Math 299 **Recitation 8** **Oct 24, 2013**

1. PROPOSITION: The sum of the first n odd numbers is n^2.

a. Try some examples: $1+3 \stackrel{?}{=} 2^2$, $1+3+5 \stackrel{?}{=} 3^2$, etc.

b. State the Proposition formally in symbols, making clear what is the n^{th} case $P(n)$. How to write a formula for the n^{th} odd number? Include implicit quantifiers: for which n should the Proposition hold?

SOLUTION: For all $n \geq 1$, we have $P(n)$: $1+3+5+\cdots+(2n-1) = n^2$.

c. Prove the Proposition by mathematical induction.

- *Base Case (Anchor):* Prove $P(1)$ directly. SOLUTION: $P(1) : 1 = 1^2$ is obvious.
- *Induction Step (Chain Step):* Start by writing the inductive hypothesis at the top, then write the inductive conclusion at the bottom, then work on the body of the proof.
  - *Inductive Hypothesis:* For some specific n, we assume the formula is true for the case $P(n)$.

    SOLUTION: We assume $P(n)$: $1+3+5+\cdots+(2n-1) = n^2$.
  - *Body of Proof:* Use the inductive hypothesis and algebra to deduce the inductive conclusion.

    SOLUTION: Note that the $(n+1)^{\text{st}}$ odd number is $2(n+1)-1 = 2n+1$. We have:

$$\begin{aligned} 1+3+5+\cdots+(2n+1) &= [1+3+5+\cdots+(2n-1)] + (2n+1) \\ &= n^2 + (2n+1) \quad \text{by the inductive hypothesis} \\ &= n^2+2n+1 = (n+1)^2 \end{aligned}$$

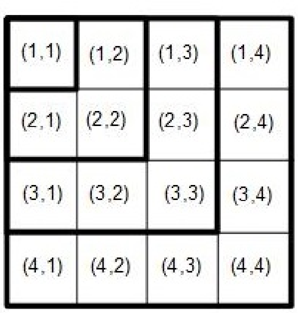
  - *Inductive conclusion:* The formula is true for the next case $P(n+1)$.

    SOLUTION: Therefore we know $P(n+1)$: $1+3+5+\cdots+(2(n+1)-1) = (n+1)^2$.
- Explain why the Anchor and Chain Steps imply the final conclusion: $P(n)$ is true for all $n \geq 1$.

  SOLUTION: We know $P(0)$ is true, and $P(0) \Rightarrow P(1)$, so we know $P(1)$ is true. Also $P(1) \Rightarrow P(2)$, so $P(2)$ is true. We can continue similarly up to any desired $P(n)$.

d. There is beautiful pictorial demonstration of the Proposition: an $n \times n$ square can be dissected into L-shaped pieces of area $1, 3, 5, \ldots, 2n-1$. Draw this on the back. Could it be made into a formal proof?

SOLUTION:



We can make this into a bijective proof by dissecting the set

$$[n] \times [n] = \{(i,j) \in \mathbb{N}\times\mathbb{N} \mid 1 \leq i,j \leq n\}$$

into disjoint subsets $S_1, S_2, \ldots, S_n$, where:

$$S_k = \{(k,1),(k,2),\ldots,(k,k)\} \ \cup \ \{(1,k),(2,k),\ldots,(k,k)\},$$

with $|S_k| = 2k-1$. Then: $[n]\times[n] = S_1 \cup \cdots \cup S_n$, so:

$$n^2 = |[n]\times[n]| = |S_1| + \cdots + |S_n| = \sum_{k=1}^{n}(2k-1).$$

2. The *Fibonacci Numbers* are a sequence of whole numbers $F_1, F_2, F_3, \ldots$ starting with $F_1 = F_2 = 1$, and conbtinuing by the recursive rule: $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$:

$$F_1 = 1, \quad F_2 = 1, \quad F_3 = 1+1 = 2, \quad F_4 = 1+2 = 3, \quad F_5 = 2+3 = 5, \quad F_6 = 3+5 = 8, \ldots$$

a. Make a table of the Fibonacci numbers:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
F_n	1	1	2	3	5	813	21	34	55	89	144	233	377	610

What is the pattern of odd versus even F_n's?

b. PROPOSITION: F_n is even when $n = 3m$ is a multiple of 3. Try proof by induction: why is it hard?

SOLUTION: This is hard because $F_{3(m+1)} = F_{3m+3}$ does not immediately precede F_{3m}, so we cannot use the inductive hypothesis right away. We must argue as follows:

Anchor: For $m = 1$, clearly $F_3 = 2$ is even. Chain: Assume $F_{3m} = 2k$, an even number. Then:

$$F_{3m+3} = F_{3m+2} + F_{3m+1} = (F_{3m+1} + F_{3m}) + F_{3m+1} = 2F_{3m+1} + F_{3m} = 2F_{3m+1} + 2k,$$

which shows F_{3m+3} is even.

c. Reformulate: For any $m \geq 1$, F_{3m-1} and F_{3m-2} are odd, and F_{3m} is even. Prove this by induction on the variable m. Does this work better?

SOLUTION: Anchor: For $m = 1$ we have *three* base numbers: $F_{3-2} = F_1 = 1$ and $F_{3-1} = F_2 = 1$ are odd, and $F_3 = 2$ is even.

