Math 254 H 4/13/2020 Differential Forms

A more natural way to differentiate ....

Preliminary: covector / linear function

Vector space V $\qquad \varphi : V \to \mathbb{R}$

$\qquad\qquad\qquad\qquad\quad v \mapsto \varphi(\vec{v})$

vectors $\vec{v}$ $\qquad$ dual vector space $V^* = \{\text{covectors } \varphi\}$

add $(\varphi + \psi)(\vec{v}) = \varphi(\vec{v}) + \psi(\vec{v})$

scalar mult $(c\,\varphi)(\vec{v}) = (c\,\varphi(\vec{v}))$

mult in $\mathbb{R}$

Use dot product to describe covectors:

$\varphi(\vec{v}) = \vec{c} \cdot \vec{v}$ = "height of $\vec{v}$ in direction $\vec{c}$" (scaled by $|\vec{c}|$)

$\varphi = \vec{c}^{\,*}$

Derivative of $f : \mathbb{R}^n \to \mathbb{R}$: gives linear approximation of $f(\vec{v})$ at each point $\vec{a}$

covector

$f(\vec{a} + \vec{h}) \approx f(\vec{a}) + Df_{\vec{a}}(\vec{h})$

center $\quad$ increment

$Df_{\vec{a}} = (\nabla f(\vec{a}))^*$

$Df_{\vec{a}} : \mathbb{R}^n \to \mathbb{R}$

linear fun = covector

$Df_{\vec{a}}(\vec{h}) = \text{grad} f \cdot \vec{h}$

∇f

• $\nabla f = \vec{F}$ vector field = vector at each point

$$\nabla f(\vec{a}) = \vec{F}(\vec{a})$$

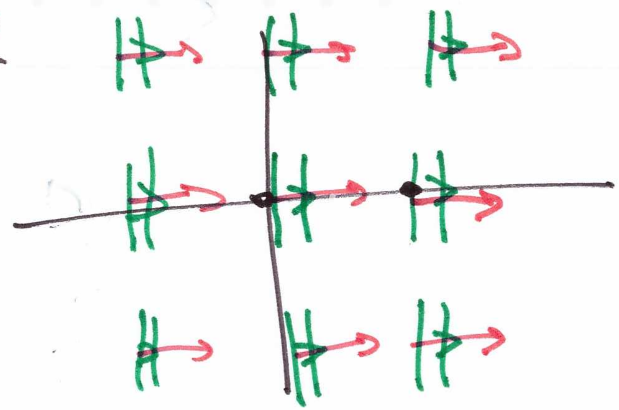
Instead: Df = covector field = covector at each point
= differential form

notation $Df = df$ (differential form) so $df_{\vec{a}} : \mathbb{R}^n \to \mathbb{R}$

$$df_{\vec{a}}(\vec{h}) = Df_{\vec{a}}(\vec{h}) = \nabla f(\vec{a}) \cdot \vec{h} = \left[\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n}\right] \cdot \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$$

• $\mathbb{R}^2$, $f(x,y) = x$

df

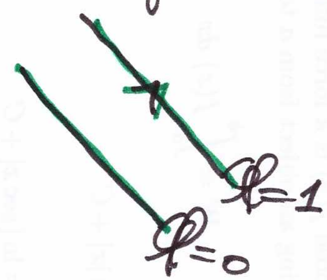


$\boxed{df = dx}$

$dx_{\vec{a}}(h_1, h_2) = h_1$

$\nabla f = \nabla x = (1,0)$

picture of covector $\ell : \mathbb{R}^2 \to \mathbb{R}$

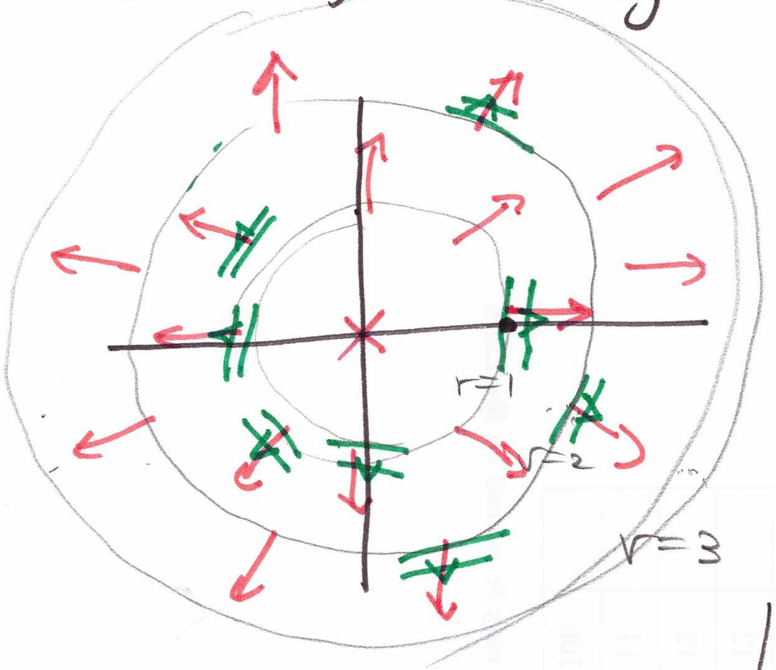


$\ell(\vec{v}) = 0$ all $\vec{v}$
draw as •

6 $f(x,y) = r(x,y) = \sqrt{x^2+y^2}$

$f : \mathbb{R}^2 \to \mathbb{R}$

$df_{\vec{a}} : \mathbb{R}^2 \to \mathbb{R}$



$df_{\vec{a}}(h_1, h_2)$

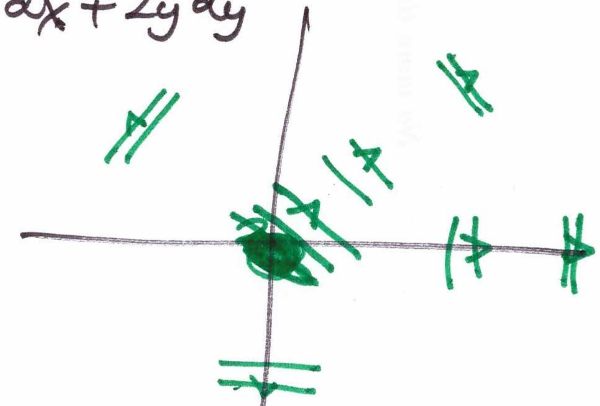
$= \nabla f(\vec{a}) \cdot (h_1, h_2)$

$= \frac{\partial f}{\partial x}(\vec{a})\, h_1 + \frac{\partial f}{\partial y}(\vec{a})\, h_2$

$\nabla r(\vec{a}) = \frac{\vec{a}}{|\vec{a}|}$

6 $f(x,y) = x^2+y^2 = r^2$

$dr = 2x\,dx + 2y\,dy$



d

$\boxed{df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.}$

Not every differential form φ appears as df for function f.
(since not every $\vec{F}$ vect field is ∇f for some f)

Any diff form α, can define its line integral over a curve.

Line integral of diff form φ over $\vec{c}(t)$ $0 \le t \le 1$

$\varphi = p\,dx + q\,dy$, $\vec{c}(t) = (x(t), y(t))$

$$\int_{\vec{c}} \varphi = \int_{t=0}^{1} \varphi_{\vec{c}(t)}(\vec{c}\,'(t))\,dt = \int \vec{F} \cdot d\vec{c}$$

$\varphi = F^*$

$$= \int_{t=0}^{1} p(x(t), y(t))\, x'(t) + q(x(t), y(t))\, y'(t)\, dt$$