Affine Schubert Varieties and Circular Complexes

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Schubert varieties have been exhaustively studied with a plethora of techniques: Coxeter groups, explicit desingularization, Frobenius splitting, etc. Many authors have applied these techniques to various other varieties, usually defined by determinantal equations. It has turned out that most of these apparently different varieties are actually Schubert varieties in disguise, so that one may use a single unified theory to understand many large families of spaces.

The most powerful result in this direction was given by Lusztig [9], [10, §11], as a footnote to his work on canonical bases. He showed that the variety of nilpotent representations of a cyclic quiver (including nilpotent conjugacy classes of matrices) is isomorphic to an open subset of a Schubert variety for the loop group $\widehat{GL}_n$. In this paper, we attempt to describe the affine Schubert varieties (§1) and Lusztig's isomorphism (§2) in the simplest terms possible.

We then apply this isomorphism to an interesting example, the variety of circular complexes, recovering many of the results of Mehta and Trivedi [12]. (The reader may skip to this application in §3 immediately after reading §1.) Our technique is similar to that of Lakshmibai and Magyar [8]: it is as a chapter in the "ubiquity of Schubert varieties."

1 Affine flag variety and Weyl group

We begin by describing the loop group and its flag variety as a classical group. For more details of the material of this section see Pressley-Segal [13], Kac-Raina [4], Slodowy [16], Kazhdan-Lusztig [5], Kumar [6, Appendix C], Shi [15], Bjorner-Brenti [2], and Eriksson-Eriksson [3].

1.1 Loop group and affine flag variety

Let $\mathbf{k}$ be an arbitrary field, and $F := \mathbf{k}((t))$, the field of formal Laurent series $f(t) = \sum_{i \geq N} a_i t^i$ with $a_i \in \mathbf{k}$; and $A := \mathbf{k}[[t]]$, the ring of formal Taylor series. For such $f(t) \neq 0$, we let $\mathrm{ord}(f)$ be the smallest integer N for which $a_N \neq 0$.

Fix a positive integer n, and define $G = \hat{\mathrm{GL}}_n(\mathbf{k}) := \mathrm{GL}_n(F)$, the group of invertible $n \times n$ matrices with coefficients in F. We call this the *loop group* because for $\mathbf{k} = \mathbb{C}$ we may think of G as a completion of the group of polynomial maps from the circle $S^1 \subset \mathbb{C}^\times$ to $\mathrm{GL}_n(\mathbb{C})$.

Let $G_j := \{g \in G \mid \mathrm{ord}\det g = j\}$, so that $G_j G_k = G_{j+k}$, and for any $\sigma \in G_1$, we have $G_j = \sigma^j G_0 = G_0 \sigma^j$, and $G = \coprod_{j \in \mathbb{Z}} G_j$. This should be thought of as the decomposition of G into connected components. (For $\mathbf{k} = \mathbb{C}$, and g a polynomial map, the number $\operatorname{ord}\det g$ is the winding number of the loop $\det g : S^1 \to \mathbb{C}^\times$, and the G_j are the connected components of G in the appropriate compact-open topology.)

Let $V := F^n$, a vector space over F with a natural action of G. Let $e_1, \ldots, e_n$ denote the standard F-basis of V, and for $c \in \mathbb{Z}$, define $e_{i+nc} := t^c e_i$. (Thus, $\{e_i\}_{i\in\mathbb{Z}}$ is a $\mathbf{k}$-basis of V, in the sense appropriate to a topological vector space with the t-adic topology.)

An *A-lattice* $\Lambda \subset V$ is the A-submodule $\Lambda = Av_1 \oplus \cdots \oplus Av_n$, where $\{v_1, \ldots, v_n\}$ is an F-basis of V. We may write $\Lambda = \mathrm{Span}_{\mathbf{k}}\langle v_i\rangle_{i\geq 1}$, the space of infinite $\mathbf{k}$-linear combinations of the vectors $v_{i+nc} := t^c v_i$. Consider the family of *standard A-lattices*:

$$E_j := \mathrm{Span}_A\langle e_j, e_{j+1}, \ldots e_{j+n-1}\rangle = \mathrm{Span}_{\mathbf{k}}\langle e_i\rangle_{i\geq j}.$$

