Math 254H 3/27/2020

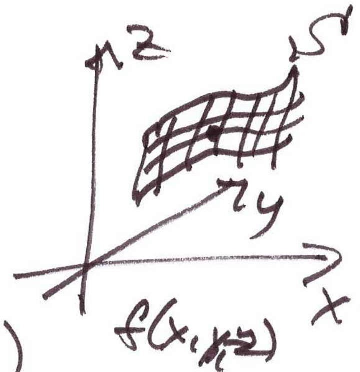
How to integrate over a parametric surface

$$P : S^* \longrightarrow S \quad \text{surface in space}$$

$S^* \subset \mathbb{R}^2$ parameter region

$S \subset \mathbb{R}^3$

$$P(u,v) = (x,y,z)$$



Integrate a function $f(x,y,z)$ (density)

area element

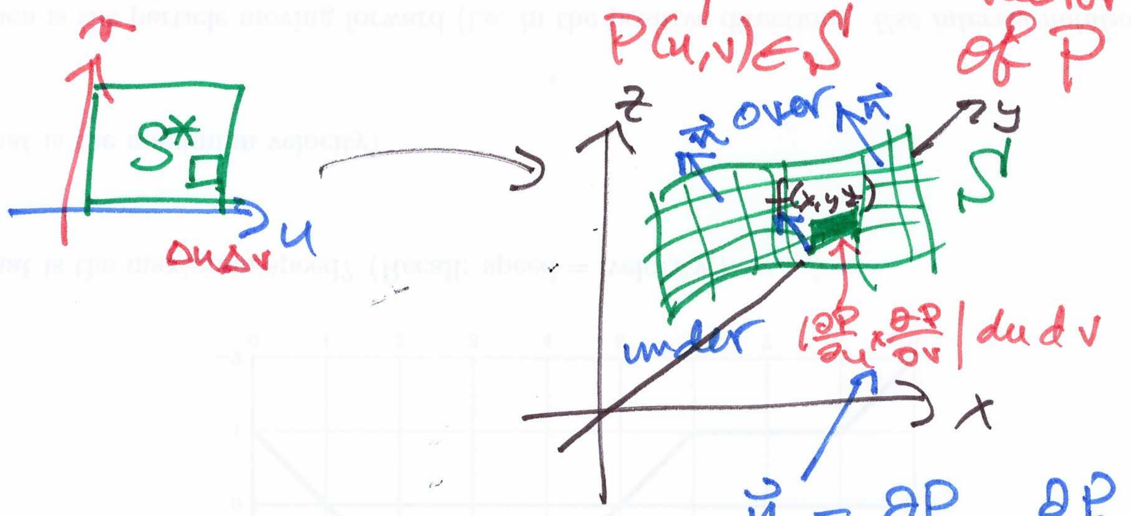
$$(\text{total mass}) \quad \iint_S f\, dS \overset{def}{=} \iint_{D^*} f(P(u,v)) \left|\frac{\partial P}{\partial u} \times \frac{\partial P}{\partial v}\right| du\,dv$$

scalar integral

density at point $P(u,v) \in S$

stretching factor of P

area of small rectangle in (u,v)



$$\vec{n} = \frac{\partial P}{\partial u} \times \frac{\partial P}{\partial v}$$

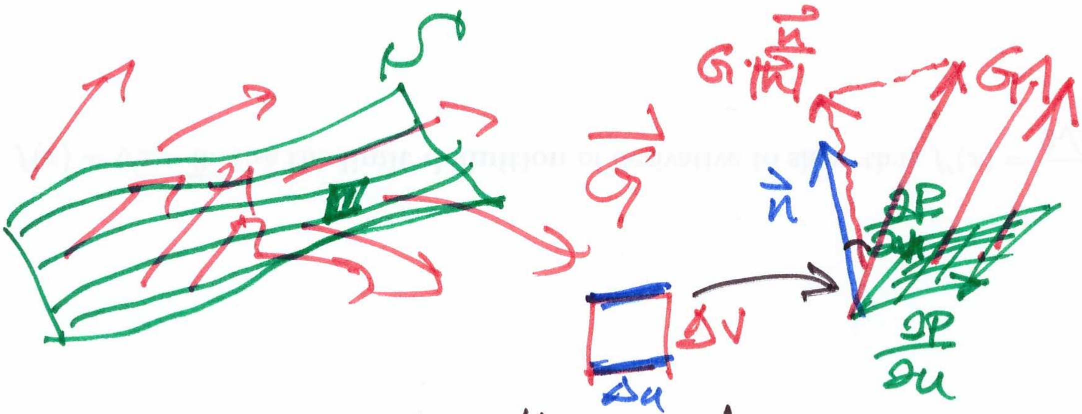
Last time (Curl theorem)

Total curl $\vec{F}$ over S

Vector field $\vec{G}$:

Flux of $\vec{G}$ through S:

(~~net~~ flow of $\vec{G}$ across S in direction of normal vector)



Flow ~~[illegible]~~ of G through small ~~p~~ parallelogram of S?