Chain: Assume F_{3m-2}, F_{3m-1} are odd, F_{3m} is even. We must examine *three* succeeding numbers: $F_{3(m+1)-2} = F_{3m+1}$, $F_{3(m+1)-1} = F_{3m+2}$, and $F_{3(m+1)} = F_{3m+3}$.

$$\begin{aligned} F_{3m+1} &= F_{3m} + F_{3m-1} &&= \text{even} + \text{odd} &&= \text{odd} \\ F_{3m+2} &= F_{3m+1} + F_{3m} &&= \text{odd} + \text{even} &&= \text{odd} \\ F_{3m+3} &= F_{3m+2} + F_{3m+1} &&= \text{odd} + \text{odd} &&= \text{even.} \end{aligned}$$

The first line uses only the inductive hypothesis; the second line uses the first line and the inductive hypothesis; the third line uses only the first two lines.

This proof has about the same complexity as in (b), but it proves a stronger result.

d. Challenge problem: For any whole number ℓ, the Fibonacci number F_ℓ evenly divides F_n whenever ℓ divides n. That is, for any $\ell, m \in \mathbb{N}$, we have that $F_{\ell m}$ is a multiple of F_ℓ. Start with $\ell = 4$, $m \geq 1$.

3. The product of two functions fg is defined by $(fg)(x) = f(x)g(x)$. (Do not confuse fg with the composition $f \circ g$.) In calculus, the Product Rule gives the derivative: $(fg)' = f'g + fg'$. This extends to a rule for the product of three functions:

$$(fgh)' = f'gh + fg'h + fgh'.$$

For example:

$$\begin{aligned}\big[(x{+}1)(x^2)(\sin x)\big]' &= (x{+}1)'(x^2)(\sin x) + (x{+}1)(x^2)'(\sin x) + (x{+}1)(x^2)(\sin x)' \\ &= (1)(x^2)(\sin x) + (x{+}1)(2x)(\sin x) + (x{+}1)(x^2)(\cos x).\end{aligned}$$

In fact, we have a Product Rule for any number of functions:

PROPOSITION: For any differentiable real functions $f_1, \ldots, f_n$, the derivative of their product is:

$$(f_1 f_2 \cdots f_n)' \;=\; (f_1' f_2 \cdots f_n) + (f_1 f_2' \cdots f_n) + \cdots + (f_1 f_2 \cdots f_n').$$

Prove this by induction, assuming the original, ordinary Product Rule.

SOLUTION: Since the statement makes no sense for $n = 0, 1$, the base case *is* the orginal Product Rule for $n = 2$: $(f_1 f_2)' = f_1' f_2 + f_1 f_2'$.

Chain: Assume the formula for a n functions: $(f_1 f_2 \cdots f_n)' = (f_1' f_2 \cdots f_n) + \cdots + (f_1 f_2 \cdots f_n')$. To compute the derivative for $n{+}1$ functions, apply the original Product Rule $(fg)' = f'g + fg'$ to the functions: $f = f_1 f_2 \cdots f_n$ and $g = f_{n+1}$:

$$\begin{aligned}(f_1 \cdots f_n f_{n+1})' &= (fg)' = f'g + fg' \\ &= (f_1 f_2 \cdots f_n)' f_{n+1} + (f_1 f_2 \cdots f_n) f_{n+1}' \\ &= \left[(f_1' f_2 \cdots f_n) + \cdots + (f_1 f_2 \cdots f_n')\right] f_{n+1} + (f_1 f_2 \cdots f_n) f_{n+1}' \\ &= (f_1' f_2 \cdots f_n f_{n+1}) + \cdots + (f_1 f_2 \cdots f_n' f_{n+1}) + (f_1 f_2 \cdots f_n f_{n+1}').\end{aligned}$$

Here we used the inductive hypothesis for the third equality. We obtain the right-hand side of the desired formula for $(n{+}1)$ functions, so we are done.

4. Consider the inequality: $n^2 \leq 2^n$, where n is a whole number.

a. Try this for $n = 1, \ldots, 7$ to see when the inequality holds.

b. Prove the inequality by induction. Hint: Prove the equivalent equality $n \leq \sqrt{2}^n$. Start with the correct base case.

SOLUTION: First we prove $n \leq \sqrt{2}^n$. Start with the base $n = 4$, for which we have $n = 4 = \sqrt{2}^4 = \sqrt{2}^n$.

For the Chain step, assume $n \leq \sqrt{2}^n$ for some $n \geq 4$, and compute:

$$n{+}1 \leq \sqrt{2}^n{+}1 = \sqrt{2}^n(1{+}\tfrac{1}{\sqrt{2}^n}) \leq \sqrt{2}^n(1{+}\tfrac{1}{\sqrt{2}^4}) = 2^n(\tfrac{5}{4}) \leq \sqrt{2}^{n+1},$$

Here we used that $n \geq 4$, so $\frac{1}{\sqrt{2}^n} \leq \frac{1}{\sqrt{2}^4} = \frac{5}{4}$.

This concludes the induction, so $n \leq \sqrt{2}^n$ for all $n \geq 4$. Both sides are positive reals, so we can square them to get: $n^2 \leq (\sqrt{2}^n)^2 = \sqrt{2}^{2n} = 2^n$ for $n \geq 4$.