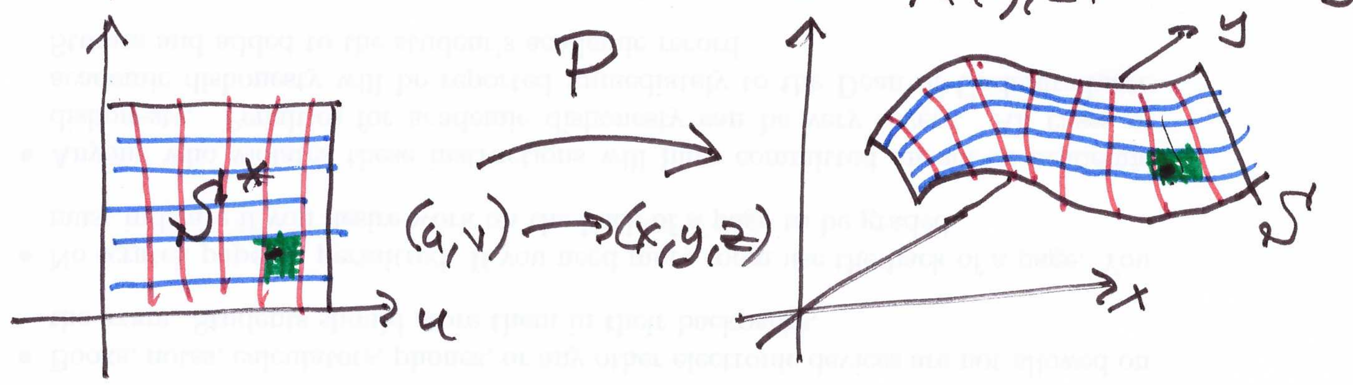
Math 254H 3/25/2020

Goal: Integrate over surface $S \subset \mathbb{R}^3$
2-dim
parametrize

$f(x,y,z)$ = density

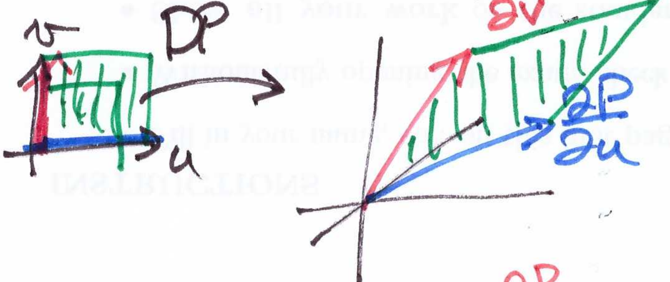


$$\iint_S f\, dS \overset{def}{=} \iint_{D^*} f(P(u,v)) \underbrace{\left|\frac{\partial P}{\partial u} \times \frac{\partial P}{\partial v}\right|}_{\text{stretching factor}} du\, dv$$

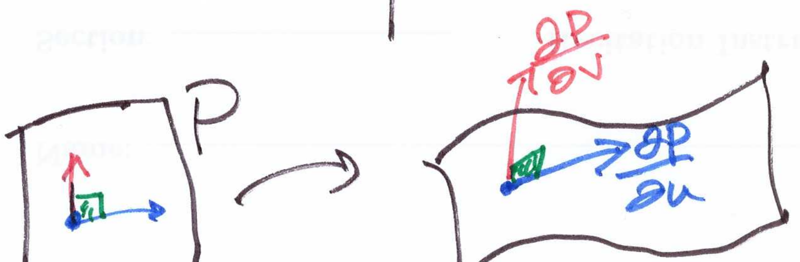
density, increment of area

Jacobian

$DP_{(u,v)} : \mathbb{R}^2 \to \mathbb{R}^3$ 3x2 matrix



$$\left[\frac{\partial P}{\partial u} \middle| \frac{\partial P}{\partial v}\right] = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}$$



