Math 299 Supplement: Real Number Axioms Nov 18, 2013

Algebra Axioms. In Real Analysis, we work within the axiomatic system of real numbers: the set $\mathbb{R}$ along with the addition and multiplication operations $+, \cdot$, and the inequality relation $<$. We do not need to list or describe the elements of $\mathbb{R}$ directly; rather, anything we want to know about $\mathbb{R}$ will follow from Axioms 1–10.

We start with the axioms of the addition and multiplication operations, which include the commutative group axioms. For any $a, b, c \in \mathbb{R}$, we have:

1.	Closure: $a + b \in \mathbb{R}$	$1'$.	Closure: $ab \in \mathbb{R}$
2.	Associativity: $(a+b)+c = a+(b+c)$	$2'$.	Associativity: $(ab)c = a(bc)$
3.	Identity element: $\exists 0,\ a + 0 = a$	$3'$.	Identity element: $\exists 1 \neq 0,\ a1 = a$
4.	Inverse: $\forall a, \exists\, b,\ a + b = 0$. Denote $b = -a$	$4'$.	Inverse: $\forall a \neq 0, \exists\, b,\ ab = 1$ Denote $b = a^{-1}$
5.	Commutativity: $a + b = b + a$	$5'$.	Commutativity: $ab = ba$

6. Distributivity: $a(b + c) = ab + ac$

We define new operations in terms of the basic ones: subtraction $a - b$ means $a + (-b)$; and division a/b or $\frac{a}{b}$ means $a \cdot b^{-1}$.

We also have the axioms of inequality. For any $a, b, c, d \in \mathbb{R}$, we have:

7. Trichotomy: Exactly one of the following is true: $a < b,\ a = b,\ a > b$.
8. Compatibility of $<$ with $+$: If $a < b$ and $c < d$, then $a + c < b + d$.
9. Compatibility of $<$ with $\cdot$: If $a < b$ and $0 < c$, then $ac < bc$.

We define $a > b$ to mean $b < a$, and $a \leq b$ to mean $a < b$ or $a = b$.

Completeness. The final axiom gives a precise meaning to the idea that the real numbers have no holes or gaps, but rather form a continuum.

First, some definitions. We say a number $b \in \mathbb{R}$ is an *upper bound* for a set $S \subset \mathbb{R}$ whenever $x \leq b$ for all $x \in S$. Furthermore, ℓ is the *least upper bound* (or *supremum*) of S means $\ell \leq b$ for every upper bound b of S. We denote this as $\ell = \mathrm{lub}(S)$ or $\sup(S)$. Intuitively, the least upper bound is the “rightmost edge” of S on the real number line.

EXAMPLES: (i) Let $S = \mathbb{N} \subset \mathbb{R}$. Then S is an unbounded set having *no* upper bounds, and hence no least upper bound.

(ii) Let $S = \{0.9, 0.99, 0.999, \ldots\}$. Then every number $b \geq 1$ is an upper bound of S, and $\ell = \sup(S) = 1$ is the least upper bound. It makes no difference whether 1 is in the set or not: $S \cup \{1\}$ has the same upper bounds as S, and $\sup(S \cup \{1\}) = 1$.

(iii) Let $S = \{x \in \mathbb{R} \mid x^2 < 2\}$. Some upper bounds for S are upper approximations to $\sqrt{2}$, like $a = 1.5, 1.42, 1.415, \ldots$. The least upper bound is $\sup(S) = \sqrt{2}$ itself, which is a way of producing this irrational number without assuming it exists.

10. Axiom of Completeness: Any set $S \subset \mathbb{R}$ which has an upper bound, also has a least upper bound in the reals: $\sup(S) \in \mathbb{R}$.

Note that this axiom fails for the rational numbers $\mathbb{Q}$, and this is their main difference from the real numbers. For example, the set in Example (iii) above has upper bounds in $\mathbb{R}$ and in $\mathbb{Q}$, but it has a least upper bound *only* in $\mathbb{R}$: in the rationals $\mathbb{Q}$, there is a “hole” where $\sup(S) = \sqrt{2}$ would be, since $\sqrt{2}$ is *not* rational.

Algebra Propositions. All the usual facts of algebra (including inequalities) can be deduced from Axioms 1–9. Throughout, we implicitly use Axioms $2, 2'$ to write expressions like $a + b + c$ instead of $(a + b) + c$, and abc instead of $(ab)c$.

