.e. $t \in [0,1]$, $|d_n(t)| \leq \delta_n \to 0$ as $n \to \infty$, then the solution $u^{\delta_n}(\cdot; r_n, \mu_n) \in L^\infty(0,1)$ of

$$\int_0^t \tilde{k}(t,s;r_n)u(s)\,ds + [\alpha(t;r_n) + \mu_n(t)]\,u(t) = \tilde{f}^{\delta_n}(t;r_n), \quad \text{a.e. t} \in [0,1], \tag{36}$$

converges to $\bar{u}$ in $L^\infty(0,1)$ as $n \to \infty$, provided that $r_n = r_n(\delta_n)$ is chosen so that

$$\frac{\delta_n}{r_n(\delta_n)} \to 0$$

and $r_n(\delta_n) \to 0$ as $n \to \infty$. In (36), $\tilde{f}^{\delta_n}$ is defined by

$$\tilde{f}^{\delta_n}(t;r_n) := \int_0^{r_n} f^{\delta_n}(t+\rho)\,d\eta_{r_n}(\rho;t).$$

Proof. ¿From (19), the true solution $\bar{u}$ of (1) satisfies

$$\int_0^t \tilde{k}(t,s;r_n)\bar{u}(s)\,ds + [\alpha(t;r_n)+\mu_n(t)]\,\bar{u}(t)$$
$$= \int_0^{r_n} f(t+\rho)\,d\eta_{r_n}(\rho;t) \;+\; \mu_n(t)\bar{u}(t), \qquad (37)$$
$$-\int_0^{r_n}\int_0^{\rho} k(t+\rho,s+t)\,[\bar{u}(s+t)-\bar{u}(t)]\;ds\,d\eta_{r_n}(\rho;t),$$

for a.e. $t\in[0,1]$. Setting $y^{\delta_n}(t;r_n,\mu_n) := u^{\delta_n}(t;r_n,\mu_n)-\bar{u}(t)$, for $u^{\delta_n}(\cdot;r_n,\mu_n)$ the solution of (36), we see that $y^{\delta_n}(t;r_n,\mu_n)$ solves

$$y_n(t) = \frac{-1}{\alpha(t;r_n)+\mu_n(t)}\int_0^t \tilde{k}(t,s;r_n)y_n(s)\,ds + \tilde{E}(t;r_n) + \tilde{F}(t;r_n), \qquad (38)$$

for a.e. $t\in[0,1]$ and $n=1,2,\ldots$, where

$$\tilde{E}(t;r_n) := \frac{\int_0^{r_n} d_n(t+\rho)\,d\eta_{r_n}(\rho;t)}{\alpha(t;r_n)+\mu_n(t)}, \qquad (39)$$

$$\tilde{F}(t;r_n) \;:=\; \frac{\int_0^{r_n(t)}\int_0^{\rho} k(t+\rho,s+t)\,[\bar{u}(s+t)-\bar{u}(t)]\;ds\,d\eta_{r_n}(\rho;t) - \mu_n(t)\bar{u}(t)}{\alpha(t;r_n)+\mu_n(t)}, \qquad (40)$$

a.e. $t\in[0,1]$, and where $\tilde{E}(\cdot;r_n)=0$ for all n in the case of noise-free data f.

A Taylor expansion may be used to write (for suitable $\nu_{t,s}(\rho)$),

$$\tilde{k}(t,s;r_n) \;=\; \int_0^{r_n} k(t+\rho,s)\,d\eta_{r_n}(\rho;t)$$
$$= \int_0^{r_n} [k(t,s)+\rho D_1 k(t+\nu_{t,s}(\rho),s)]\;d\eta_{r_n}(\rho;t),$$

a.e. $0\le s\le t\le 1$, so that

$$\tilde{k}(t,s;r_n) = a_0(t;r_n)k(t,s) + \int_0^{r_n}\rho D_1 k(t+\nu_{t,s}(\rho),s)\,d\eta_{r_n}(\rho;t), \qquad (41)$$

for a.e. $0\le s\le t\le 1$, where a_0 is defined in (29). Similarly, (21) and (22) give

$$\alpha(t;r_n) = a_1(t;r_n)(1+h(t;r_n)), \quad \text{a.e. t} \in [0,1],$$

where $|h(t;r_n)|\le \frac{3}{2}\|k\|_{1,\infty}\|r_n\|_\infty$ for a.e. $t\in[0,1]$, so that

$$\alpha(t;r_n)+\mu_n(t) = a_1(t;r_n)\left(1+\frac{\mu_n(t)}{a_1(t;r_n)}+h(t;r_n)\right), \quad \text{a.e. t} \in [0,1].$$

