Math 299 **Midterm 1 Review** **Sep 9, 2013**

Exam topics

- Basic structures: sets, lists, functions
  - Sets { }: write all elements, or define by condition
  - Set operations: $A \cup B$, $A \cap B$, $A \setminus B$, A^c
  - Lists ( ): Cartesian product $A \times B$
  - Functions $f : A \to B$ defined by any input-output rule
  - Injective function: $\forall a_1, a_2 \in A$: $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$
  - Surjective function: $\forall b \in B$, $\exists a \in A$ with $f(a) = b$
  - Enumeration. Powerset: number of subsets $S \subset [n]$ is 2^n.
    Functions: number of $f : [k] \to [n]$ is n^k.
  - A, B have same cardinality: there is a bijection $f : A \to B$
  - A is countable: there is a bijection $f : \mathbb{N} \to A$
    Diagonal argument: $\mathbb{N} \times \mathbb{N}$ and $\mathbb{Q}$ are countable

- Formal logic
  - Statements: definitely true or false
  - Conditional (open) statement $P(x)$: true/false depends on variable x
  - Logical operations: *and*, *or*, *not*, *implies*
  - Truth tables and logical equivalence
  - Implication $P \Rightarrow Q$ equivalent to: contrapositive $not(Q) \Rightarrow not(P)$;
    independent from: converse $Q \Rightarrow P$, inverse $not(P) \Rightarrow not(Q)$
  - Negate implication: $not(P \Rightarrow Q)$ is equivalent to: P *and* $not(Q)$
  - Quantifiers close a conditional statement: $\forall x, P(x)$: for all x, $P(x)$ is true;
    $\exists x, P(x)$: there exists some x such that $P(x)$ is true.
  - Negate quantifiers: $not(\forall x, P(x))$ is equivalent to: $\exists x, not(P(x))$
  - Logical equivalences and set equations
  - Logic in mathematical notation versus in words

1. Consider the following statement, which may be true or false:

"A necessary condition for the equation $ax^2 + bx + c = 0$
to have a solution is: $a \neq 0$ and $b^2 - 4ac \geq 0$."

a. Rephrase this as an if-then implication, putting in unstated quantifiers.

b. Write the contrapositive of the statement.

c. Write the converse of the statement.

d. Write the inverse of the statement.

e. Write the negation of the statement (simplified by moving the *not* as far into the statement as possible).

f. Which of the above statements (a)-(e) are equivalent to each other?

g. In fact, the original statement is false. Disprove it (prove the negation).

2. Consider the equations below involving arbitrary sets $A, B, C \subset X$.

- Translate each side of the equation into a conditional statement combining "$x \in A$," "$x \in B$," "$x \in C$."
- Write a truth table for the combined statements on the two sides: if they are equivalent, the set equation is true; otherwise it is false.

a. $(A \cup B)^c \stackrel{?}{=} A^c \cup B^c$

b. $(A \cup B) \setminus (A \cap B) \stackrel{?}{=} (A \setminus B) \cup (B \setminus A)$

c. $A \cup (B \cap C) \stackrel{?}{=} (A \cap B) \cup (A \cap C)$

d. $A \times (B \cup C) \stackrel{?}{=} (A \times B) \cup (A \times C)$

3. In my paperback copy of *The Hobbit*, let W be the set of words used in the text, and let $P = \{1, \ldots, 245\}$ be the set of page numbers. Let $f : W \to P$ be the function which takes each word to the first page where it appears.

Explain why f probably does or does not have the following properties:

a. Is f is injective?

b. Is f surjective?

c. Is f bijective?

d. The index of a book lists keywords, and next to each, the page numbers where the keyword appears. Does this data naturally define a function?

Solutions

1a. Start by rephrasing:

"The conditions $a \neq 0$ and $b^2-4ac \geq 0$ are necessary
for $ax^2+bx+c=0$ to have a solution."

The statement "Q is necessary for P" is equivalent to "if P then Q":

"If $ax^2+bx+c=0$ has a solution, then $a \neq 0$ and $b^2-4ac \geq 0$."

The constants a, b, c are implicitly any real numbers, and "has a solution" means there is some $x \in \mathbb{R}$ which makes the equation true:

"For any $a, b, c \in \mathbb{R}$, if there is some $x \in \mathbb{R}$ with $ax^2+bx+c=0$,
then $a \neq 0$ and $b^2-4ac \geq 0$."

