Counting Sets and Functions

We will learn the basic principles of combinatorial enumeration: counting all possible objects of a specified kind.

Question 1. The first objects to count are functions whose domain is an interval of integers, $f : \{1, 2, \ldots, n\} \to C$, where C is a given finite set. We will use the notation $[n] = \{1, \ldots, n\}$, so we are dealing with $f : [n] \to C$. These can be formally modeled more neatly than general functions: we can present the data of f simply by listing its values:

$$f = (f(1), f(2), \ldots, f(n)) \in \underbrace{C \times \cdots \times C}_{n \text{ factors}} = C^n.$$

Conversely, any list $(c_1, \ldots, c_n) \in C^n$ represents a function with $f(i) = c_i$ for $i = 1, \ldots, n$. Hence, the number of functions is equal to the number of lists in C^n, namely:

PROPOSITION 1: The number of all possible functions $f : [n] \to C$ is $|C|^n$.

For example, the number of functions $f : [3] \to \{0, 1\}$ is $2^3 = 8$, namely:

$$(0,0,0),\ (1,0,0),\ (0,1,0),\ (0,0,1),\ (1,1,0),\ (1,0,1),\ (0,1,1),\ (1,1,1).$$

A list like $(1, 0, 1)$ represents the function with $f(1) = 1$, $f(2) = 0$, $f(3) = 1$.

Question 2. Next, we wish to count all subsets $S \subseteq [n]$. For example, there are 8 subsets $S \subseteq [3]$:

$$S = \{\},\ \{1\},\ \{2\},\ \{3\},\ \{1,2\},\ \{1,3\},\ \{2,3\},\ \{1,2,3\}.$$

Surprisingly, we can reduce this question to the previous one through the Bijection Principle of combinatorics: if we can transform one data structure into another by an invertible mapping (a bijection), then the two types of data have the same number of possibilities. Formally: suppose we have sets $\mathcal{A}, \mathcal{B}$ and mappings (functions) $\phi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{A}$ which are inverses, meaning that they undo each other:

$$\psi(\phi(a)) = a \ \text{ for all } a \in \mathcal{A}, \ \text{ and } \ \phi(\psi(b)) = b \ \text{ for all } b \in \mathcal{B}.$$

Then ϕ and ψ are bijections, and $|\mathcal{A}| = |\mathcal{B}|$.

In our case, we can define the *Indicator Transform*, an invertible mapping ϕ which changes subsets $S \subseteq [n]$ into functions $f : [n] \to \{0, 1\}$. That is, if we let $\mathcal{A}$ be the set of all such subsets S, and $\mathcal{B}$ the set of all such functions f, we define a mapping $\phi : \mathcal{A} \to \mathcal{B}$ by:

$$\phi(S) = f, \quad \text{where} \quad f(i) = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

We call f the *indicator function* of S. The inverse mapping $\psi : \mathcal{B} \to \mathcal{A}$ takes each function to a corresponding subset:

$$\psi(f) = S = \{i \in [n] \mid f(i) = 1\}.$$

In the example above, each set $S \subseteq [3]$ has its characteristic function $f : [3] \to \{0,1\}$ listed in the corresponding place above it in the previous example.

PROPOSITION 2: The number of all possible subsets $S \subseteq [n]$ is 2^n.

Proof: If we show that ϕ is a bijection, this will imply $|\mathcal{A}| = |\mathcal{B}|$, and we already know that $|\mathcal{B}|$, the number of functions $f : [n] \to \{0,1\}$, is 2^n by Prop. 1.

We check that ϕ, ψ are inverses, using their definitions above:

$$\psi(\phi(S)) = \psi(f) = \{i \mid f(i) = 1\} = \{i \mid i \in S\} = S.$$

Also, $\phi(\psi(f)) = \phi(\{i \mid f(i) = 1\}) = f'$, where $f'(j) = 1$ whenever $j \in \{i \mid f(i) = 1\}$. That is, $f'(j) = 1$ whenever $f(j) = 1$, and otherwise $f'(j) = f(j) = 0$, so $f' = f$ and $\phi(\psi(f)) = f$. This shows that ϕ is invertible, and hence a bijection. Q.E.D.

Question 3. Now we count *injective* functions.