If we pick $\mu_n(t)\ge 0$, a.e. $t\in[0,1]$, satisfying (34), then (25)–(28) guarantee that, for all r_n sufficiently small,

$$\frac{1}{\alpha(t;r_n)+\mu_n(t)} = \frac{1}{a_1(t;r_n)}\,(1+g(t;r_n))\,,$$

where $\|g(\cdot;r_n)\|_\infty = \mathcal{O}(r_n)$. Conditions (25)–(28) on η_{r_n} also give us that

$$\frac{a_1(t,r_n)}{a_0(t,r_n)} = \kappa r_n(1+\mathcal{O}(r_n)), \quad \text{a.e. t} \in [0,1], \tag{42}$$

where

$$\kappa = \frac{\sum_{\ell=1}^{L} c_\ell \sigma_\ell}{\sum_{\ell=1}^{L} \sigma_\ell} > 0.$$

The kernel in (38) may then be estimated by

$$\begin{aligned} \frac{\tilde{k}(t,s;r_n)}{\alpha(t;r_n)+\mu_n(t)} &= \frac{a_0(t;r_n)}{a_1(t;r_n)}\left(1+g(t;r_n)\right)k(t,s) + D(t,s;r_n) \\ &= \frac{1}{\kappa r_n}(1+\mathcal{O}(r_n))k(t,s) + D(t,s;r_n), \end{aligned} \tag{43}$$

for a.e. $0 \le s \le t \le 1$, where

$$\begin{aligned} |D(t,s;r_n)| &:= \left| \frac{1+g(t;r_n)}{a_1(t;r_n)} \int_0^{r_n} \rho D_1 k(t+\nu_{t,s}(\rho),s)\, d\eta_{r_n}(\rho;t) \right| \\ &\le \frac{1+|g(t;r_n)|}{a_1(t;r_n)} \int_0^{r_n} \rho\, |D_1 k(t+\nu_{t,s}(\rho),s)|\, d\eta_{r_n}(\rho;t) \\ &\le \|k\|_{1,\infty}[1+|g(t;r_n)|] \\ &\le 2\|k\|_{1,\infty}, \end{aligned}$$

for r_n sufficiently small and a.e. $0 \le s \le t \le 1$. Thus

$$\frac{\tilde{k}(t,s;r_n)}{\alpha(t;r_n)+\mu_n(t)} = \frac{k(t,s)}{\varepsilon_n} + K_n(t,s), \quad 0 \le s \le t \le 1,$$

where $\varepsilon_n := \kappa r_n \to 0$ as $n \to \infty$, and

$$K_n(t,s) := \frac{\tilde{\gamma}(t;r_n)k(t,s)}{\kappa r_n} + D(t,s;r_n)$$

is bounded, a.e. $0 \le s \le t \le 1$, as $n \to \infty$. In addition, for all r_n sufficiently small and for a.e. $t \in [0,1]$,

$$\begin{aligned} |\tilde{E}(t;r_n)| &\le \frac{\int_0^{r_n} |d_n(t+\rho)|\, d\eta_{r_n}(\rho;t)}{a_1(t;r_n)}\left(1+|g(t;r_n)|\right) \\ &\le 2\delta_n \frac{a_0(t;r_n)}{a_1(t;r_n)} \\ &\le 4\delta_n/(\kappa r_n), \end{aligned}$$

