Lecture: Mon 8/29/05

1. $\mathbb{N} := \{0, 1, 2, \ldots\}$ natural numbers (whole numbers)

   - "God made the whole numbers; all the rest is the work of man."
   - operations $\implies$ solving equations $\implies$ inverse operations $\implies$ new number systems
   - accounting $\implies$ algebra:
     8 sheep, 3 born, how many?
     addition operation: $x = 8 + 3 = 11$
   - 11 sheep, 5 male, how many female?
     $y + 5 = 11$, $y = 11 - 5 = 6$ (inverse to + op)
   - 11 sheep, king wants 15 for taxes, how many left?
     $z + 15 = 11$, $z = 11 - 15 = -4$, debt of 4 sheep
     new type of number, has meaning in original context
   - $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ integers (from German *Zahl*)

2. $\mathbb{Q}$ rational numbers

   - 300 peasants, 15 sheep each in taxes, how much revenue?
     multiplication operation: $u = 300 \times 15 = 4500$
   - 4500 revenue, 160 soldiers, how much for each?
     $4500/60 = 225/8 = 28 + \frac{1}{8}$ fraction
   - $\mathbb{Q} = \{\, a/b \text{ with } a, b, \in \mathbb{Z}\,,\ b \neq 0 \,\}$ rational numbers (fractions)

   - make definitions for $\mathbb{Q}$ in terms of known terms for $\mathbb{Z}$
     equality of fractions: $a/b = c/d \iff ad = bc$
     addition of fractions: $a/b + c/d := (ad + bc)/bd$

3. $\mathbb{R}$ real numbers

   - square field has area, 200 yd^2, side is how long?
     $s^2 = 200$, $s = \sqrt{200} = 10\sqrt{2}$.
   - Proposition: $\sqrt{2} \notin \mathbb{Q}$: that is, $(a/b)^2 \neq 2$ for all $a/b \in \mathbb{Q}$
   - Lemma: a^2 even $\implies$ a even
     Proof of Lemma: if even $a = 2n$, then $a^2 = 4n^2$ even; if odd $a = 2n+1$, then $a^2 = 4n^2+4n+1 = 2(2n^2+2n)+1$ odd.
   - Proof of Proposition: Suppose $a/b \in \mathbb{Q}$ in lowest terms, so a, b are not both even. Suppose $(a/b)^2 = 2$, so that $a^2 = 2b^2$ is even. By the Lemma, $a = 2n$ is even, so $2b^2 = (2n)^2 = 4n^2$ and $b^2 = 2n^2$ is even. By the Lemma, b is also even, so we could not have *any* solution a/b in lowest terms.

- solve $x^2 = c \implies$ solve $ax^2 + bx + c = 0$
  Complete-the-square trick: Rewrite eqn as $x^2+(b/a)x+(c/a) = 0$. If $2d = b/a$ then:

  $$(x+d)^2 - d^2 + \frac{c}{a} = x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

  Now we can solve $(x+d)^2 = d^2 - c/a$, so $x = -d \pm \sqrt{d^2 - c/a}$ for $d = b/2a$. Work out the usual quadratic formula.

4. $\mathbb{C} = \{\, a + ib \ \text{ for } \ a, b \in \mathbb{R} \,\}$, complex numbers

   - solve $x^2 + 1 = 0$ gives new number $i = \sqrt{-1}$
   - can define operations on numbers $a + bi$ for $a, b \in \mathbb{R}$ in terms of known operations on $\mathbb{R}$.
     $(a + bi)(c + di) = ac + i^2bd + iad + ibc = (ac - bd) + i(ad + bc)$
   - can now solve $x^2 = -a$: $x = \sqrt{-a} = i\sqrt{a}$ for $a \geq 0$.
   - can now solve $ax^2 + bx + c = 0$ for *any* $a, b, c \in \mathbb{R}$ (even if no real solution): quadratic formula
   - Fundamental Theorem of Algebra: Any polynomial equation

     $$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$$

     with coefficients $a_0, a_1, \ldots, a_n \in \mathbb{C}$ has at least one solution $x = a + bi \in \mathbb{C}$.
   - Thus, the process of finding more general number systems to solve equations ends with $\mathbb{C}$.