Math 254H Mon 3/23/2020

Parametric surface $P : \mathbb{R}^2 \to \mathbb{R}^3$

$$P(u,v) = (x(u,v),\ y(u,v),\ z(u,v))$$

$$P : S^* \longrightarrow S$$

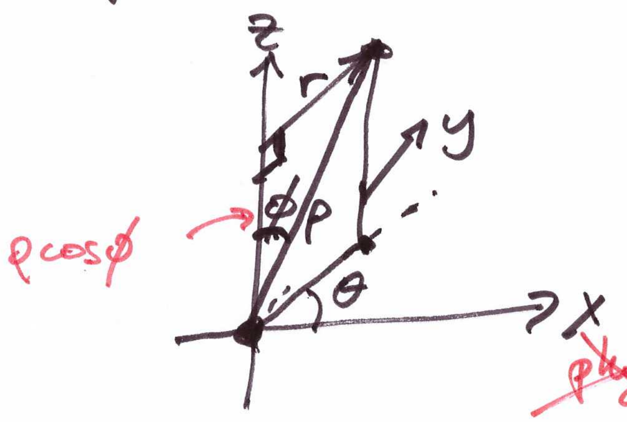
$$S^* \subset \mathbb{R}^2, \quad S \subset \mathbb{R}^3$$

$(u,v) \in \mathbb{R}^2$ parameter region

$(x,y,z) \in \mathbb{R}^3$ surface in space

Spherical coordinates $Sph : \mathbb{R}^3 \to \mathbb{R}^3$

$$(\rho, \theta, \phi) \to (x,y,z)$$



ρ = distance from $\vec{O}$

θ = turn in xy-plane

~~physics~~ ϕ = tilt from z-axis

$r = \rho \sin\phi$

$$(r\cos\theta,\ r\sin\theta,\ \rho\cos\phi)$$

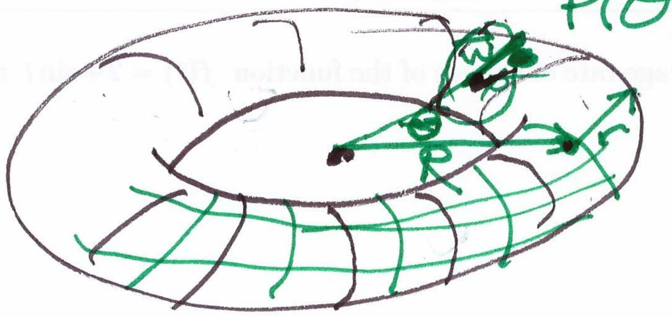
$$= (\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)$$

Sphere: radius $\rho = 2$

$$P(\theta,\phi) = (2\sin\phi\cos\theta,\ 2\sin\phi\sin\theta,\ 2\cos\phi)$$

Torus:

$P(\theta,\phi) = (x,y,z)$



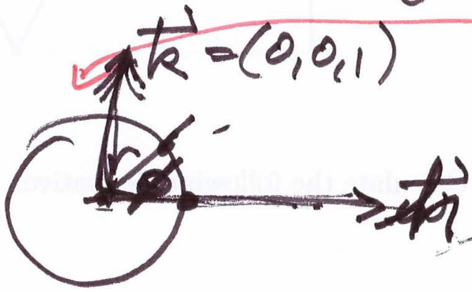
Donut core radius R

tube radius r

$$P(\theta,\phi) = (R\cos\theta, R\sin\theta, 0) + r\vec{w}$$

θ = turn angle in xy-plane

ϕ = tilt above xy-plane



$$(r\cos\phi)\vec{h} + r\sin(\phi)\vec{k}$$

$$\vec{h} = (\cos\theta, \sin\theta, 0)$$

unit radial vector (horizontal)

$$P(\theta,\phi) = \left(\begin{array}{c|c|c} R\cos\theta & R\sin\theta & 0 \\ + r\cos\phi\cos\theta & + r\cos\phi\sin\theta & r\sin\phi \end{array} \right)$$