PROPOSITION 3: (i) The number of possible injective functions $f : [n] \to C$ is:

$$|C|\,(|C|-1)\,(|C|-2)\cdots(|C|-n+1).$$

(ii) The number of possible bijective functions $f : [n] \to [n]$ is: $n! = n(n-1)\cdots(2)(1)$.
(iii) The number of possible injective functions $f : [k] \to [n]$ is: $n(n-1)\cdots(n-k+1)$.

Proof. (i) An injective function corresponds to $(f(1), \ldots, f(n))$ with all the entries different from each other. We can choose $f(1)$ to be any element of C, giving $|C|$ possible choices; then for $f(2)$, we can choose any element of C except $f(1)$, giving $|C| - 1$ possibilites; and similarly $|C| - 2$ possibilities for $f(3)$, etc. The number of possible combined choices for f is the product of the individual possibilities, which gives the desired formula.

(ii) From part (i), we see that the number of injective functions $f : [n] \to [n]$ is $n(n-1)\cdots(n-n+1) = n!$. But every injective function is bijective: the image of f has the same size as its domain, namely n, so the image fills the codomain $[n]$, and f is surjective and thus bijective.

(iii) In part (i), replace the domain by $[k]$ and the codomain by $[n]$. Q.E.D.

Question 4. Our last problem is to count the number of subsets $S \subseteq [n]$ with a fixed size $|S| = k$. For example, the number of 3-element subsets $S \subseteq [5]$ is 10:

$$\{1,2,3\},\ \{1,2,4\},\ \{1,2,5\},\ \{1,3,4\},\ \{1,3,5\},\ \{1,4,5\},\ \{2,3,4\},\ \{2,3,5\},\ \{2,4,5\},\ \{3,4,5\}.$$

We define the symbol $\binom{n}{k}$, pronounced "n choose k", to be the answer to the counting problem, so by definition $\binom{5}{3} = 10$. We call these the *choose numbers* or *binomial coefficients.*

PROPOSITION 4: For any integers $0 \leq k \leq n$, we have:

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

We will not give a formal proof, but rather examine the above example to see why the formula works. Consider the following table, which contains all the injective functions $f : [3] \to [5]$, each listed in the column corresponding to its image set $S = \{f(1), f(2), f(3)\}$.

$\{1,2,3\}$	$\{1,2,4\}$	$\cdots$	$\{2,4,5\}$	$\{3,4,5\}$
$(1,2,3)$	$(1,2,4)$	$\cdots$	$(2,4,5)$	$(3,4,5)$
$(1,3,2)$	$(1,4,2)$	$\cdots$	$(2,5,4)$	$(3,5,4)$
$(2,1,3)$	$(2,1,4)$	$\cdots$	$(4,2,5)$	$(4,3,5)$
$(2,3,1)$	$(2,4,1)$	$\cdots$	$(4,5,2)$	$(4,5,3)$
$(3,1,2)$	$(4,1,2)$	$\cdots$	$(5,2,4)$	$(5,3,4)$
$(3,2,1)$	$(4,2,1)$	$\cdots$	$(5,4,2)$	$(5,4,3)$

The columns correspond to subsets, which by definition are counted by the unknown value $\binom{5}{3}$. The rows correspond to bijections $g : [3] \to [3]$, and there are 3! of these by Prop. 3(ii). The total number of injections in the table is (5)(4)(3) by Prop. 3(iii). Now, the number of columns times the number of rows equals the total number of entries in the table, so we have:

$$\binom{5}{3} \cdot 3! = (5)(4)(3)\,,$$

which immediately gives the desired formula $\binom{5}{3} = \frac{(5)(4)(3)}{3!}$.

In a general proof, we would define an invertible mapping $\phi : \mathcal{S} \times \mathcal{B} \to \mathcal{I}$, where $\mathcal{S}$ is the set of all k-element subsets $S \subseteq [n]$; $\mathcal{B}$ is the set of all bijections $g : [k] \to [k]$; and $\mathcal{I}$ is the set of all injections $f : [k] \to [n]$. This would guarantee $|\mathcal{S}| \cdot |\mathcal{B}| = |\mathcal{I}|$, that is: $\binom{n}{k} \cdot k! = n(n-1) \cdots (n-k+1)$, giving the desired formula. If you want a challenge, try to define this mapping ϕ and its inverse ψ.