Component of G in direction $\vec{n}$

$$\vec{G} \cdot \frac{\vec{n}}{|\vec{n}|} = \text{strength of flow}$$

unit vector

$$|\vec{n}| = \left|\frac{\partial P}{\partial u} \times \frac{\partial P}{\partial v}\right| = \text{area of paralelogram}$$

Total flow: Flux integral

$$\iint_S \vec{G} \cdot \underline{d\vec{S}} \stackrel{\text{def}}{=} \iint_{S^*} \vec{G}(P(u,v)) \cdot \frac{\vec{n}}{|\vec{n}|} |\vec{n}| \, du\, dv$$

$$= \iint_{S^*} \vec{G}(P(u,v)) \cdot \underbrace{\vec{n}\, du\, dv}_{d\vec{S}}$$

normal element of surface → $d\vec{S}$
vector

Curl Theorem: Given $\vec{F}$, surface S & boundary $\vec{c}$

circulation of $\vec{F}$ around $\vec{c}$ = integral of rate of circulation of $\vec{F}$ "enclosed by $\vec{c}$"

$\oint \vec{F}(\vec{c}) \cdot d\vec{c}$

= integral of directional curl of $\vec{F}$ parallel to S, curl $\vec{F} \cdot \vec{n}$ (around normal axis $\vec{n}$)

$$= \iint_S (\operatorname{curl} \vec{F}) \cdot d\vec{S}$$

Flux of curl $\vec{F}$ through S

scalar integral ~~ov~~ over a curve:

$$\int_{\vec{c}} f\, dc$$

$$\parallel$$

$$\int_{t=0}^{1} \underbrace{f(\vec{c}(t))}_{\text{density}} \underbrace{|\vec{c}\,'(t)|\, dt}_{\text{length increment}}$$

Main Theorems for $\mathbb{R}^3$

(in analogy to $\mathbb{R}^2$)

Gradient Theorem: Total change of f over a curve $\vec{c}$ = Integral of rate of change of f along $\vec{c}$

$$f(\vec{c}(1)) - f(\vec{c}(0)) = \int \nabla f(\vec{c}) \cdot \vec{c}\,' \, dt = \int \nabla f(\vec{c}) \cdot d\vec{c}$$

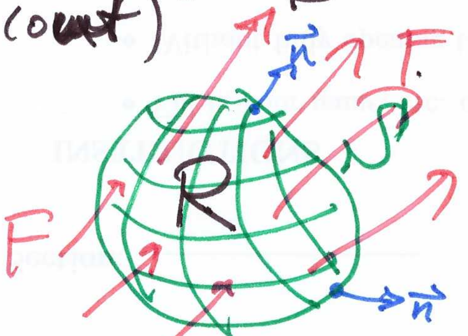
Curl Theorem

$$\oint \vec{F}(\vec{c}) \cdot d\vec{c} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

$\vec{c}$ closed, boundary of S

Divergence Theorem S closed

flux of $\vec{F}$ across S (out) = integral of rate of flux of $\vec{F}$ enclosed by S



S = boundary of solid R

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_R \left(\text{rate of flux of } F \right) dR$$

volume increment of R

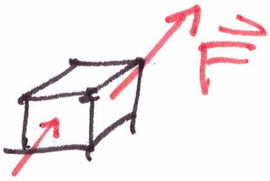
scalar   scalar

$$= \iiint_{R^*} (\operatorname{div} \vec{F}) \det[DP] \, du\,dv\,dw$$

mult of scalars

Need to interpret (compute)

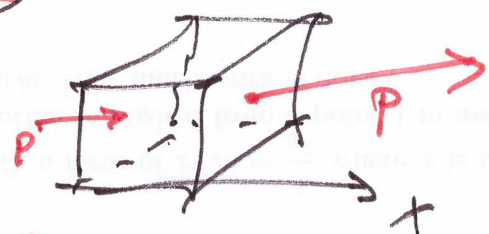
$\operatorname{div}\vec{F}$ = rate of flux of $\vec{F}$ near (x,y,z)



$$= \lim_{\Delta x \Delta y \Delta z \to 0} \frac{\left(\begin{array}{l}\text{flux of } \vec{F} \\ \text{through small box} \\ \text{size } \Delta x\, \Delta y\, \Delta z\end{array}\right)}{vol = \Delta x\, \Delta y\, \Delta z}$$

$$= \underbrace{\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial z}}_{\text{scalar}}$$

$\vec{F} = (p, q, r)$



$\frac{\partial p}{\partial x}$ positive contribution to div