Toroidal coordinates

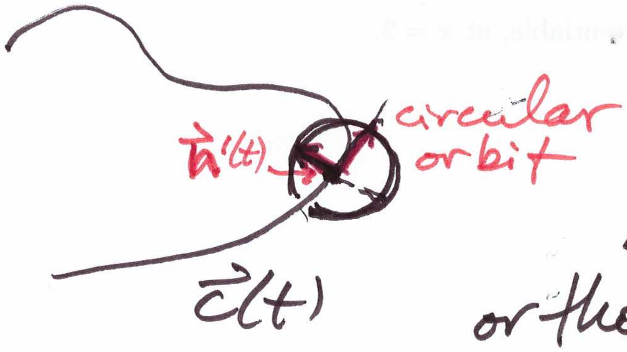
Fix R, vary r, θ, ϕ

$$\mathrm{Tor}(r,\theta,\phi) = (R\cos\theta + r\cos\phi\cos\theta, \ldots)$$

parametrizes space with torus shells going out from core at radius R.

Similar for arbitrary curve, not just circular core!

Frenet Frame: $\vec{c}(t) = (x(t), y(t), z(t))$ curve (core)



Need basis for normal plane orthogonal to $\vec{c}\,'(t)$

Frenet coords:

$$F(t, r, \phi) = \vec{c}(t) + r\cos\phi\,\vec{n}(t) + r\sin\phi\,\vec{b}(t)$$

t: along $\vec{c}(t)$; r: radius from core; ϕ: tilt

Move at constant speed,
then acceleration $\perp$ velocity.

Analytically: unit tangent $\vec{u}(t) = \dfrac{\vec{c}\,'(t)}{|\vec{c}\,'(t)|}$

$$|\vec{u}(t)| = 1$$

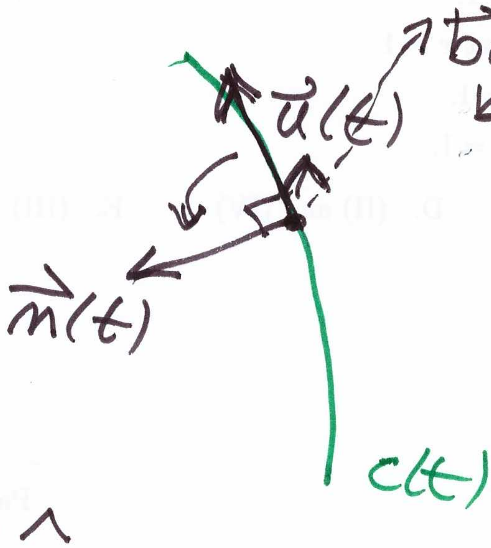
$$\vec{u}(t) \cdot \vec{u}(t) = 1$$

$$\frac{d}{dt} \qquad \vec{u}\,'(t) \cdot \vec{u}(t) + \vec{u}(t) \cdot \vec{u}\,'(t) = 0$$

$$\cancel{2}\, \underline{\vec{u}\,'(t)} \cdot \vec{u}(t) = 0$$

$$\underline{\vec{u}\,'(t)} \perp \vec{u}(t) = \frac{\vec{c}\,'(t)}{|\vec{c}(t)|}$$

unit normal $\vec{n}(t) = \dfrac{\vec{u}\,'(t)}{|\vec{u}\,'(t)|}$



binormal ?

$$\vec{b}(t) = \vec{u}(t) \times \vec{n}(t)$$

$$|\vec{b}(t)| = 1$$

$$= |\vec{u}|\,|\vec{n}|\sin\theta$$

Goal: Integrate over parametric surface S: ⑤

$P: S^* \to S \subset \mathbb{R}^3$

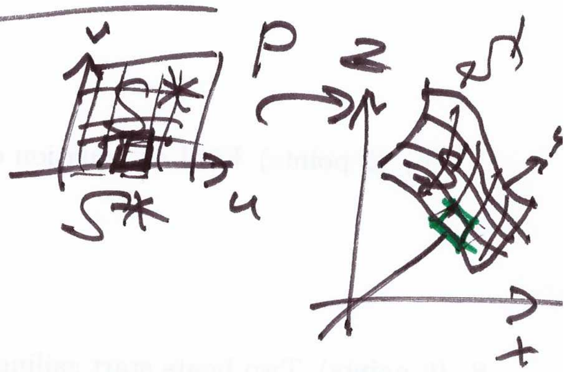
(param region $\mathbb{R}^2$) (surface)