In symbols:

$$\forall a, b, c \in \mathbb{R},\ (\exists x \in \mathbb{R},\ ax^2+bx+c=0) \Rightarrow (a \neq 0 \text{ and } b^2-4ac \geq 0).$$

b. The above statement $P \Rightarrow Q$ is equivalent to its contrapositive:

"For any $a, b, c \in \mathbb{R}$, if not $(a \neq 0$ and $b^2-4ac \geq 0)$,
then there is no $x \in \mathbb{R}$ with $ax^2+bx+c=0$."

Simplifying further:

"For any $a, b, c \in \mathbb{R}$, if $a=0$ or $b^2-4ac<0$,
then $ax^2+bx+c \neq 0$ for all $x \in \mathbb{R}$."

In symbols:

$$\forall a, b, c \in \mathbb{R},\ (a=0 \text{ or } b^2-4ac<0) \Rightarrow (\forall x \in \mathbb{R},\ ax^2+bx+c \neq 0).$$

c. The original statement $P \Rightarrow Q$ (which is false) is logically independent of the converse $Q \Rightarrow P$ (which is true):

"The conditions $a \neq 0$ and $b^2-4ac \geq 0$ are sufficient
for $ax^2+bx+c=0$ to have a solution."

Or:

"For any $a, b, c \in \mathbb{R}$, if $a \neq 0$ and $b^2-4ac \geq 0$,
then there is some $x \in \mathbb{R}$ with $ax^2+bx+c=0$."

1e&g. The original statement $\forall a,b,c$: $P \Rightarrow Q$ from part (a) has the negation: $\exists a,b,c$: P *and not*(Q):

"There is some $a, b, c \in \mathbb{R}$ and $x \in \mathbb{R}$ with $ax^2+bx+c=0$,
and with $a=0$ or $b^2-4ac<0$."

To prove this existence statement, it is enough to find one example of a, b, c which satisfy the conditions. We can just take $a = b = c = 0$. Then there *is* some $x \in \mathbb{R}$ (for example $x = 0$) which satisfies $ax^2 + bx + c = 0$, we *do* have $a = 0$, and it is *irrelevant* whether $b^2 - 4ac < 0$.

2a. The left side is:

$$\begin{aligned}(A \cup B)^c &= \{x \in X \mid \textit{not}(x \in A \cup B)\} \\ &= \{x \in X \mid \textit{not}(x \in A \textit{ or } x \in B)\} \\ &= \{x \in X \mid x \notin A \textit{ and } x \notin B\}\end{aligned}$$

The right side is:

$$\begin{aligned}A^c \cup B^c &= \{x \in X \mid x \in A^c \textit{ or } x \in B^c\} \\ &= \{x \in X \mid x \notin A \textit{ or } x \notin B\}\end{aligned}$$

The conditions defining these two sides are clearly not equivalent. For example, if $x \in A$ is True and $x \in B$ is False, then the left condition ($x \notin A$ *and* $x \notin B$) is False, but the right condition ($x \notin A$ *or* $x \notin B$) is True.

Thus an element x can be in the right set but not in the left set, so the two sets are *not* equal. However, a closer examination reveals: $(A \cup B)^c \subseteq A^c \cup B^c$.

2d. The left side is:

$$\begin{aligned}A \times (B \cup C) &= \{(x, y) \mid x \in A \textit{ and } y \in B \cup C\} \\ &= \{(x, y) \mid x \in A \textit{ and } (y \in B \textit{ or } y \in C)\}\end{aligned}$$

The right side is:

$$\begin{aligned}(A \times B) \cup (A \times C) &= \{(x, y) \mid (x, y) \in A \times B \textit{ or } (x, y) \in A \times C\} \\ &= \{(x, y) \mid (x \in A \textit{ and } y \in B) \textit{ or } (x \in A \textit{ and } y \in C))\}\end{aligned}$$

Now, a truth table shows that for any statements P, Q, R, we have the equivalence:

$$P \textit{ and } (Q \textit{ or } R) \iff (P \textit{ and } Q) \textit{ or } (P \textit{ and } R).$$

Applying this to $P =$ "$x \in A$", $Q =$ "$x \in B$", $R =$ "$x \in C$", we find that the conditions defining the left and right sets are equivalent. That is, x is in the left-side set if and only if x is in the right side set, so the sets are equal.

3a&c. The function is not injective. A couterexample: let w_1, w_2 be the first two distinct words on page 1. Then $w_1 \neq w_2$, but $f(w_1) = f(w_2) = 1$. Since f is not injective, it cannot be bijective.

3b. For the function to be surjective, for any page $p \in P$, there must be a word $w \in W$ with $f(p) = w$. That is, each page must have a new word which did not appear before. This is unlikely for the last few pages, so the function is probably not surjective.

3d. It would be natural to define $g(w)$ to be the page numbers where w appears, but this is not a function because (unlike $f(w)$) there is not a single output for each input.