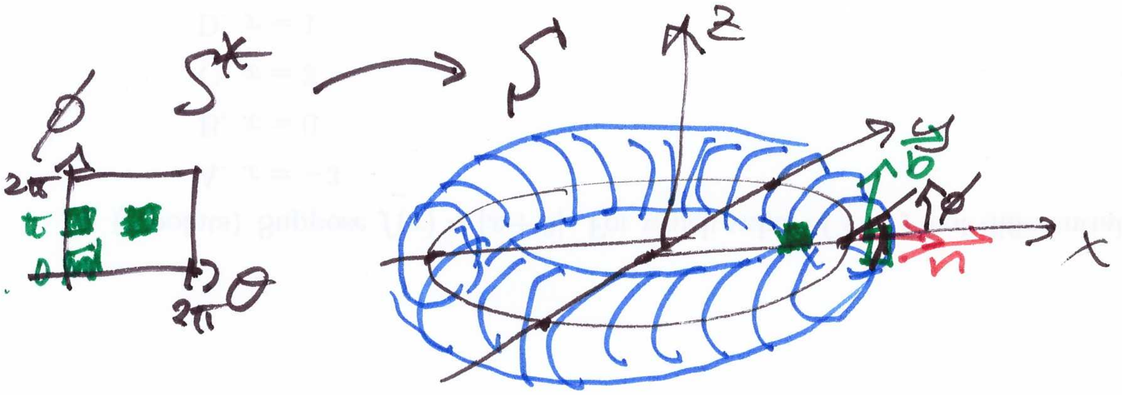
S^*

stretching factor = area of $\left(\frac{\partial P}{\partial v}, \frac{\partial P}{\partial u}\right)$ parallelogram

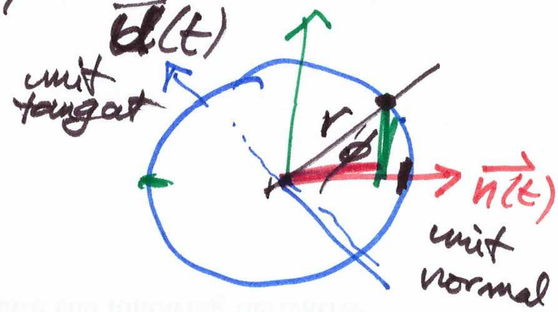
length of cross product $= \left|\frac{\partial P}{\partial u} \times \frac{\partial P}{\partial v}\right|$

Torus: tube around circular core

$$\vec{c}(t) = R(\cos\theta, \sin\theta, 0)$$



$$T(\theta, \phi) = R(\cos\theta, \sin\theta, 0)$$
$$+ r\cos\phi\ \vec{n}(t)$$
$$+ r\sin\phi\ \vec{b}(t)$$



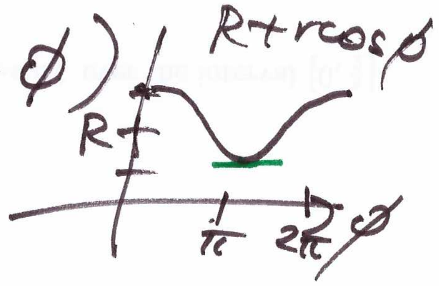
$$= R(\cos\theta, \sin\theta, 0)$$
$$+ r\cos\phi(\cos\theta, \sin\theta, 0)$$
$$+ r\sin\phi(0, 0, 1)$$

"toroidal parameter map"

$$\frac{\partial T}{\partial \theta} = \begin{bmatrix} -\sin\theta\,(r\cos\phi + R) \\ \cos\theta\,(r\cos\phi + R) \\ 0 \end{bmatrix}$$

$$\frac{\partial T}{\partial \phi} = \begin{bmatrix} -r\sin\phi\,\cos\theta \\ -r\sin\phi\,\sin\theta \\ r\cos\phi \end{bmatrix}$$

$$\left| \frac{\partial T}{\partial \theta} \times \frac{\partial T}{\partial \phi} \right| = r\,(R + r\cos\phi)$$



Area : density $\delta = 1$

$$\iint_S 1\,dS = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{2\pi} \underline{r(R + r\cos\phi)}\,d\phi\,d\theta$$

no θ

$$= \int_{\theta=0}^{2\pi} 1\,d\theta \int_{\phi=0}^{2\pi} r(R + r\cos\phi)\,d\phi$$

$$= 2\pi \quad rR(2\pi) = 4\pi^2 rR$$

FYI: How to generalize surface integral to k-dim $S \subset \mathbb{R}^n$ n-space

$k \le n$

$$P : \mathbb{R}^k \longrightarrow \mathbb{R}^n$$

$$P(u_1, \ldots, u_k) = (x_1(u_1 \ldots u_k), \ldots, x_n(u_1 \ldots u_k))$$

$$\text{Area} = \underbrace{\int \cdots \int}_{\substack{u_1 \ldots u_k \\ S^k}} \begin{pmatrix}\text{stretching} \\ \text{factor}\end{pmatrix} du_1 \cdots du_k$$

stretching factor

$$DP = \left[\frac{\partial P}{\partial u_1} \Bigg| \cdots \Bigg| \frac{\partial P}{\partial u_k} \right]$$

