Math 299 **Recitation 2** **Sept 5, 2013**

PROBLEM 1.

(a) A coin is flipped twice. Describe the possible outcomes as the elements of a set. What is the cardinality of the set?

Let H denote 'heads' and T denote 'tails'. Note that the order matters in the sense that getting H first and then T is different from T and then H. Then the set of possible outcomes can be described as

$$\{(H,H),(H,T),(T,H),(T,T)\}.$$

This has cardinality 4.

(b) Two identical coins are each flipped once. Describe the possible outcomes as the elements of a set. What is the cardinality of the set?

For this problem order does not matter since we cannot distinguish between the coins. Consequently, the set of possible outcomes can be described as

$$\{\{H,H\},\{H,T\},\{T,T\}\}.$$

Note that $\{H,H\}=\{H\}$, and similarly for T, so the set of outcomes can be equivalently expressed as

$$\{\{H\},\{H,T\},\{T\}\}.$$

This has cardinality 3.

(c) A penny and a dime are each flipped once. Describe the possible outcomes as the elements of a set. What is the cardinality of the set?

In this case order matters since we can distinguish between the coins. Since there are two distinguishable coins, this has the same answer as part (a).

PROBLEM 2.

(a) Fix real numbers a,b,c. Use set notation to describe the set of solutions $(x,y)\in\mathbb{R}^2$ to the equation $y=ax^2+bx+c$.

This set can be described as

$$\{(x,y)\in\mathbb{R}^2 | y = ax^2+bx+c\}.$$

(b) Let $S = \{a, b, c\}$. Use set notation to describe the set of functions $f : S \to \{0, 1\}$. What is the cardinality of this set?

Let F denote the set of functions $f : S \to \{0, 1\}$. There are multiple ways to describe F; here is one. Let $f \in F$. Then f is determined uniquely by specifying which elements of S are sent to 1. For example, if we say $\{a, c\}$ are the elements which are sent to 1, then the associated function f sends a and c to 1 and b to zero. That is, each element of F can be identified with a unique subset of S, and vice versa. So we can describe F as

$$\{T |\ T \subseteq S\}.$$

This has cardinality 8. (Note, $8 = 2^{|S|}$. Can you explain why?)

PROBLEM 3. A line segment in the plane can be described by its endpoints $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $\mathbb{R}^2$. The set of all line segments in the plane can be described by the set of all possible pairs of endpoints:

$$L = \{\{a, b\} \mid a, b \in \mathbb{R}^2,\ a \neq b\}.$$

Note: The elements of L *identify* line segments, but they are not line segments themselves. This is similar to the relationship between a person and his or her name: the name *identifies* the person, but strictly speaking the name is not the person.

(a) Why did we include the requirement $a \neq b$?

The condition $a \neq b$ means that we do not consider points in $\mathbb{R}^2$ to be line segments. (However, sometimes it is convenient to extend the notion of line segments to include points. Then points can be viewed as a degenerate type of line segment.)

(b) Consider the set of ordered pairs:

$$S = \{(a, b) \mid a, b \in \mathbb{R}^2, a \neq b\}.$$

Is this the same as L? Can you come up with a geometric interpretation for the elements of the set S? What kind of object is can be specified by a pair of points (a, b)?

The set S is not the same set as the set L because S consists of ordered pairs (a, b), while L consists of unordered pairs $\{a, b\}$.

The set S can be interpreted geometrically as the set of directed line segments in the plane (i.e., finite length arrows). (Note that S contains the set of *vectors* in $\mathbb{R}^2$, but it also contains much more. This is because vectors are typically required to be based at the origin, so they would have $a = 0$ (or $b = 0$, depending on your conventions).)

(c) Describe the set T of triangles in the plane $\mathbb{R}^2$ in an analogous way.