$$\iint_S f \, \underbrace{dS}_{\text{increment of area}}$$

$P(u,v) = (x, y, z)$

density function $f: \mathbb{R}^3 \to \mathbb{R}$

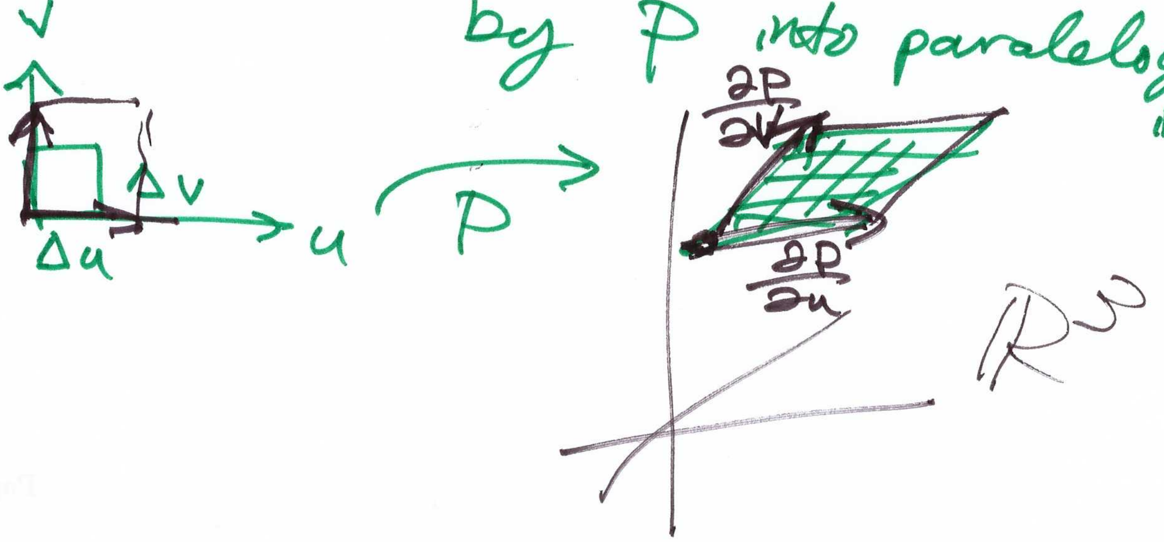
Area: $f = 1$



$$A = \iint_S dS$$

$$= \iint_{S^*} \underbrace{\left(\begin{array}{c}\text{stretching} \\ \text{factor}\end{array}\right)}_{\left|\frac{\partial P}{\partial u} \times \frac{\partial P}{\partial v}\right|} du\, dv$$

How much small rectangle $\Delta u \Delta v$ is stretched by P into paralelogram in $\mathbb{R}^3$



$P(u,v)$ is approximately linear near any point.

$$\left(\begin{array}{c}\text{stretching factor}\\ \text{of } P \text{ near } \cancel{(x,y,z)}\ (u,v)\end{array}\right) = \left(\begin{array}{c}\text{stretching factor}\\ \text{of } DP_{\cancel{(x,y,z)}\ (u,v)}\end{array}\right)$$

$$DP_{\cancel{(x,y,z)}\ (u,v)} = \begin{bmatrix} \Delta x(u,v) \\ \Delta y(u,v) \\ \Delta z(u,v) \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} \dfrac{\partial P}{\partial u} & \dfrac{\partial P}{\partial v} \end{array}\right]}_{3\times 2}$$

Jacobian

$=$ Area of parallelogram spanned by $\dfrac{\partial P}{\partial u}$ & $\dfrac{\partial P}{\partial v}$ $\subset \mathbb{R}^3$

$$\overset{!}{=} \left| \frac{\partial P}{\partial u} \times \frac{\partial P}{\partial v} \right|$$ length of cross product!

$\subset \mathbb{R}^3$