Math 310.003 **Complex Number Geometry** **Fall 2018**

The complex numbers $\mathbb{C}$ are all pairs of real numbers $(a,b) \in \mathbb{R}^2$, endowed with the usual vector addition, as well as a non-obvious multiplication:

$$(a,b)+(c,d)=(a{+}c,b{+}d), \qquad (a,b)(c,d)=(ac{-}bd,ad{+}bc).$$

This defines a field with identity element $1_{\mathbb{C}}=(1,0)$, and we shall identify any real number a with $(a,0)$. (This defines a one-to-one homomorphism $\mathbb{R}\to\mathbb{C}$.) Letting $i=(0,1)$, we have $i^2=-1=(-1,0)$. Since $(a,b)=a+bi$, we call a and b the real and imaginary components.

To every complex number (a,b), we associate a linear transformation of the plane by multiplying an arbitrary vector (x,y) by (a,b):

$$L_{(a,b)}:\mathbb{R}^2\to\mathbb{R}^2, \qquad L_{(a,b)}(x,y)=(a,b)(x,y)=(ax{-}by,ay{+}bx).$$

Extra Credit Problems

1. Write the 2×2 matrix of the linear mapping $L_{(a,b)}$, with respect to the standard basis $(1,0),(0,1)$. (Recall any linear L is associated to a matrix $[L]$ defined by the column vectors $L(1,0)=(p,q)$ and $L(0,1)=(s,t)$, so $[L]=\left[\begin{smallmatrix} p & s \\ q & t\end{smallmatrix}\right]$. Then we have $L(x,y)=\left[\begin{smallmatrix} p & s \\ q & t\end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ y\end{smallmatrix}\right]$. In particular, $L_{(1,0)}$ should be the identity matrix.)

2. Explain why the mapping $f:\mathbb{C}\to M_2(\mathbb{R})$ given by $f(a,b)=[L_{(a,b)}]$ must be a one-to-one homomorphism, using the distributive and associative properties of complex multiplication. Conclude that the complex numbers are isomorphic to the subring of matrices resulting from all $L_{(a,b)}$.

3. Characterize $L_{(0,1)}$ in geometric terms as a mapping of the plane to itself, by seeing its effect on the two basis vectors $(1,0)$ and $(0,1)$. How does multiplying by i move the plane?

4. Similarly, characterize $L_{(a,b)}$ in geometric terms. Hint: Write complex numbers in polar coordinates as:

$$(x,y)=(r\cos\theta, r\sin\theta) \quad \text{and} \quad (a,b)=(c\cos\alpha, c\sin\alpha).$$

Use trig identities to understand how the basis vectors are affected by the mapping.