Math 132-12 & 16 **Fall 2003**

Homework Project Due Fri 9/12

1. Compute the derivative of each function $f(x)$ below by algebraically computing the limit

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\,.$$

The strategy is always to cancel a factor of h from the top and bottom. Some algebraic formulas you will need:

$$a^2 - b^2 = (a-b)(a+b), \qquad a^3 - b^3 = (a-b)(a^2 + ab + b^2).$$

a. $f(x) = (2x+1)^2$. **b.** $f(x) = \sqrt{2x+1}$. **c.** $f(x) = x^3$. **d.** $f(x) = \sqrt[3]{x}$.

e. (Extra Credit) How far can you go in computing the derivative of $f(x) = \sin(x)$? (I mean compute with explanations, not just by quoting the answer from some authority.)

2. With your calculator, numerically estimate the following derivatives

$$f'(x) \cong \frac{\Delta f}{\Delta x}$$

using the indicated increments $\Delta x = h$.

a. $f(x) = \sqrt[3]{x}$, $x = 2$, with h small enough to get 3 decimal places of accuracy. Compare with the exact answer from 1(d).

b. $f(x) = \sqrt[3]{x}$, successively $x = 0.0,\ 0.1,\ 0.2,\ \ldots, 1.9,\ 2.0$, always taking $h = 0.1$. Write the results in a table.

c. $f(x) = \sin(x)$, successively $x = 0.0,\ 0.1,\ 0.2,\ \ldots, 1.9,\ 2.0$, always taking $h = 0.1$. Write the results in a table.

3. Use Excel or another spreadsheet to plot the following graphs.

a. Plot the points in your table from 2(b) for the approximate derivative function $(\sqrt[3]{x})'$. In the same picture, plot the graph of the exact derivative from 1(d). How do they compare? Why are they not exactly equal?

b. Plot the points in your table from 2(c) for the derivative function $\sin'(x)$. Make a guess of a formula for $\sin'(x)$ based on this graph.

4. Do Ch. 2.1, Exercise 34, p. 118. This covers the physical meaning of derivative ($f'(x)$ is the rate of change of $f(x)$ at each x), as well as the graphical meaning: the function giving the slopes of the graph $y = f(x)$ above each point x on the x-axis.

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Test I Review

Derivatives In this chapter we have learned several meanings of the derivative $f'(x) = \frac{df}{dx}$.

PHYSICAL: The derivative is the instantaneous *rate of change* of $f(x)$ with respect to x, or how fast $f(x)$ increases per unit increase in x. The units of the derivative are ($f(x)$-units) per (x-units): for example, if $f(x)$ is in km and x is in hr, the derivative is in km/hr.

NUMERICAL: The rate of change is approximately the change in $f(x)$ divided by the change in x, over some small x-interval of length $\Delta x = h$. The small interval starts at x and ends at $x+h$, so the function starts at $f(x)$ and ends at $f(x+h)$. Then we let the increment h tend to 0 (or to be as small as possible given our data). In formulas:

$$f'(x) = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Note that in this limit, x is a constant while $h \to 0$.

ALGEBRAIC: If $f(x)$ is given by a formula, we get a limit having the form $\frac{0}{0}$ when you substitute $h = 0$. We can compute the value of this limit exactly by algebraic simplifications which cancel the vanishing terms in the numerator and denominator.

Example: Evaluate the derivative $f'(x)$ for $f(x) = 1/x$. In the following, x is a constant number (the point at which we find the rate of change), while h is an arbitrary increment, tending to zero.

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
(*) &= \lim_{h \to 0} \frac{\frac{1}{(x+h)} - \frac{1}{(x)}}{h} = \lim_{h \to 0} \left(\frac{1}{x+h} - \frac{1}{x} \right) \cdot \frac{1}{h} \\
&= \lim_{h \to 0} \left(\frac{x}{(x+h)x} - \frac{(x+h)}{(x+h)x} \right) \cdot \frac{1}{h} \\
&= \lim_{h \to 0} \left(\frac{x - (x+h)}{(x+h)x} \right) \cdot \frac{1}{h} = \lim_{h \to 0} \left(\frac{-h}{(x+h)x} \right) \cdot \frac{1}{h} \\
&= \lim_{h \to 0} \frac{-1}{(x+h)x} = \frac{-1}{(x+0)x} = -\frac{1}{x^2}
\end{aligned}$$

You could check this numerically by plugging some particular x into $(*)$, say $x = -2$, and then plugging in $h = \pm 0.1$, ± 0.01, etc. into expression $(*)$, which would yield numbers approaching $-1/x^2 = -0.25$.