$n \times k$ matrix

$\|$

k-dim volume of paralelopiped spanned by $\frac{\partial P}{\partial u_1} \ldots \frac{\partial P}{\partial u_k} \subset \mathbb{R}^n$

$$\underbrace{DP^T \cdot DP}_{k \times k \text{ matrix}} = \underbrace{\begin{bmatrix} \frac{\partial P}{\partial u_1} \\ \vdots \\ \frac{\partial P}{\partial u_k} \end{bmatrix}}_{k \times n} \cdot \underbrace{\left[\frac{\partial P}{\partial u_1} \Bigg| \cdots \Bigg| \frac{\partial P}{\partial u_k} \right]}_{n \times k}$$

$$\left[\frac{\partial P}{\partial u_i} \cdot \frac{\partial P}{\partial u_j} \right]_{i,j=1}^{k}$$

$$\begin{matrix}\text{stretch} \\ \text{factor}\end{matrix} = \sqrt{\det DP^T \cdot DP}$$

Curl Theorem: In $\mathbb{R}^3$

total circulation of $\vec{F}$ around $\vec{c}(t)$ = integral of rate of circulation enclosed by $\vec{c}$ (over surface S with boundary $\vec{c}$)

scalar

$$\oint \vec{F}(\vec{c})\cdot d\vec{c}$$

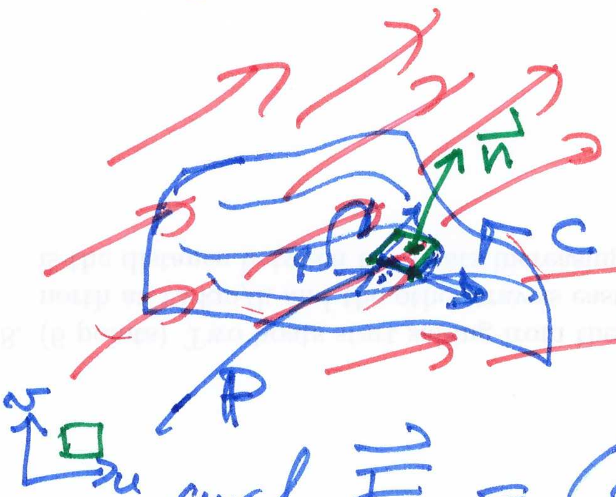
=

vector

$$\iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\,dS$$

← unit normal vector

?? $\left|\frac{\partial P}{\partial u}\times\frac{\partial P}{\partial v}\right| du\,dv$



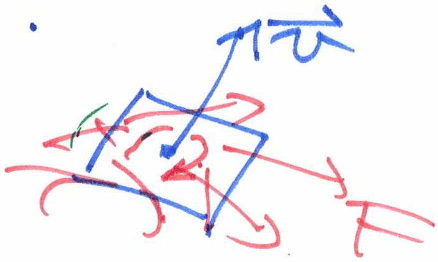
$$\text{curl}\,\vec{F} = (\text{curl}_{yz}F,\ -\text{curl}_{xz}F,\ \text{curl}_{xy}F)$$

directional curl

$$\text{curl}_{\vec{v}}F = (\text{curl}\,\vec{F})\cdot\vec{v}$$

$|\vec{v}|=1$

plane perp to v
around axis v



$$\vec{n} = \frac{\frac{\partial P}{\partial u}\times\frac{\partial P}{\partial v}}{\left|\frac{\partial P}{\partial u}\times\frac{\partial P}{\partial u}\right|}$$

unit normal to S at $P(u,v)$

integral of rate of circulation of F over surface S

$$= \iint_{S^*} \underbrace{\frac{(\operatorname{curl}\vec{F})\cdot\left(\frac{\partial P}{\partial u}\times\frac{\partial P}{\partial v}\right)}{\cancel{\left|\frac{\partial P}{\partial u}\times\frac{\partial P}{\partial v}\right|}}}_{\text{rate of circulation of F parallel to S}} \underbrace{\cancel{\left|\frac{\partial P}{\partial u}\times\frac{\partial P}{\partial v}\right|}\, du\, dv}_{\text{area increment}}$$

$$= \iint_{S^*} \underbrace{(\operatorname{curl}\vec{F})\cdot\left(\frac{\partial P}{\partial u}\times\frac{\partial P}{\partial v}\right)}_{\text{scalar}} du\, dv$$

$$\overset{!}{=} \oint_{C} \vec{F}(\vec{c})\cdot d\vec{c}$$

Curl Theorem in Space!