Euler's Method is the most basic numerical method for solving differential equations, such as those which appear in natural science. Consider an equation satisfied by an unknown function $y = y(t)$:

$$\frac{dy}{dt} = f(t, y) \qquad \text{or} \qquad y'(t) = f(t, y(t)),$$

where $f(t, y)$ is a given smooth function, along with a known initial value $y(0)$. The Picard-Lindelof Theorem guarantees a unique solution $y(t)$, which can be approximated as follows.

We fix a time increment Δt, and from $y(0)$ we recursively compute approximate values $y(\Delta t)$, $y(2\Delta t)$, $y(3\Delta t), \ldots$. Given $y(t)$ at the current point, we step to the next point by following the tangent line from $(t, y(t))$:

$$y(t{+}\Delta t) \;\approx\; y(t) + y'(t)\Delta t \;=\; y(t) + f(t, y(t))\Delta t.$$

For smaller and smaller $\Delta t \to 0$, the approximatie solutions $y(t)$ converge to the exact solution of the equation $y'(t) = f(t, y(t))$.

1. The *logistic equation* describes the growth of a self-reproducting population which is constrained by a maximum environmental capacity $y = 1$:

$$\frac{dy}{dt} \;=\; y(1 - y).$$

a. Explain why this equation describes roughly exponential growth when the population y is small, but stalling growth as it nears the environmental capacity $y = 1$. Are there any constant solutions $y(t) = C$?

b. Using a spreadsheet or other software, implement Euler's Method for this equation, with the initial condition $y(0) = 0.1 = 10\%$ capacity, and graph the result. How long does it take to reach 90% of capacity, the time t_1 with $y(t_1) = 0.9$? Experiment with different Δt until your estimate for t_1 stabilizes to 1 decimal place.

2. Approximate $y(t) = \sin(t)$ by solving the spring equation (Hooke's law):

$$y'' = -y \quad \text{with} \quad y(0) = 0,\; y'(0) = 1.$$

This is a second-order differential equation of the form:

$$y'' = F(y) \quad \text{with given} \quad y(0), y'(0).$$

(Since the right side does not involve t, this is an *autonomous* equation.) To solve, we compute lists of both $y(t)$ and $y'(t)$, updating y using its slope y', and updating y' using its slope $(y')' = F(y)$.

a. Implement Euler's Method, updating the pairs $y(t), y'(t)$ for $t = 0, \Delta t, 2\Delta t, \ldots$ by:

$$\begin{aligned} y(t{+}\Delta t) &= y(t) + y'(t)\,\Delta t \\ y'(t{+}\Delta t) &= y'(t) + F(y(t))\,\Delta t. \end{aligned}$$

Graph the resulting $y = y(t)$ over $t \in [0, \pi]$. How small must you take Δt to get $y(\pi) \approx \sin(\pi) = 0.0$ correct up to 1 decimal place?

b. The errors in the above method accumulate quickly. We can coax errors to cancel each other using the *Midpoint Method* (or *Leapfrog Method*). Instead of the initial value $y'(0)$, we use a staggered initial value $y'(\frac{\Delta t}{2}) = y'(0) + F(y(0))\frac{\Delta t}{2}$. Starting from $y(0), y'(\frac{\Delta t}{2})$, we compute successive pairs $y(t), y'(t+\frac{\Delta t}{2})$ by the formulas:

$$\begin{aligned} y(t+\Delta t) &= y(t) + y'(t+\tfrac{\Delta t}{2})\,\Delta t \\ y'(t+\tfrac{3\Delta t}{2}) &= y'(t+\tfrac{\Delta t}{2}) + F(y(t+\Delta t))\,\Delta t. \end{aligned}$$

Note how each new value of y and y' is computed using only previously known values.

PROBLEM: Implement this, adjusting Δt to get $y(t) \approx \sin(t)$ correct to 2 decimals.

3. Numerical planetary model. Next we consider methods for plotting the trajectory of an object moving in the xy-plane under the inverse-square force field:

$$\mathbf{F}(x,y) = (p(x,y), q(x,y)) = -\frac{(x,y)}{(x^2+y^2)^{3/2}}.$$

Letting $\mathbf{c}(t) = (x(t), y(t))$ be the position at time t, and assuming the object has unit mass, Newton's Second Law of Motion says the acceleration is equal to the force:

$$\mathbf{c}''(t) = \mathbf{F}(\mathbf{c}(t)) \qquad \text{or} \qquad \begin{cases} x''(t) &= p(x(t), y(t)) \\ y''(t) &= q(x(t), y(t)) . \end{cases}$$

To solve this system of second-order differential equations, we again use the Midpoint Method. Given an intial position $\mathbf{c}(0) = (x(0), y(0))$ and initial velocity $\mathbf{c}'(0) = (x'(0), y'(0))$, we compute the staggered initial velocity $(x'(\frac{\Delta t}{2}), y'(\frac{\Delta t}{2}))$ as before. Starting with $x(0), y(0), x'(\frac{\Delta t}{2}), y'(\frac{\Delta t}{2})$, we compute successive quadruples:

$$x(t),\ y(t),\ x'(t+\tfrac{\Delta t}{2}), y'(t+\tfrac{\Delta t}{2}),$$

updating by:

$$\begin{aligned} x(t+\Delta t) &= x(t) + x'(t+\tfrac{\Delta t}{2})\,\Delta t \\ y(t+\Delta t) &= y(t) + y'(t+\tfrac{\Delta t}{2})\,\Delta t \\ x'(t+\tfrac{3\Delta t}{2}) &= x'(t+\tfrac{\Delta t}{2}) + p(x(t+\Delta t), y(t+\Delta t))\,\Delta t \\ y'(t+\tfrac{3\Delta t}{2}) &= y'(t+\tfrac{\Delta t}{2}) + q(x(t+\Delta t), y(t+\Delta t))\,\Delta t. \end{aligned}$$

a. Implement this, and graph the resulting trajectory for $(x(0), y(0)) = (1, 0)$, $(x'(0), y'(0)) = (0, 0.6)$. (Use a scatter-plot graph of the points $(x(t), y(t))$, connected by line segments.) Compute out for a full orbit, until $(x(t), y(t))$ returns to its initial position (up to an approximation error). Adjust Δt to get the return point correct to at least 1 decimal place.

b. Check that the above orbit approximately obeys Kepler's First Law of planetary motion: that the planets orbit in ellipses with the sun at one focus. That is, check that your orbit is approximately equal to a curve $\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = 1$, where $2a$ is the horizontal axis of the ellipse, $2b$ is the vertical axis, and $c = \sqrt{a^2 - b^2}$ is the distance from the center to the focus along the horizontal axis. Find the correct values of a, b, and plot the graph of the ellipse together with the orbit.

c. *Extra Credit:* Try to spot-check Kepler's other conclusions. Second Law: A planet moves so its radius from the sun sweeps equal areas over equal times. Third Law: The period of a planet's orbit is proportional to the $3/2$ power of its major axis. (You would need to compare two different orbits for this.)