Math 418H **Fall 2005**

Lecture: Mon 11/7

1. Fundamental Theorem of Algebra

- *Theorem:* Any polynomial
$$f(z) = a_0 + a_1 z + \cdots + a_n z^n \in \mathbb{C}[z]$$
of degree $n \geq 1$ has at least one complex root $z = \alpha$ with $f(\alpha) = 0$.

- *First step:* We give a proof by contradiction. Suppose $f(z)$ were a polynomial with *no roots*. Then its reciprocal $g(z) := 1/f(z)$ would be analytic everywhere. Furthermore:
$$\lim_{|z|\to\infty} |f(z)| = \lim_{|z|\to\infty} |a_n z^n| = \infty\,,$$
meaning that $f(z)$ has large radius if z is far from the origin. Thus $\lim_{|z|\to\infty} g(z) = 0$, meaning that $g(z)$ has small radius when z is far from the origin.

- *Second step, Liouville's Theorem:* Let $g(z)$ be a function which is complex analytic on the whole plane, with $\lim_{|z|\to\infty} g(z) = 0$. Then $g(z)$ can only be the zero constant function: $g(z) = 0$ for all z.

  Proof: Consider any particular $\gamma \in \mathbb{C}$, and take a very large circle $\mathcal{C}(r, \gamma)$ with radius r and center γ. Given $\epsilon > 0$, by assumption we can take radius large enough so that $|g(z)| < \epsilon$ for z on the circle $\mathcal{C}(r, \gamma)$. By Cauchy's Mean Value Theorem, the value $g(\gamma)$ at the center is the average of the value of $g(z)$ on the circle $\mathbb{C}(r, \gamma) = c(t)$:
$$g(\gamma) = \operatorname*{Avg}_{c\in\mathcal{C}(r,\gamma)} g(c) := \frac{1}{2\pi r}\int g(c(t))\ |c'(t)|\ dt\,.$$
  Taking lengths and applying the triangle inequality,
$$|\operatorname*{Avg}_{\mathcal{C}} F(c)\,| \leq \operatorname*{Avg}_{\mathcal{C}}\ |F(c)|\,,$$
  we have:
$$|g(\gamma)| = |\operatorname*{Avg}_{\mathcal{C}(r,\gamma)} g(c)\,| \leq \operatorname*{Avg}_{\mathcal{C}(r,\gamma)} |g(c)| \leq \epsilon\,.$$
  Since this is true for any $\epsilon > 0$, we must have $|g(\gamma)| = 0$. This holds for each $\gamma \in \mathbb{C}$.

- *Third step:* Since $g(z) = 1/f(z)$ is not the zero constant function, we have a contradiction. Thus there cannot exist any non-vanishing polynomial $f(z) \in \mathbb{C}[z]$.

- Paraphrasing: Liouville's Theorem says that if a non-constant analytic function becomes very small as $|z| \to \infty$, then $g(z)$ must compensate for this by having non-anlytic points somewhere (for example, blowing up to infinity). Hence, if an analytic $f(z)$ becomes very large as $|z| \to \infty$ (as does a polynomial), then $f(z)$ must compensate for this by vanishing somewhere, i.e., having roots.
- This is a pure existence proof: it shows that a root-free polynomial function $f(z)$ would lead to an analytic function $g(z)$ violating the Cauchy Mean Value Theorem. The proof gives no clue how to find a root for a given $f(z)$: we will give an algorithm for this next time.

2. Factoring polynomials

- *Proposition:* Every monic complex polynomial $f(z)$ of degree n can be uniquely factored in $\mathbb{C}[z]$ as a product of n linear functions.

$$f(z) = (z - \alpha_1) \cdots (z - \alpha_n) .$$

  That is, the irreducible polynomials of $\mathbb{C}[z]$ are linear.

  Proof: By the Fundamental Theorem, $f(z)$ has a root $z = \alpha$ and thus a linear factor: $f(z) = (z - \alpha_1)\, f_1(z)$, where $f_1(z)$ has degree $n-1$. Repeat this for $f_1(z)$ until all factors are linear.

- *Proposition:* Every monic real polynomial $f(z)$ of degree n can be uniquely factored in $\mathbb{R}[z]$ as a product of linear and quadratic functions:

$$f(z) = (z - \alpha_1) \cdots (z - \alpha_k)\, q_1(z) \cdots q_\ell(z) ,$$

  where $\alpha_j \in \mathbb{R}$, $q_j(z) \in \mathbb{R}[z]$ has degree 2, and $k + 2\ell = n$. That is, the irreducible polynomials of $\mathbb{R}[z]$ are linear and quadratic.

  Proof: A real polynomial $f(z) \in \mathbb{R}[z]$ can be factored into complex linear factors as above. But if $f(\alpha) = 0$, then $f(\overline{\alpha}) = \overline{f(\alpha)} = 0$, so the non-real roots come in complex conjugate pairs. Each such pair $\alpha \neq \overline{\alpha}$ with $\alpha = a + bi$ gives a real factor:

$$(z - \alpha)\,(z - \overline{\alpha}) = z^2 + (\alpha + \overline{\alpha})\, z + \alpha\overline{\alpha} = z^2 + 2az + (a^2 + b^2) \in \mathbb{R}[z] .$$

  These factors are irreducible in $\mathbb{R}[z]$ since their roots $\alpha, \overline{\alpha}$ are not in $\mathbb{R}$ by assumption.

- *Example:* Let $f(z) = z^4 + 1$, having roots $\alpha_1 = \mathrm{cis}(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}(1 + i)$, $\alpha_2 = \mathrm{cis}(\frac{3\pi}{4}) = \frac{\sqrt{2}}{2}(-1 + i)$, and their conjugates $\overline{\alpha}_1$, $\overline{\alpha}_2$. Factoring:

$$\begin{aligned} f(z) &= (z - \alpha_1)(z - \overline{\alpha}_1)(z - \alpha_2)(z - \overline{\alpha}_2) \\ &= (z^2 + (\alpha_1 + \overline{\alpha}_1)\, z + \alpha_1\overline{\alpha}_1)\ (z^2 + (\alpha_2 + \overline{\alpha}_2)\, z + \alpha_2\overline{\alpha}_2) \\ &= (z^2 + \sqrt{2}\, z + 1)\ (z^2 - \sqrt{2}\, z + 1) \end{aligned}$$