for r_n sufficiently small, from the assumptions of the theorem. Further,

$$\begin{aligned} &|\tilde{F}(t;r_n)| \\ &\quad \le \left(\frac{\int_0^{r_n}\int_0^{\rho} k(t+\rho,s+t)\, |\bar{u}(s+t)-\bar{u}(t)|\; ds\, d\eta_{r_n}(\rho;t) + \mu_n(t)|\bar{u}(t)|}{a_1(t;r_n)} \right)(1+|g(t;r_n)|) \end{aligned}$$

$$\leq 2\|k\|_\infty \|\bar{u}\|_{1,\infty} \frac{\int_0^{r_n} \int_0^{\rho} s\, ds\, d\eta_{r_n}(\rho;t)}{a_1(t;r_n)} + 2\|\bar{u}\|_\infty \frac{\mu_n(t)}{a_1(t;r_n)}$$
$$= \mathcal{O}(r_n)$$

for r_n sufficiently small and a.e. $t \in [0,1]$.

Thus $y^{\delta_n}(\cdot, r_n, \mu_n)$ satisfies the conditions of Lemma 4.1, i.e., $y^{\delta_n}(\cdot, r_n, \mu_n)$ satisfies

$$y_n(t) \;=\; -\int_0^t \left(\frac{k(t,s)}{\varepsilon_n} + K_n(t,s)\right) y_n(s)\, ds + E_n(t), \quad \text{a.e. t} \in [0,1],$$

where $E_n(t) := \tilde{E}(t;r_n) + \tilde{F}(t;r_n)$ satisfies $\|E_n\|_\infty \to 0$ as $n \to \infty$ under the assumptions in the theorem. The result then follows from Lemma 4.1. ▮

4.2. Convergence with variable r and fixed μ

We turn to a second result which allows r_n to vary with t and places fewer constraints on η_{r_n} than did Theorem 4.1; however, this improvement occurs at the expense of requiring a fixed (r_n-dependent) construction of μ_n.

THEOREM 4.2. *Assume that $\bar{u} \in C[0,T]$ with $\bar{u}(0) = 0$. Let $\{r_n(\cdot)\} \subset \mathcal{P}$ be given with $r_n \to 0$ as $n \to \infty$, uniformly on $[0,1]$, and assume that the sequence $\{\eta_{r_n}\}$ of measures is defined as usual from $\{r_n\}$ using Definition 4.1.*

If $\mu_n = \mu_n(\cdot; r_n)$ is given by

$$\mu_n(t;r_n) := C_\mu \|r_n\|_\infty^p \int_0^{r_n(t)} d\eta_{r_n}(\rho;t), \quad t \in [0,1], \tag{44}$$

for some $p \in (0, 1/2]$ and $C_\mu > 0$, then the solution $u(\cdot; r_n, \mu_n) \in L^\infty(0,1)$ of (35) converges to $\bar{u}(\cdot)$ in $L^\infty(0,1)$ as $n \to \infty$. Moreover, if f^{δ_n} satisfies the conditions of Theorem 4.1, then the solution $u^{\delta_n}(\cdot; r_n, \mu_n)$ of (36) converges to $\bar{u}$ in $L^\infty(0,1)$, provided that $r_n(\cdot) = r_n(\cdot; \delta_n)$ is chosen so that

$$\frac{\delta_n}{\|r_n(\cdot;\delta_n)\|_\infty^p} \to 0$$

and $\|r_n(\cdot;\delta_n)\|_\infty \to 0$ as $n \to \infty$.

Proof. As in the proof of Theorem 4.1, the error function $y^{\delta_n}(\cdot; r_n) = u^{\delta_n}(\cdot; r_n, \mu_n) - \bar{u}(\cdot)$ satisfies (38) for $n = 1, 2, \ldots$, where now $\mu_n(t) = \mu_n(t; r_n)$ is given by (44). Further, from (41) we may write

$$\tilde{k}(t,s;r_n) = a_0(t;r_n)\left[k(t,s) + H(t,s;r_n)\right], \qquad \text{a.e. } 0 \le \text{s} \le \text{t} \le 1, \tag{45}$$

