Math 299 **Supplement: Sets and Functions** **Sep 4, 2013**

All mathematical objects are formally defined in terms of sets and lists. Also, real-world objects usually have a natural mathematical model in these terms.

DEFINITION: A *set* is any collection of objects, either listed out or defined by some condition. Order of elements is irrelevant, and no repeated elements are allowed. We enclose a set with curly brackets { }.

EXAMPLES:

- $A = \{1, 3, 5\} = \{3, 5, 1\}$.
- $\mathbb{N} = \{\text{all natural numbers}\} = \{0, 1, 2, 3, \ldots\}$.
- $\mathbb{R} = \{\text{all real numbers}\}$.
- $B = \{x \in \mathbb{R} \text{ such that } x^2 = 2\} = \{\sqrt{2}, -\sqrt{2}\}$.
- $C = \{x \in \mathbb{N} \text{ such that } x^2 = 2\} = \{\}$, the empty set, since there is no natural-number solution to $x^2 = 2$.
- $D = \{x \in \mathbb{R} \text{ such that } x^2 > 1\} = \{x \in \mathbb{R} \text{ such that } x > 1 \text{ or } x < -1\}$.

DEFINITION: A *list* is an ordered enumeration of any objects, with repeated entries allowed. We enclose a list with round parentheses ( ).

EXAMPLES:

- $(1, 3, 5) \neq (1, 1, 3, 5) \neq (1, 5, 1, 3)$.
- An infinite sequence is a list. The even numbers $a_n = 2n$ form the sequence:
$$(a_n)_{n=1}^{\infty} = (a_1, a_2, a_3, \ldots) = (2, 4, 6, \ldots)$$

DEFINITION: The *Cartesian product* $A \times B$ is the set of all pairs (a, b) with a drawn from A and b drawn from B:

$$A \times B = \{(a, b) \text{ for all } a \in A,\ b \in B\}.$$

We use the times symbol $\times$ because of the counting formula: $|A \times B| = |A| \cdot |B|$.

Sample definitions.

Coordinate geometry. A point in the coordinate plane is a list of two real-number coordinates, $P = (x, y)$, such as the origin $(0, 0)$. The plane is just the set of all such points:

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \text{ for all } x, y \in \mathbb{R}\}.$$

The line L through the points $(1, 0)$ and $(0, 1)$ is the set of points which satisfy a certain linear equation:

$$L = \{(x, y) \in \mathbb{R}^2 \ \text{ s.t. } \ x + y = 1\}.$$

The points $(1, 0)$, $(\frac{1}{3}, \frac{2}{3})$, $(-\sqrt{2}, \sqrt{2}{-}1)$ are all elements of L.

Playing cards. Cards in a standard deck are distinguished by two pieces of data: a face value $\text{A}, 2, 3, \ldots, 10, \text{J}, \text{Q}$, or K; and a suit $\spadesuit, \clubsuit, \heartsuit$, or $\diamondsuit$. It is natural to model a card as a pair list: e.g. the queen of hearts corresponds to $c = (\text{Q}, \heartsuit)$. Then the deck is the cartesian product $D = F \times S$, where:

$$F = \{\text{A}, 2, 3, \ldots, 10, \text{J}, \text{Q}, K\} \qquad \text{and} \qquad S = \{\spadesuit, \clubsuit, \heartsuit, \diamondsuit\}.$$

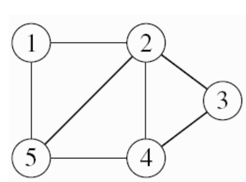
Note that D is naturally a set rather than a list, because it is the same deck regardless of how it is shuffled.-

A hand H is an unordered collection of cards from the deck: that is, a subset $H \subset D$. To study poker odds, we put all possible 5-card hands into a set:

$$P = \{H \subset D \ \text{ s.t. } \ |H| = 5\}.$$

Notice that each element $H \in P$ is itself a set, and each element $c \in H$ is itself a list (a pair). All kinds of real-world objects can be modeled by some such layer-cake of sets and lists.

Combinatorial graphs. These formal objects (not to be confused with the graph of a function) model real-world situations in which certain pairs of discrete objects are related or attached to each other. For example:



In this picture, the objects are vertices or nodes $1, 2, \ldots, 5$; and 1 & 2 are attached, 1 & 5 are attached, etc., but 1 & 3 are *not* attached. We formalize this as a set of vertices $V = \{1, 2, 3, 4, 5\}$, and a set of edges E whose elements are unordered pairs of attached vertices: $e = \{v, w\}$. That is:

$$E = \{\{1, 2\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{4, 5\}\}.$$

In general, we define a graph G as any pair of sets (V, E) in which the elements of E are two-element subsets of V: $e = \{v, w\}$ with $v, w \in V$.

Ordered pair definition of a function. A function $f : A \to B$ is a "rule" taking each input $a \in A$ to its output $b = f(a) \in B$. But if two rules give identical outputs for each input, then they define the same function. We can formalize this by saying that a function is simply its "table of values":

$$f \;=\; \{(a, b) \in A \times B \;\text{ s.t. }\; b = f(a)\} \;=\; \{(a, f(a)) \text{ for all } a \in A\}.$$

For example, the function $f(n) = n^2 + n$ on whole numbers $n \in \mathbb{N}$ is defined by the table:

n	0	1	2	3	$\cdots$
$f(n)$	0	2	5	12	$\cdots$

which can be thought of as the set of all pairs of inputs and outputs:

$$\begin{aligned} f &= \{(0,0), (1,2), (2,5), (3,12), \ldots\} \\ &= \{(n, n^2{+}n) \text{ for all } n \in \mathbb{N}\}. \end{aligned}$$

Note that the function $g(n) = n(n+1)$ is defined by a different formula, but gives the same values, the same set of pairs, and hence the same function: $f = g$.

What is formal mathematics? The above set-and-list definitions seem to forget the "essence" of the objects they model. For example, a card is a marked piece of paper, not a pair like $(\mathrm{Q}, \heartsuit)$. Yet this data identifies a particular card, so the formal definition just what we need to model and solve any problem about card-hands. Similarly, the plane coordinates (x, y) don't describe *what* a geometric point is, but they determine *where* it is, and this is all that is relevant to any geometric problem.

For a function on the real numbers, the set-of-pairs definition gives the same formal object as a set of points in the plane: for example, the function $\ell(x) = -x + 1$ gives:

$$\ell = \{(x, y) \in \mathbb{R}^2 \;\text{ s.t. }\; y = -x + 1\},$$

which is just the line L in a previous example. Why this coincidence? Because the line is the *graph* of the function! The function and the graph are defined by the *same data*, so formally, the function *is* its graph.

In each case, the formal definition in terms of sets and lists does not describe the "true meaning" of the object, but contains all its identifying data. For formal mathematics, we reason with this set-theory data, and leave the intuitive meaning for informal discussion.