Weekly Homework 5 – Hint

Here is a style model for the proof of the Conservative Vector Field Theorem.

State the full theorem at the beginning. Divide the proof into sections corresponding to implications among the four conditions. For example (i) $\Rightarrow$ (ii) means: If (i) $\mathbf{F}$ is conservative then (ii) $\mathbf{F}$ is path-independent. Prove the chain (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv), then in reverse (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

Proof: (i) $\Rightarrow$ (ii): Suppose $\mathbf{F}$ is conservative, meaning $\mathbf{F} = \nabla f$ for some potential function $f(x, y)$. Let $\mathbf{c}_1, \mathbf{c}_2$ be any two paths between the same endpoints, so that $\mathbf{c}_1(0) = \mathbf{c}_2(0)$ and $\mathbf{c}_1(1) = \mathbf{c}_2(1)$. Then we have:

$$
\begin{array}{rcll}
\int \mathbf{F}(\mathbf{c}_1) \cdot d\mathbf{c}_1 & = & \int \nabla f(\mathbf{c}_1) \cdot d\mathbf{c}_1 & \text{by assumption,} \\
& = & f(\mathbf{c}_1(1)) - f(\mathbf{c}_1(0)) & \text{by the Gradient Theorem,} \\
& = & f(\mathbf{c}_2(1)) - f(\mathbf{c}_2(0)) & \text{since } \mathbf{c}_1 \text{ and } \mathbf{c}_2 \text{ have same endpoints,} \\
& = & \int \nabla f(\mathbf{c}_1) \cdot d\mathbf{c}_1 & \text{by the Gradient Theorem,} \\
& = & \int \mathbf{F}(\mathbf{c}_2) \cdot d\mathbf{c}_2 & \text{by assumption.}
\end{array}
$$

Thus $\int \mathbf{F}(\mathbf{c}_1) \cdot d\mathbf{c}_1 = \int \mathbf{F}(\mathbf{c}_2) \cdot d\mathbf{c}_2$, which means $\mathbf{F}$ is path-independent.

Method Notes: Assume the hypothesis (setup) at the beginning, spelling out its definition. The conclusion is deduced at the end, along with its definition as a formula A = B. In between, you prove the formula, starting with the left side A and transforming it step by step into the right side B, justifying each step by a known theorem or assumption.

In your rough draft, you may work from both sides of the formula to meet in the middle, but in your final proof, arrange all the equalities to go from the left side to the right side.