where

$$|H(t,s;r_n)| \;=\; \frac{\left|\int_0^{r_n(t)} \rho D_1 k(t + \nu_{t,s}(\rho), s) d\eta_{r_n}(\rho;t)\right|}{a_0(t;r_n)}$$

$$\leq \|k\|_{1,\infty}\|r_n\|_\infty,$$

for a.e. $0 \leq s \leq t \leq 1$. Employing a Taylor expansion as in (20), we obtain

$$\alpha(t;r_n) = C_\mu \|r_n\|_\infty^p \, a_0(t;r_n)\, \tilde{h}(t;r_n), \quad \text{a.e. t} \in [0,1],$$

for C_μ and p given in (44) and where, for a.e. $t \in [0,1]$,

$$\begin{aligned}
\left|\tilde{h}(t;r_n)\right| &= \frac{\left|\int_0^{r_n(t)}\int_0^\rho [1+\rho D_1 k(t+\xi,t+\zeta)+sD_2k(t+\xi,t+\zeta)]\; ds\, d\eta_{r_n}(\rho;t)\right|}{C_\mu\|r_n\|_\infty^p\, a_0(t;r_n)} \\
&\leq \frac{1}{C_\mu\|r_n\|_\infty^p\, a_0(t;r_n)}\left[\|r_n\|_\infty a_0(t;r_n)+\frac{3}{2}\|k\|_{1,\infty}\int_0^{r_n(t)}\rho^2\, d\eta_{r_n}(\rho;t)\right] \\
&\leq \frac{\|r_n\|_\infty^{1-p}}{C_\mu}\left[1+\frac{3}{2}\|k\|_{1,\infty}\|r_n\|_\infty\right] \\
&= \mathcal{O}(\|r_n\|_\infty^{1-p})
\end{aligned}$$

as $\|r_n\|_\infty \to 0$. Thus using this estimate and the definition of $\mu_n(\cdot;r_n)$ given in (44),

$$\begin{aligned}
\frac{1}{\alpha(t;r_n)+\mu_n(t;r_n)} &= \frac{1}{C_\mu\|r_n\|_\infty^p\, a_0(t;r_n)}\cdot\frac{1}{1+\tilde{h}(t;r_n)} \\
&= \frac{1+\gamma_p(t;r_n)}{C_\mu\|r_n\|_\infty^p\, a_0(t;r_n)}, \quad \text{a.e. t} \in [0,1], \qquad (46)
\end{aligned}$$

for $\|r_n\|_\infty$ sufficiently small, where $\|\gamma_p(\cdot;r_n)\|_\infty = \mathcal{O}(\|r_n\|_\infty^{1-p})$ as $\|r_n\|_\infty \to 0$.

Using (45) and (46), it follows that the solution y_n of (38) satisfies

$$\begin{aligned}
y_n(t) &= -\frac{1+\gamma_p(t;r_n)}{C_\mu\|r_n\|_\infty^p}\int_0^t [k(t,s)+H(t,s;r_n)]\; y_n(s)\, ds + \tilde{E}(t;r_n)+\tilde{F}(t;r_n) \\
&= -\int_0^t \left[\frac{k(t,s)}{C_\mu\|r_n\|_\infty^p}+G(t,s;r_n)\right] y_n(s)\, ds + \tilde{E}(t;r_n)+\tilde{F}(t;r_n), \qquad (47)
\end{aligned}$$

for a.e. $t \in [0,1]$, where $\tilde{E}$ and $\tilde{F}$ are given by (39) and (40), respectively (with $\tilde{E} \equiv 0$ when noise-free data f is used), and where for a.e. $0 \leq s \leq t \leq 1$,

$$|G(t,s;r_n)| = \frac{1}{C_\mu\|r_n\|_\infty^p}\, |k(t,s)\gamma_p(t;r_n)+H(t,s;r_n)\,[1+\gamma_p(t;r_n)]\,|\,.$$

The terms in G are of order $\|r_n\|_\infty^{1-2p}$ and $\|r_n\|_\infty^{1-p}$, respectively, and since $p \in (0,1/2]$, it follows that $\|G(\cdot,\cdot;r_n)\|_\infty = \mathcal{O}(1)$ as $n \to \infty$.