Note that $E_j = \sigma^j E_1$, where we use the shift operator $\sigma(e_i) := e_{i+1}$, or as a matrix:

$$\sigma = \begin{pmatrix} 0 & 0 \cdots & 0 & t \\ 1 & 0 \cdots & 0 & 0 \\ 0 & 1 \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 \cdots & 1 & 0 \end{pmatrix} \in G_1$$

The *affine Grassmannian* $\mathrm{Gr}(V)$ is the space of all A-lattices of V. Clearly $\mathrm{Gr}(V)$ is a homogeneous space with respect to the obvious action of G, and the stabilizer of the standard lattice E_1 is $P_{\hat{0}} := GL_n(A)$, the subgroup of matrices with coefficients in A and with determinant having ord $= 0$. Thus $\mathrm{Gr}(V) \cong G/P_{\hat{0}}$, and the connected components of the Grassmannian are $\mathrm{Gr}_j(V) := G_0 \cdot E_j = G_j \cdot E_1 \cong G_j/P_{\hat{0}}$. In fact, $\mathrm{Gr}_j(V) := \{\Lambda \mid \mathrm{vdim}(\Lambda) = j\}$, where we define the *virtual dimension*

$$\mathrm{vdim}(\Lambda) := \dim_{\mathbf{k}}(\Lambda/\Lambda \cap E_1) - \dim_{\mathbf{k}}(E_1/E_1 \cap \Lambda).$$

The *complete affine flag variety* $\mathrm{Fl}(V)$ is the space of all flags of lattices $\Lambda_\bullet = (\Lambda_1 \supset \cdots \supset \Lambda_n)$ such that $\Lambda_n \supset t\Lambda_1$ and $\dim_{\mathbf{k}}(\Lambda_j/\Lambda_{j+1}) = 1$. There always exists an F-basis $\{v_1, \ldots, v_n\}$ of V such that $\Lambda_j = \mathrm{Span}_A\langle v_j, \ldots, v_n, tv_1, \ldots, tv_{j-1}\rangle = \mathrm{Span}_{\mathbf{k}}\langle v_i\rangle_{i\geq j}$, where i runs over all integers not less than j, and $v_{i+nc} := t^c v_i$. The *standard flag* is $E_\bullet := (E_1 \supset \cdots \supset E_n)$, whose stabilizer B is the subgroup of matrices $b \in P_{\hat{0}}$ which are lower-triangular modulo t:

$$B := \{b = (b_{ij}) \in GL_n(A) \mid \mathrm{ord}(b_{ij})>0 \;\; \forall\, i<j\}.$$

Thus, $\mathrm{Fl}(V) \cong G/B$, with connected components $\mathrm{Fl}_j(V) := G_j \cdot E_\bullet \cong G_j/B = \{\Lambda_\bullet \mid \mathrm{vdim}(\Lambda_1) = j\}$. Furthermore, the projection $\mathrm{Fl}(V) \to \mathrm{Gr}(V)$, $\Lambda_\bullet \mapsto \Lambda_1$ is a bundle whose fiber is the space of complete flags in the n-dimensional $\mathbf{k}$-vector space $\Lambda_1/t\Lambda_1$.

1.2 Affine Weyl group

We first discuss $\widetilde{W}$, the Weyl group of the disconnected group G; and then W, the Weyl group of the connected component G_0.

Let S_∞ be the group of bijections $\pi : \mathbb{Z} \to \mathbb{Z}$, and let $\sigma : i \mapsto i{+}1$ be the shift bijection. Let $\widetilde{W} \subset S_\infty$ be the subgroup of bijections which commute with the nth power of σ: that is, $\widetilde{W} := \{\pi \in S_\infty \mid \pi\tau = \tau\pi\}$, where $\tau := \sigma^n : i \mapsto i{+}n$.

