Math 299 **Recitation 5** **Sept. 26, 2013**

PROBLEM 1. Say whether True or False for each of the following and if the following statement is not true, fix the statement to be true.

1. _____ : "Earning a final grade of C from MTH 299 class is a necessary and sufficient condition for passing MTH 299".
2. _____ : "Competing all the requirements of a given degree program is a necessary and sufficient condition for earning the degree".
3. _____ : "x is a square" is a sufficient condition for "x is a rectangle".
4. _____ : "x is an equilateral rectangle" is a necessary condition for "x is a square".
5. _____ : "Being divisible by 4 is necessary for being an even number".
6. _____ : "A function f is continuous at a point a" is necessary for "The limit of the funciton f exists as x approaches to a".
7. _____ : " x is an element in a set $A \cap B^c$ " is sufficient for " x is an element in a set $A \setminus B$ ".
8. _____ : "A function f is invertible" is necessary for "A function f is injective".

Answer: T, T, T, T, F (sufficient), F (sufficient), T (but it is necessary and sufficient), F (sufficient).

PROBLEM 2. Fill in the blank with *necessary, sufficient*, or *necessary and sufficient.*

"For integers a, b, a and b are perfect squares " is ____________________ for "ab is a perfect square ".
(Note: Perfect square is an integer that is the square of an integer.)

(1) Restate the above statement as "*if* $\cdots$, *then* $\cdots$".

"a and b are perfect squares" is sufficient for "ab is a perfect square".

If a and b are perfect squares, then ab is a perfect square

(2) Write the *Hypothesis* and the *Conclusion*.

Hypothesis: a and b are perfect squares.

Conclusion: ab is a perfect square.

(3) Prove the statement with mathematical rigor. If the answer is not necessary or not sufficient, find an counterexample.

Note that if $x = n^2$ for some integer n, the integer x is a perfect square. Assume that a and b are perfect squares and are integers. By the definition of a perfect square, there are integers k and l such that $a = k^2$ and $b = l^2$. Then

$$ab = k^2 l^2 = (kl)^2.$$

Since kl is an integer, by the defintion of a perfect square, ab is a perfect square.

Note that the necessary condition is one which must hold for a conclusion to be true. When we consider the case $a = 5$ and $b = 5$, even if a and b are not perfect squares, ab is a perfect square. Therefore, "a and b are perfect squares" is not a necessary condition for "ab is a perfect square."

PROBLEM 3. (Bonus Problem) Fill in the blank with *necessary, sufficient*, or *necessary and sufficient.*

"$x \in (A \cap B)$ " is ______________________ for "$x \in (A^c \cup B)$".

(1) Restate the above statement as "*if* $\cdots$, *then* $\cdots$".

(2) Write the *Hypothesis* and *Conclusion.*

(3) Prove the statement with mathematical rigor. If the answer is not necessary or not sufficient, find a counterexample.

PROBLEM 4. Use quantifiers to express the following statements.

A. Every student in MTH299 knows what negation of statement is.

: Let A be a set of students who register MTH 299. $\forall x \in A$ (x knows what negation of statement is.)

B. Every student at MSU experiences moments of joy.

: Let B be a set of students at MSU. $\forall a \in B$ (a experiences moments of joy).

C. I know someone who seems never to have experienced any moments of shyness in their whole life.

: Let C be a set of people in the world and time be denoted by t. $\exists p \in C\ \forall t$ (p did not experience shyness at time t).

D. Every student at MSU is at least 18 years old.

: Let D be a set of students at MSU. $\forall y \in D$ (y is at least 18 years old).

E. If you disagree with a statement D, write negation of the statement D and again express it with quantifiers.

: "I know someone at MSU who is under 18 years old".

$\exists y \in D$ (y is under 18 years old).

F. For any natural number n, there is a prime number m larger than n.

: $\forall n \in \mathbb{N}$, $\exists m$ (m is a prime numebr and $m > n$).

G. There is an integer M larger than the number of contacts saved in any MSU student's cell phone.

: Let X be a set of students at MSU. $\exists M \in \mathbb{Z}\ \forall x \in X$ ($M > n(x)$ where $n(x)$ is the number of contacts saved in x's cell phone).

H. There are at least two students at MSU whose lastnames are "Sage".

: Let Y be a set of studetns at MSU. $\exists x, y \in Y$ (x and y have "Sage" as the lastname and $x \neq y$).

PROBLEM 5. (A substitute problem in case that lecture doesn't cover up to negation of quantifiers) Quantifiers in sentences are one of the linguistic constructs that are hard for computers to handle in general. Here is a nice pair of example dialogues :

1. A: "How was the birthday party after I left?"

   B: "It was really fun. Everybody had a drink."

2. A: "How was the birthday party after I left?"

   B: "It was really fun. Everybody watched a movie."

Did everybody have their own drink, or did they share the same drink? Did they watch a same movie or a different movie (with their smart phone)? Write the precise situation in the sentence B with quantifiers. The order of quantifiers are important to interpretate the situation correctly.

Answer: First, everyone had their own drink at the party. But they watched the same movie. Let P be a set of people who were still at the party after I left, that is,

$$P = \{p : p \text{ is a person who was at the party after I left the party.}\}$$

Then, first conversation is expressed as follow: " $\forall p \in P$, $\exists$ a drink that p had." . The second conversation is "$\exists$ a movie M such that $\forall p \in P$, p watched the movie M".

PROBLEM 5. An expression involving quantifieres is in *positive form* if none of the quantifiers is negated. Thus $\neg\forall x,\ P(x)$ is not in positive form, but the equivalent expression $\exists x,\ \neg P(x)$ is in positive form. Negate each of the following statements and express it in positive form.

1. $\forall x \in \mathbb{N}\ \exists y \in \mathbb{N},\ x + y = 1.$

2. $\forall x > 0\ \exists y < 0,\ x + y = 0.$

3. $\exists x \in \mathbb{R}\ \forall \epsilon > 0,\ -\epsilon < x < \epsilon.$

4. $\forall x, y \in \mathbb{N}\ \exists z \in \mathbb{N},\ x + y = z^2.$

PROBLEM 6. Go Spartan! There is a tailgate party at the parking lot of a stadium. A group of seven students, Alex, Bob, Cathy, David, Ellena, Fernando, and George, are going to the tailgate together. They will drink soda subject to the following:

If Fernando drinks coke, then Alex doesn't drink it.

Ellena doesn't drink coke only if George drinks it.

If Cathy drinks coke, then so does Fernando.

"David doesn't drink coke" is sufficient for "Bob drinks it".

"Cathy drinks coke" is necessary for "David does drink coke".

Which one of the following could be a complete and accurate list of students who drink coke? Remember that the conditional statement (original statement) is logically equivalent to its contrapositive. In addition, use the fact that if A implies B and B implies C, then A implies C.

(1) Alex and Ellena

(2) Cathy and David

(3) Bob and Fernando

(4) Ellena, David, and George

(5) Bob, Ellena, Fernando, and George.

Answer: (5).