Thus we may apply Lemma 4.1 with $K_n(t,s) = G(t,s;r_n)$, $\varepsilon_n = C_\mu\|r_n\|_\infty^p$, and $E_n(\cdot)$, $F(\cdot)$ given in (30) by

$$\begin{aligned}
E_n(t) &= \frac{\int_0^{r_n(t)}\int_0^\rho k(t+\rho,s+t)\,[\bar{u}(s+t)-\bar{u}(t)]\; ds\, d\eta_{r_n}(\rho;t)+\alpha(t;r_n)\bar{u}(t)}{\alpha(t;r_n)+\mu_n(t;r_n)} \\
&\quad + \frac{\int_0^{r_n(t)} d_n(t+\rho)\, d\eta_{r_n}(\rho;t)}{\alpha(t;r_n)+\mu_n(t;r_n)}, \\
F(t) &= -\bar{u}(t),
\end{aligned}$$

for a.e. $0 \le t \le 1$. In bounding E_n, we have from (46) that for a.e. $t \in [0,1]$,

$$\begin{aligned}
&\frac{\left|\int_0^{r_n(t)} \int_0^{\rho} k(t+\rho, s+t)\,[\bar u(s+t) - \bar u(t)]\; ds\, d\eta_{r_n}(\rho;t) + \alpha(t;r_n)\bar u(t)\right|}{\alpha(t;r_n) + \mu_n(t;r_n)} \\
&\qquad \le \frac{3\alpha(t;r_n)}{\alpha(t;r_n) + \mu_n(t;r_n)}\|\bar u\|_\infty \\
&\qquad \le \frac{3\alpha(t;r_n)}{\mu_n(t;r_n)}\|\bar u\|_\infty \\
&\qquad = 3\tilde h(t;r_n)\|\bar u\|_\infty \\
&\qquad = \mathcal{O}(\|r_n\|_\infty^{1-p}).
\end{aligned}$$

Finally, the last term in $E_n(t)$ is bounded for $0 \le t \le 1$ by

$$\begin{aligned}
\left|\frac{\int_0^{r_n(t)} d_n(t+\rho)d\eta_{r_n}(\rho;t)}{\alpha(t;r_n) + \mu_n(t;r_n)}\right| &\le \delta_n \frac{a_0(t;r_n)}{\mu_n(t;r_n)} \\
&= \delta_n \frac{a_0(t;r_n)}{C_\mu \|r_n\|_\infty^p a_0(t;r_n)}.
\end{aligned}$$

Thus, under the conditions of the theorem we have $\|E_n\|_\infty \to 0$ as $n \to \infty$. An application of Lemma 4.1 then gives that $u^{\delta_n}(\cdot;r_n,\mu_n)$ converges to $\bar u$ in $L^\infty(0,1)$ as $n \to \infty$. ∎

5. CONTINUOUS LOCAL REGULARIZATION WITH GENERAL MEASURES

Finally we turn to a generalization of the theory developed earlier, here allowing the measure η_r to take a more general form. The discrete measure defined in preceding sections will be a special case of the general measure we construct here; in addition, the theory of this section will apply to measures of the form given in Example 5.1 below. Although the generalization of the theory in Sections 3–4 leads to more technical definitions and theoretical results, the proofs of these results are quite similar to those given in the preceding sections (see also [35]). We will state the relevant theorems below without proof.

In the generalization we will let $f \in \mathcal{F}_D$, the admissible set of data functions, where it is assumed that all $g \in \mathcal{F}_D$ are bounded Borel functions on $[0,1]$.