For $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{Z}^n$, define an element of $\widetilde{W}$, $\tau^{\mathbf{c}} : i \mapsto i{+}nc_{(i \bmod n)}$. This gives an embedding of the additive group $\mathbb{Z}^n \subset \widetilde{W}$. Furthermore, we have the embedding $S_n \subset \widetilde{W}$ with $\bar{\pi}(i + nc) := \bar{\pi}(i) + nc$ for $\bar{\pi} \in S_n$. Then we may write any element $\pi \in \widetilde{W}$ as $\pi = \bar{\pi}\tau^{\mathbf{c}}$ for unique $\bar{\pi} \in S_n$, $\mathbf{c} \in \mathbb{Z}^n$, and we have $\bar{\pi}_1\tau^{\mathbf{c}_1}\bar{\pi}_2\tau^{\mathbf{c}_2} = \bar{\pi}_1\bar{\pi}_2\tau^{\bar{\pi}_2^{-1}(\mathbf{c}_1)+\mathbf{c}_2}$. That is, $\widetilde{W} = S_n \ltimes \mathbb{Z}^n$, a semi-direct product. The normal subgroup $\mathbb{Z}^n$ is the kernel of the homomorphism $\widetilde{W} \to S_n$ which takes each $\pi : \mathbb{Z} \to \mathbb{Z}$ to a permutation of cosets $\bar{\pi} : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$.

Thus, an element $\pi \in \widetilde{W}$ is equivalent to a sequence of integers $[\pi(1), \ldots, \pi(n)]$ such that $i \mapsto \bar{\pi}(i)$ defines a permutation of $\mathbb{Z}/n\mathbb{Z}$. For example, in this one-line notation $\sigma = [2, 3, \cdots, n{+}1]$ and $\tau^{\mathbf{c}} = [1 + nc_1, 2 + nc_2, \ldots, n + nc_n]$.

We obtain another useful notation by embedding $\widetilde{W} \subset G$. If we let $\pi = \bar{\pi}\tau^{\mathbf{c}} \in W$ act F-linearly on V by $\pi(e_i) := e_{\pi(i)}$, the corresponding matrix is the *affine permutation matrix* (a_{ij}) with $a_{\bar{\pi}(i),i} = t^{c_i}$. For example, σ becomes the matrix in G_1 of the previous section, $\tau^{\mathbf{c}} = \mathrm{diag}(t^{c_1}, \cdots, t^{c_n})$, and $\tau = \tau^{(1,\cdots,1)} = \mathrm{diag}(t, \cdots, t)$,

Using this embedding, we may show that $\widetilde{W} \cong N_G(\hat{T})/\hat{T}$, where $\hat{T}$ is the subgroup of diagonal matrices with entries in A.

Considering the embedding $\widetilde{W} \subset G$, we define

$$W_j := \widetilde{W} \cap G_j = \left\{ \pi \in \widetilde{W} \;\middle|\; \sum_{i=1}^{n} \pi(i) - i = j \right\}, \qquad W := W_0.$$

Thus W is a normal subgroup of $\widetilde{W}$, and we have $W = S_n \ltimes \mathbb{Z}_0^n$, where $\mathbb{Z}_0^n := \mathbb{Z}^n \cap W = \{\mathbf{c} \mid \sum_{i=1}^n c_i = 0\}$. That is, $W \cong \widehat{S}_n$, the affine Weyl group of extended Dynkin type $\widehat{A}_{n-1}$.

Define the simple reflections $s_0, \ldots, s_{n-1}$ in W by $s_i(i) = i{+}1$, $s_i(i{+}1) = i$, and $s_i(j) = j$ for $j \not\equiv i,\ i{+}1 \bmod n$. We shall sometimes denote $s_n := s_0$. Then $s_0, \ldots, s_{n-1}$ are involutions generating W and satisfying the Coxeter relations $(s_i s_{i+1})^3 = \mathrm{id}$ for $0 \le i \le n{-}1$, and $(s_i s_j)^2 = \mathrm{id}$ otherwise. We have a semi-direct product $\widetilde{W} = \langle\sigma\rangle \ltimes W$. Here σ acts on W via the outer automorphism: $\sigma s_i \sigma^{-1} = s_{i+1}$.