PROPOSITION 1 (Mulitplication by zero): $0a = 0$.
Proof. By Axioms 3 and 6, we have: $0a = (0 + 0)a = 0a + 0a$. Adding $-0a$ to the left and right sides of this equality, we get: $0a - 0a = 0a + 0a - 0a$, which we can simplify by Axioms 4 and 3 to $0 = 0a$ as desired.

PROPOSITION 2 (Multiplication with signs):

$$\text{(i) } -(-a) = a, \quad \text{(ii) } (-a)b = -(ab), \quad \text{(iii) } (-a)(-b) = ab.$$

Proof. (i) By Axiom 4, $0 = (-a) - (-a)$. Adding a to both sides gives: $a = a + (-a) - (-a) = 0 - (-a) = -(-a)$.
(ii) We have: $(-a)b + ab = (-a + a)b = 0b = 0$ by Prop. 1. Switching the sides: $0 = (-a)b + ab$, and adding $-(ab)$ to both sides: $-(ab) = (-a)b + ab - (ab) = (-a)b + 0 = (-a)b$.
(iii) Applying (ii) twice, we have: $(-a)(-b) = -(a(-b)) = -((-b)a) = -(-(ba)) = -(-(ab))$. Thus $(-a)(-b) = -(-(ab)) = ab$ by (i).

PROPOSITION 3. If $a < b$, then $-b < -a$.
Proof. Let $a < b$. Using Axiom 8, we add $-a-b$ to both sides, getting: $a+(-a-b) < b+(-a-b)$. Simplifying the left and right sides by Axioms 2, 3, 4, 5 gives $-b < -a$.

PROPOSITION 4. $0 < 1$.
Proof. Surprisingly, this is not immediate. Suppose for a contradiction that $0 \not< 1$. By Axiom 7, this means $0 \geq 1$, but Axiom $3'$ says $0 \neq 1$. Thus $0 > 1$, and by Prop. 3, $0 < -1$, so by Axiom 9, $0(-1) < (-1)(-1)$. By Prop. 1, $0(-1) = 0$, and by Prop. 2, $(-1)(-1) = (1)(1) = 1$, which means $0 < 1$. But we already saw $0 > 1$, so this contradicts the uniqueness part of Axiom 7. This contradiction shows $0 < 1$.

PROPOSITION 5 (Transitivity of $<$): If $a < b$ and $b < c$, then $a < c$.
Proof. Suppose $a < b < c$. By Axiom 8, we can subtract a from the first inequality to get $0 < b - a$, and subtract b from the second inequality to get $0 < c - b$. Adding these two inequalities: $0 < (b - a) + (c - b) = c - a$. Adding a to ths inequality gives $a < (c - a) + a = c$.

DEFINITION: $|x| = x$ if $x \geq 0$, and $|x| = -x$ if $x < 0$.

PROPOSITION 6: For $a, b \in \mathbb{R}$: (i) $|ab| = |a|\,|b|$. (ii) $|a + b| \leq |a| + |b|$;
(iii) (Triangle inequality) For $x, y, z \in \mathbb{R}$, $|x - z| \leq |x - y| + |y - z|$.

Proof. (i) If $a, b \geq 0$, then $ab > 0$ by Axiom 9, and by definition $|ab| = ab = |a|\,|b|$. If $b < 0 \leq a$, then $a, -b \geq 0$ and $-(ab) = a(-b) > 0$, so $ab < 0$; thus $|ab| = -ab = a(-b) = |a|\,|b|$ by Prop. 2. Similarly for $a < 0 \leq b$. If $a, b < 0$, then $ab = (-a)(-b) > 0$ and $|ab| = (-a)(-b) = |a|\,|b|$. Axiom 7 guarantees that we have considered all possible cases.
(ii) We have $|x| = \max\{x, -x\}$, so $|a + b| = \max\{a + b, -a - b\}$, whereas we easily see: $|a| + |b| = \max\{a + b, a - b, -a + b, -a - b\}$. The larger set clearly has a larger maximum, so $|a + b| \leq |a| + |b|$.
(iii) This follows from (ii) taking $a = x - y$, $b = y - z$, so that $a + b = x - z$.

Limits. Consider an infinite sequence $(a_n)_{n=1}^{\infty} = (a_1, a_2, a_3, \ldots)$ with $a_i \in \mathbb{R}$.