Definition 5.1. For given $r \in \mathcal{P}$, we say that the one-parameter family $\mathcal{N}_r = \{\eta_r(\cdot;t), t \in [0,1]\}$ is an r-suitable family of measures if, for a.e. $t \in [0,1]$, $\eta_r(\cdot;t)$ is a finite positive Borel measure on $[0, r(t)]$, and the family $\mathcal{N}_r$ is such that the following three conditions hold:

(1) $\int_0^{r(t)} \rho\, d\eta_r(\rho;t) > 0$, a.e. t $\in [0,1]$;

(2) the quantities $\tilde k(\cdot,\cdot;r)$, $\alpha(\cdot;r)$, and $\tilde f(\cdot;r)$, defined by (12), (13), and (14), respectively, are Lebesgue measurable on their respective domains (for all $f \in \mathcal{F}_D$); and

(3) $t \mapsto \dfrac{\int_0^{r(t)} d\eta_r(\rho;t)}{\int_0^{r(t)} \rho\, d\eta_r(\rho;t)} \in L_2(0,1)$.

As indicated above, given $r \in \mathcal{P}$ the family $\mathcal{N}_r = \{\eta_r(\cdot;r), t \in [0,1]\}$, where η_r is given by Definition 3.3, is an r-suitable family of measures. Another example of an r-suitable family of measures is as follows.

EXAMPLE 5.1. Let $r \in \mathcal{P}$ be given, and let $\omega_r \in L^\infty(\,(0,\|r\|_\infty)\times(0,T)\,)$ be given with $0 < \underline{\omega}_r \le \omega_r(\rho,t)$, a.e. $\rho \in (0,\|r\|_\infty)$, $t \in (0,T)$. Define $\mathcal{N}_r = \{\eta_r(\cdot;r), t \in [0,1]\}$, where

$$\int_0^{\|r\|_\infty} g(\rho) d\eta_r(\rho;t) := \int_0^{\|r\|_\infty} g(\rho)\omega_r(\rho,t)d\rho, \quad \text{a.e. t} \in [0,1],$$

for g a bounded Borel function. Then $\mathcal{N}_r$ is an an r-suitable family of measures.

When η_r takes a more general form than that assumed in previous sections, we still obtain the existence of solutions $u(\cdot;r,\mu)$ of equation (11). The next theorem states this result precisely, with proof similar to that of Theorem 3.1.

THEOREM 5.1. *Given $r \in \mathcal{P}$, let $\mathcal{N}_r = \{\eta_r(\cdot;t), t \in [0,1]\}$ be an r-suitable family of measures and let μ be given as in Definition 3.2. Assume that k and $f \in \mathcal{F}_D$ satisfy Hypothesis 3.1. Then for $\|r\|_\infty$ sufficiently small, there is a unique solution $u(\cdot;r,\mu) \in L^2(0,1)$ of equation (11) which depends continuously on $f \in L^\infty(0,1)$.*

Analogous to Theorem 4.1, we obtain convergence of regularized solutions to the true solution $\bar{u}$, provided the regularization parameter r is "nearly constant" (this requirement can be seen to follow from condition (48) below), while the functional regularization parameter μ need only satisfy a weak size condition (49) relative to r.

THEOREM 5.2. *Assume that $\bar{u} \in C^1[0,1]$. Let $\{r_n\} \subset \mathcal{P}$ be a given sequence with $r_n \to 0$ as $n \to \infty$, uniformly on $[0,1]$, and for each $n = 1,2,\ldots$, let $\mathcal{N}_{r_n}$ denote a family $\{\eta_{r_n}(\cdot;t), t \in [0,1]\}$ of r_n-suitable measures. Assume that there is $\kappa_n : \mathbb{R}^+ \to \mathbb{R}^+$ for which*

$$\frac{a_1(t,r_n)}{a_0(t,r_n)} = \kappa_n(\|r_n\|_\infty)\left(1 + \mathcal{O}(\|r_n\|_\infty)\right), \quad \text{a.e. t} \in [0,1], \tag{48}$$

as $\|r_n\|_\infty \to 0$, where, for some $m_\kappa > 0$, κ_n satisfies $\kappa_n(\|r_n\|_\infty) \ge m_\kappa \|r_n\|_\infty$ for all n sufficiently large, and $\kappa_n(\|r_n\|_\infty) \to 0$ as $n \to \infty$.