Any triangle is determined by its three vertices. As in part (a), a natural requirement is to not allow points or line segments to be triangles. Then the set of triangles in the plane can be described as

$$\big\{\{a, b, c\} \mid a, b, c \in \mathbb{R}^2, \text{ and } a, b, c \text{ are not colinear}\big\}.$$

Here is an alternate description. Note that a, b, c are not colinear exactly when the vectors $b - a$ and $c - a$ do not lie in the same line. Let $M(a, b, c)$ be the matrix with columns given by the vectors $b - a$ and $c - a$. Then the vectors $b - a$ and $c - a$ do not lie in the same line precisely when the determinant of $M(a, b, c)$ is not zero:

$$\det(M(a, b, c)) \neq 0.$$

Therefore, the set T can be described as

$$\big\{\{a, b, c\} \mid a, b, c \in \mathbb{R}^2, \text{ and } \det(M(a, b, c)) \neq 0\big\}.$$

PROBLEM 4. Here is an example of an interesting problem that we can formulate in terms of combinatorial graphs (see Homework 9/4 handout).

(R) Suppose that we are throwing a party. What is the smallest number of people which guarantees that either some group of m people all know each other, or some group of n people are all strangers to each other?

Given m and n, the answer to this problem is known as the **Ramsey number** $R(m,n)$. Ramsey's Theorem asserts that this is a finite number. For example, suppose there are at least 2 people. If these 2 know each other, then they make a group of 2 who know each other; otherwise, this same group of 2 are strangers. This shows $R(2,2) = 2$.

(a) Try to find $R(2,3)$: that is, how many people are needed to guarantee that some group of 3 all know each other, or some group of 2 are strangers? Also, find $R(3,2)$.

The Ramsey number $R(2,3)$ is equal to 3. To see this, we will first show $R(2,3) \geq 3$. This follows by considering the case where there are only two people and they do not know each other. So it is not the case that 2 of them know each other (we assumed this) and it is not the case that 3 of them don't know each other (there are only 2 people!). So the number 2 does not meet the requirements of $R(2,3)$ and so $R(2,3) > 2$.

To see that $R(2,3) = 3$, suppose there are 3 people. If any two of them know each other, then we are done. So we may assume none of them know each other. But then we have 3 people who do not know each other, so we are done in this case as well.

(b) Fix positive integers m and n. Formulate Problem (R) in term of graphs. Use the following questions to help: What do the vertices correspond to in the problem? Suppose there is an edge between two vertices. What does this correspond to? What does it mean if there is no edge between two vertices? What do the numbers m, n and $R(m,n)$ correspond to in terms of the graph?

Given m, n, Ramsey's number can be defined as the minimum number $R(m,n)$ such that any graph with $R(m,n)$ vertices is guaranteed to satisfy at least one of the following conditions:

i) there is a subset S of the vertex set such that every pair of vertices in S is connected by an edge, and $|S| = m$; or

ii) there is a subset T of the vertex set such that no pair of vertices in S is connected by an edge, and $|T| = n$.

(c) Can you show $R(m,n) = R(n,m)$?

We introduce the following terms. First, we say that a graph G is a **Ramsey graph** for (m,n) if it has $R(m,n)$ vertices and it satisfies the condition given in (a) for the Ramsey number $R(m,n)$. Second, we say that the **complement** of a graph G is the graph H with the same set of vertices, but with edge set defined by saying $\{a,b\}$ is an edge in H if and only if $\{a,b\}$ is not an edge in G.

With these preliminaries, note that if G is a Ramsey graph for (m,n), then its complement is a Ramsey graph of (n,m). This shows $R(m,n) = R(n,m)$.

(d) Try to show $R(3,3) > 5$: that is, 5 people are not enough to guarantee 3 mutual acquaintances or 3 mutual strangers. Hint: It is enough to find a single acquaintance graph which breaks the guarantee.

Consider the graph with vertex set

$$V = \{1,2,3,4,5\}$$

and edge set

$$E = \{\{1,2\},\{1,3\},\{3,4\}\}.$$

This shows $R(3,3) > 5$.

(e) Now show $R(3,3) \leq 6$, meaning 6 people is enough in the case of (d). Together, these imply $R(3,3) = 6$. (This was a Putnam problem in 1953.)

It suffices to show that any graph with 6 vertices either has a triangle, or its complement has a triangle. This can be shown by considering cases as in (a), and we leave it to the reader.

It is known that $R(4,4) = 18$, but in general, the numbers $R(m,n)$ are extremely difficult to compute. The prolific mathematician Paul Erdös characterized it thus:

Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number R(5, 5). We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number R(6, 6), however, we would have no choice but to launch a preemptive attack.