DEFINITION: We say (a_n) converges to L, written $\lim_{n\to\infty} a_n = L$, meaning that for any error bound $\epsilon > 0$, there exists a threshold $N \in \mathbb{N}$ (depending on ϵ) such that $n \geq N$ forces a_n into the error interval $L - \epsilon < a_n < L + \epsilon$. In symbols:

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N},\ n > N \Rightarrow |a_n - L| \leq \epsilon.$$

THEOREM: Suppose the sequence (a_n) is increasing, with an upper bound b: that is, $a_1 \leq a_2 \leq a_3 \leq \cdots \leq b$. Then (a_n) converges to $L = \sup\{a_n \mid n \geq 1\}$.

Proof. The least upper bound $L = \sup\{a_n\}$ exists by Axiom 10. Then L is an upper bound, so $a_n \leq L < L + \epsilon$ for all n and all $\epsilon > 0$.

Since L is the *least* upper bound, we know that $L - \epsilon$ is not an upper bound of (a_n) for any $\epsilon > 0$. This can only be if $L - \epsilon < a_N$ for some N, and since $a_N \leq a_n$ for all $n \geq N$, we have:

$$L - \epsilon < a_n < L + \epsilon,$$

namely $|L - a_n| < \epsilon$. In summary, for any $\epsilon > 0$, there is some N such that $n \geq N$ implies $|L - a_n| < \epsilon$. This is precisely the definition of $\lim_{n\to\infty} a_n = L$.

PROP: If $\lim\limits_{n\to\infty} a_n = L$, $\lim\limits_{n\to\infty} b_n = M$, then (i) $\lim\limits_{n\to\infty} a_n + b_n = L + M$; (ii) $\lim\limits_{n\to\infty} a_n b_n = LM$.

Proof. By the definition of the limits in the hypothesis, for any $\epsilon_1 > 0$, there is N_1 such that $n \geq N_1 \;\Rightarrow\; |a_n - L| < \epsilon_1$; and for any $\epsilon_2 > 0$, there is N_2 such that $n \geq N_2 \;\Rightarrow\; |b_n - M| < \epsilon_2$.

(i) Now let $\epsilon > 0$ and take $\epsilon_1 = \epsilon_2 = \frac{1}{2}\epsilon$. Then taking $N = \max(N_1, N_2)$ above, for any $n \geq N$ we have:

$$\begin{array}{rcll} |(a_n + b_n) - (L + M)| & = & |(a_n - L) + (b_n - M)| & \\ & \leq & |a_n - L| + |b_n - M| & \text{by Triangle Inequality} \\ & < & \epsilon_1 + \epsilon_2 = \epsilon & \text{since } n \geq N \geq N_1, N_2. \end{array}$$

(ii)

Problems

Prove the following statements using the above Axioms and Propositions.

1. If $0 < a < b$, then $b^{-1} < a^{-1}$.

2a. There is no largest element of $\mathbb{R}$. Hint: Contradiction.

b. There is no smallest element of the positive reals $\mathbb{R}_{>0}$.

c. If $|x| < \epsilon$ for all $\epsilon > 0$, then $x = 0$. *Hint:* Contradiction.

3. The limit $\lim_{n\to\infty} a_n$ converges to at most one value. That is, if (a_n) converges to L and also to L', then $L = L'$. *Hint:* This is not just a matter of writing $L = \lim_{n\to\infty} a_n = L'$, since the whole point is to prove the limit in the middle is a well-defined, unambiguous value. Rather, write out the definition of $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = L'$ and use Prop. 5(iii) above to prove that $|L - L'| < \epsilon$ for every $\epsilon > 0$. Why does this give the conclusion?

4a. Using the formal definiton, prove:

$$\lim_{n\to\infty} \frac{\left(1 + \frac{1}{n}\right)^2 - 1}{\frac{1}{n}} = 2.$$

Hint: For the rough draft, work backward from the conclusion $|a_n - L| < \epsilon$.

b. What does the above limit mean in calculus? *Hint:* It concerns the behavior of the function $f(x) = x^2$ near $x = 1$.

5. Using the formal definition, prove: The sequence $a_n = n^2$ diverges; that is, (a_n) does not converge to any value $L \in \mathbb{R}$, meaning $\lim_{n\to\infty} a_n = L$ is *false*.