GRAPHICAL: The derivative $f'(x)$ is the slope of the tangent line to the graph $y = f(x)$ at a point $(x, f(x))$, or the slope of the graph itself when you zoom in close to this point.

Infinite Limits Another important type of limit has the form $\frac{\text{non-zero}}{0}$ when you substitute the value of the variable.
Example:

$$\lim_{x \to 3^-} \frac{1}{(x-3)^2} = ??$$

Solution: Substitution gives $\frac{1}{0}$, so the one-sided limit will have an infinite value. We must determine whether it tends to $+\infty$ or $-\infty$.

The notation $x \to 3^-$ means that x approaches 3 from below: $x < 3$. Thus $x - 3 < 0$ is a small negative number, and $(x-3)^2 > 0$ is a small positive number. Finally, $1/(x-3)^2$ is a large positive number, and the limit is $+\infty$. We could also substitute values like $x = 2.99, 2.999$, etc., to see which way the limit is going.

Rate of Change. The following problem (from p. 16 of your book) illustrates the physical, numerical and graphical meanings of the derivative applied to real-world data.

Let $f(x)$ measure the temperature (in °F) inside a wall, at a depth of x inches. The indoor temperature (at $x = 0$) is 72°; the outdoor temperature (at $x = 5$) is 0°. In between, the temperature gradually drops as it passes through gypsum wallboard, fiberglass insulation, and wood sheathing. The function is given (a) in the table below; (b) in the graph on p. 16 of your book. The insulating quality of a material is measured by the rate of temperature drop: the larger the rate, the better the insulator. We are interested in evaluating the insulting quality of the three types of wall material.

PROBLEM: Find the approximate rate of change of temperature $f(x)$ with respect to x at $x = 0.2$ (inside the gypsum), $x = 2.0$ (fiberglass), and $x = 4.5$ (wood). Base your answers on **(a)** the table; then re-do them from **(b)** the graph on p. 16.

Table of Values of $f(x)$

x	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4
$f(x)$	69	69.5	68	64	61	58	55	52	48	45	42	39	35

x	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0	4.2	4.4	4.6	4.8	5.0
$f(x)$	32	29	26	22	19	16	13	10	8.5	7	5.5	4	2.5

SOLUTION: **a.** Since the problem asks for the rate of change of $f(x)$ (degrees of temperature change per inch of wall), we are looking for derivatives: $f'(0.2)$, $f'(2.0)$, $f'(4.5)$.

For example, to get the derivative

$$f'(2.0) = \frac{df}{dx} \cong \frac{\Delta f}{\Delta x}$$

for a small increment Δx, let us take the smallest available $\Delta x = 0.2$ and compute $\frac{\Delta f}{\Delta x}$ over the interval $x = 2$ to $x = 2 + \Delta x = 2.2$. From the table, we see that $f(2.0) = 42$, $f(2.2) = 39$, so

$$\frac{\Delta f}{\Delta x} = \frac{f(2.2) - f(2.0)}{\Delta x} = \frac{39 - 42}{0.2} = -15.$$

We could also consider $\Delta x = -0.2$, corresponding to the interval $x = 1.8$ to $x = 2.0$ before our point. Then

$$\frac{\Delta f}{\Delta x} = \frac{f(1.8) - f(2.0)}{\Delta x} = \frac{45 - 42}{-0.2} = -15.$$

Thus, our estimate is: $f'(2) \cong -15$.

b. Now look at the graph. Recall that the derivative $f'(2)$ is the slope of the graph $y = f(x)$ near the point $(2, f(2)) = (2, 42)$. Close up, any smooth graph looks like a line, and the derivative is the slope of that line.

In our case, a large part of the graph near $x = 2$ looks linear, and we need to estimate the slope of this line. Taking the run $\Delta x = 2$, we see that the drop Δf from $x = 1$ to $x = 3$ is about $58 - 26 = 32°$ (i.e., a rise of $-32°$). Thus,

$$f'(2) = \text{slope} = \frac{\text{rise}}{\text{run}} \cong \frac{-32}{2} = -16.$$

This is actually a more reliable answer than in part (a), because the graph is clearly linear, and the round-off error is minimized if we take Δx and Δf large.

CONCLUSION: Computing these following the example of $f'(2.0)$ above, the final answers should be roughly: $f'(0.2) \cong -2.5°\text{F/in}$, $f'(2.0) \cong 16°\text{F/in}$, $f'(4.5) \cong 7.5°\text{F/in}$. Thus, fiberglass is about twice as good an insulator as wood, and five times as good as gypsum.