The Bruhat length $\ell(\pi)$ is defined as usual for $\pi \in W$, as the smallest number of simple reflections whose product is π; and we extend this to $\widetilde{W}$ by letting $\ell(\sigma^j\pi) := \ell(\pi)$. We have J. Shi's formula [15]:

$$\ell(\pi) = \sum_{1 \le i < j \le n} \left| \mathrm{floor}\!\left(\tfrac{\pi(j) - \pi(i)}{n}\right) \right|,$$

where floor(x) denotes the greatest integer not exceeding x. Lusztig showed:

$$\ell(\tau^{\mathbf{c}}) = (n{-}1)c_1 + (n{-}3)c_2 + \cdots + (-n{+}3)c_{n-1} + (-n{+}1)c_n\,,$$

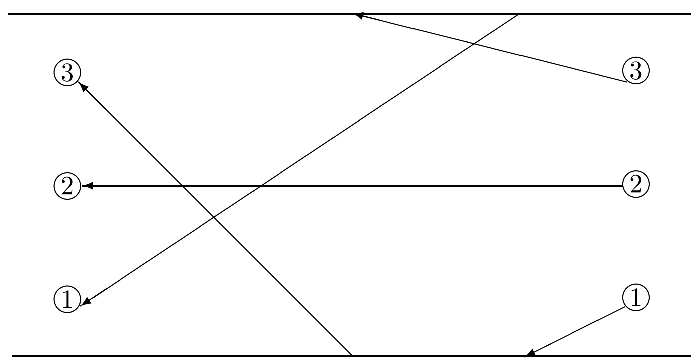
namely, the dot product of $\mathbf{c}$ with

$$2\rho^\vee := (n{-}1, n{-}3, \ldots, -n{+}3, -n{+}1) = \sum_{1\le i<j\le n} e_i - e_j.$$

1.3 Wiring diagrams

The structure of the Weyl group is further elucidated by the *loop wiring diagrams* (cf. Berenstein-Fomin-Zelevinsky [1]). Consider a cylinder $[0,1]\times S^1$. On the right end, label each point $(1, e^{2\pi\sqrt{-1}i/n})$ with the integer i, and similarly on the left end. Now, represent a permutation $\pi = \bar\pi\tau^{\mathbf{c}} \in \widetilde{W}$ by n curves, each joining a point i on the right to the point $\bar\pi(i)$ on the left, but looping counter-clockwise around the cylinder c_i times.

Example. For $n=3$, the permutation $\pi = [-2,2,6] \in W$, with $\bar\pi = \mathrm{id} \in S_3$ and $\mathbf{c} = (-1,0,1) \in \mathbb{Z}^3_0$, is represented by the picture:



Here we represent our cylinder by identifying the top and bottom borders of the picture, so that each point i on the right is connected to the same i on the left (since $\bar\pi = \mathrm{id}$). However, since $\mathbf{c} = (-1,0,1)$, the curve starting from 1 travels once clockwise around the cylinder, the curve from 2 travels straight across, and the curve from 3 travels once counter-clockwise.

We may read off much combinatorial data from this picture. Since the curves have a total of 4 crossings, we conclude that $\ell(\pi) = 4$. By listing these crossings, as well as crossings over the top and bottom margins, we obtain a reduced decomposition: $\pi = s_2 s_1 s_2 \sigma s_2 \sigma^{-1}$. That is, the leftmost crossing switches the top two curves, giving a factor s_2; the second switches the bottom two curves, s_1; again s_2; then the bottom curve crosses to the top, σ; again s_2; and finally the top curve crosses to the bottom, σ^{-1}. Using $s_i\sigma = \sigma s_{i-1}$, we have $\pi = s_2 s_1 s_2 s_0$.

1.4 Schubert varieties

By Gaussian elimination, we obtain the *Bruhat decomposition* of G into double B-cosets: $G = \coprod_{\pi\in\widetilde{W}} B\pi B$, where we consider each π as an affine permutation matrix. Hence we also have a Bruhat decomposition of the affine flag variety $\mathrm{Fl}(V) = \coprod_{\pi \in \widetilde{W}} X_\pi^\circ$ into *Schubert cells* $X_\pi^\circ := B{\cdot}\pi E_\bullet$, where $\pi E_\bullet$ is a translation of the standard flag $E_\bullet = (E_1 \supset \cdots \supset E_n)$. In particular, $\mathrm{Fl}_j(V)$ is the union of all X_π° with $\pi \in W_j$. The Schubert cells can be defined by dimension constraints called *Schubert conditions*. For $\pi \in W_j$, we have:

$$\begin{aligned} X_\pi^\circ &= \left\{ \Lambda_\bullet \in \mathrm{Fl}(V) \;\middle|\; \dim_{\mathbf{k}}(E_j/\Lambda_i \cap E_j) = \#(\mathbb{Z}_{\geq j} \backslash \pi\mathbb{Z}_{\geq i}) \;\right\} \\ &= \left\{ \Lambda_\bullet \in \mathrm{Fl}_j(V) \;\middle|\; \dim_{\mathbf{k}}(\Lambda_i/\Lambda_i \cap E_j) = \#(\pi\mathbb{Z}_{\geq i} \backslash \mathbb{Z}_{\geq j}) \;\right\} \end{aligned}$$

where $\mathbb{Z}_{\geq i}$ denotes the integers not less than i; we define $\pi\mathbb{Z}_{\geq i} := \{\pi(i), \pi(i+1), \ldots\}$; and $\backslash$ denotes set complement. Indeed, the set on the right of the equation is clearly B-invariant, and $\pi' E_\bullet$ lies in this set if and only if $\pi' = \pi$.

The *Schubert variety*, meaning the topological closure $X_\pi := \overline{X_\pi^\circ}$, is obtained by replacing $=$ in the above Schubert conditions with $\leq$. We say $\pi \leq \pi'$ in the *Chevalley-Bruhat order* if $X_\pi \subset X_{\pi'}$, and we can express this combinatorially as: $\pi \leq \pi'$ iff $\#\pi\mathbb{Z}_{\geq i} \backslash \mathbb{Z}_{\geq j} \leq \#\pi'\mathbb{Z}_{\geq i} \backslash \mathbb{Z}_{\geq j}$ for all $1 \leq i \leq n$, $j \in \mathbb{Z}$.

We explain below how X_π has the structure of a projective algebraic variety. With this structure, the varieties X_π include as special cases the familiar Schubert varieties for $\mathrm{GL}_n(\mathbf{k})$. In fact, for $\pi = \bar{\pi} \in S_n \subset \widetilde{W}$ and $\Lambda_\bullet \in X_\pi$, we have $\pi(\mathbb{Z}_{\geq 1}) = \mathbb{Z}_{\geq 1}$ and $\pi(\mathbb{Z}_{\geq n+1}) = \mathbb{Z}_{\geq n+1}$. Hence $\Lambda_1/\Lambda_1 \cap E_1 = 0$ and $\Lambda_1 \subset E_1$. Also $n = \dim_{\mathbf{k}}(\Lambda_1/\Lambda_1 \cap E_{n+1}) \leq \dim_{\mathbf{k}}(E_1/E_{n+1}) = n$, so $\Lambda_1 = E_1$ and $\Lambda_i \supset t\Lambda_1 = E_{n+1}$ for $1 \leq i \leq n$. Letting $\mathbf{k}^n = E_1/E_{n+1}$ and $V_i := \Lambda_i/E_{n+1}$, we thus find that $\Lambda_\bullet \in X_\pi$ is in natural correspondence with the complete flag

$$\mathbf{k}^n = V_1 \supset V_2 \supset \cdots \supset V_n \supset 0,$$

and the affine Schubert conditions on $\Lambda_\bullet$ are equivalent to the usual Schubert conditions

$$\dim_{\mathbf{k}}(V_i \cap \bar{E}_j) \geq \#(\pi[i,n] \cap [j,n])$$

relative to the standard flag $\bar{E}_j := E_j/E_{n+1}$. For example, $X_{\mathrm{id}} = \{E_\bullet\}$, a single point.

For a general $\pi \in \widetilde{W}$, we can find a, b so that $\mathbb{Z}_{\geq a} \supset \pi\mathbb{Z}_{\geq i} \supset \mathbb{Z}_{\geq b}$ for $1 \leq i \leq n$. Then any $\Lambda_\bullet \in X_\pi$ satisfies $E_a \supset \Lambda_i \supset E_b$, and we may embed X_π inside a partial flag variety of the finite-dimensional $\mathbf{k}$-vector space E_a/E_b. The flags in the image of this embedding must satisfy certain ordinary Schubert conditions, but they must also be stable under the nilpotent map induced on E_a/E_b by $t \in A$. This makes X_π into an algebraic variety over $\mathbf{k}$ (in fact, even defined over the integers).