Then if, for some $C_\mu \ge 0$ and $p \ge 1$, $\mu_n(\cdot)$ is selected satisfying $\mu_n(t) \ge 0$, a.e. $t \in [0,1]$, and

$$\mu_n(t) \le C_\mu \|r_n\|_\infty^p \int_0^{r_n(t)} \rho\, d\eta_{r_n}(\rho;t), \quad \text{a.e. t} \in [0,1], \tag{49}$$

for all n sufficiently large, it follows that $u(\cdot; r_n, \mu_n)$, the solution of (35), belongs to $L^\infty(0,1)$ and converges to $\bar{u}$ in $L^\infty(0,1)$ as $n \to \infty$.

Moreover, if the sequence $\{f^{\delta_n}\} \subset \mathcal{F}_D$ is given with $d_n(t) := f^{\delta_n}(t) - f(t)$, a.e. $t \in [0,1]$, $n = 1, 2, \ldots$, where for a.e. $t \in [0,1]$, $|d_n(t)| \leq \delta_n \to 0$ as $n \to \infty$, then the solution $u^{\delta_n}(\cdot; r_n, \mu_n) \in L^\infty(0,1)$ of (36) converges to $\bar{u}$ in $L^\infty(0,1)$ as $n \to \infty$, provided that $r_n(\cdot) = r_n(\cdot; \delta_n)$ is chosen so that

$$\frac{\delta_n}{\|r_n(\cdot;\delta_n)\|_\infty} \to 0$$

and $\|r_n(\cdot;\delta_n)\|_\infty \to 0$ as $n \to \infty$.

Finally, when we relax restrictions on the r regularization parameter, we obtain convergence provided we make a precise definition of the μ regularization parameter. This result is a generalization of Theorem 4.2.

THEOREM 5.3. *Assume that $\bar{u} \in C[0,T]$ with $\bar{u}(0) = 0$. Let $\{r_n(\cdot)\} \subset \mathcal{P}$ be given with $r_n \to 0$ as $n \to \infty$, uniformly on $[0,1]$, and assume that, for each $n = 1, 2, \ldots$, we have an r_n-suitable family $\mathcal{N}_r = \{\eta_{r_n}(\cdot; t), t \in [0,1]\}$ of measures. If $\mu_n = \mu_n(\cdot; r_n)$ is given by*

$$\mu_n(t; r_n) := C_\mu \|r_n\|_\infty^p \int_0^{r_n(t)} d\eta_{r_n}(\rho; t), \quad t \in [0,1],$$

for some $p \in (0, 1/2]$ and $C_\mu > 0$, then the solution $u(\cdot; r_n, \mu_n) \in L^\infty(0,1)$ of (35) converges to $\bar{u}(\cdot)$ in $L^\infty(0,1)$ as $n \to \infty$. Moreover, if f^{δ_n} satisfies the conditions of Theorem 4.1, then the solution $u^{\delta_n}(\cdot; r_n, \mu_n)$ of (36) converges to $\bar{u}$ in $L^\infty(0,1)$, provided that $r_n(\cdot) = r_n(\cdot; \delta_n)$ is chosen so that

$$\frac{\delta_n}{\|r_n(\cdot;\delta_n)\|_\infty^p} \to 0$$

and $\|r_n(\cdot;\delta_n)\|_\infty \to 0$ as $n \to \infty$.

6. CONCLUSION

We have presented theoretical convergence results for a local regularization method applied to ill-posed Volterra problems of the form (1). This method extends the ideas in [22] to the case of a local regularization parameter r which is allowed to vary over different parts of the domain, and to Volterra problems with nonconvolution kernels (which, in practice, means that t-dependent families $\{\eta_r(\cdot; t), t \in [0,1]\}$ of measures must be considered). In addition to allowing variability in r, we have considered the inclusion of a second variable regularization parameter μ which serves to further restrict the variation of regularized solutions. Numerical results indicate that, for fixed regularization parameter r, the parameter

μ may be determined using a local (sequential) Morozov discrepancy principle provided one has a good estimate of the data error and propagated error. A more extensive discussion of this sequential discrepancy principle and analysis of the convergence of a discrete local regularization method (based on the ideas of this paper) may be found in [27].

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