We can imitate all the standard geometric constructions for Schubert varieties of $\mathrm{GL}_n(\mathbf{k})$. For example, we can show $\dim_{\mathbf{k}} X_\pi = \ell(\pi)$; we can explicitly construct Bott-Samelson resolutions of X_π as configuration varieties; and we can use the usual Frobenius-splitting arguments to show that the variety X_π is normal, Cohen-Macaulay, etc.

Example. Consider as above $n = 3$, $\pi = [-2, 2, 6]$. Take $a = -2$, $b = 4$, and write:

$$X_\pi = \{\ (E_{-2} \overset{3}{\supset} \Lambda_1 \overset{1}{\supset} \Lambda_2 \overset{1}{\supset} \Lambda_3 \overset{1}{\supset} E_4)\ \big|\ t\Lambda_1 \subset \Lambda_3,\ \Lambda_2 \subset E_1\ \ \},$$

where $U \overset{d}{\supset} V$ means $U \supset V$ and $\dim_{\mathbf{k}}(U/V) = d$. All the Schubert conditions for X_π follow from the conditions specified on the right side of the equation.

Let $\mathbf{k}^6 = E_{-2}/E_4$ with basis $\{\bar{e}_{-2}, \bar{e}_{-1}, \bar{e}_0, \bar{e}_1, \bar{e}_2, \bar{e}_3\}$, and take $\bar{t} : \mathbf{k}^6 \to \mathbf{k}^6$, $\bar{e}_i \mapsto \bar{e}_{i+3} \bmod E_4$, so that $\bar{t}^2 = 0$. Then we have the isomorphism

$$X_\pi \cong \{\ (\mathbf{k}^6 \overset{3}{\supset} V_1 \overset{1}{\supset} V_2 \overset{1}{\supset} V_3 \overset{1}{\supset} 0)\ \big|\ \bar{t}(V_1) \subset V_3,\ V_2 \subset \bar{E}_1\ \ \},$$

a subvariety of a partial flag variety of $\mathrm{GL}_6(\mathbf{k})$ defined by Schubert conditions and the algebraic incidence condition $\bar{t}(V_1) \subset V_3$.

Further, we can construct a Bott-Samelson variety corresponding to the reduced word $\pi = s_2 s_1 s_2 s_0$:

$$Z_{2120} := \left\{ \begin{array}{c} (\Lambda_1, \Lambda_2, \Lambda_3, \Lambda'_3) \\ \in \mathrm{Gr}(V)^4 \end{array} \ \middle|\ \begin{array}{ccccc} & & E_2 \leftarrow E_3 & & \\ & \swarrow & \nwarrow & \nwarrow & \\ E_1 & & & & \Lambda'_3 \leftarrow tE_1 \\ & \nwarrow & \swarrow & \swarrow & \\ & & \Lambda_1 \leftarrow \Lambda_2 \leftarrow \Lambda_3 \leftarrow t\Lambda_1 & & \end{array} \right\},$$

where each arrow $U \leftarrow V$ indicates the condition $U \overset{1}{\supset} V$. We may build up this variety by starting with a single point (the standard flag) and successively adding the spaces Λ'_3, Λ_2, Λ_3, Λ_1, corresponding to the reflections s_2, s_1, s_2, s_0. Clearly Z_{2120} is an iterated $\mathbb{P}^1$-fibration (thus smooth), and it maps birationally to X_π by dropping Λ'_3. (In this case, X_π happens to be smooth itself.) Note that the pattern of inclusions defining Z_{2120} is the dual graph of the wiring diagram (turned sideways).

1.5 Partial flag variety and opposite cell

A subset $I \subset [0, n{-}1]$ corresponds to a *parabolic subgroup* $G \supset P_I \supset B$ with Weyl group $W_I := \langle s_i \rangle_{i \in I} \subset W$. For $I \ni 0$, the partial flag variety corresponding to the complement $\widehat{I} := [0, n{-}1] \backslash I$ is:

$$G/P_{\widehat{I}} \cong \left\{ (\Lambda_1 \supset \cdots \supset \Lambda_h) \ \middle|\ \begin{array}{c} \Lambda_j \text{ a lattice},\quad \Lambda_h \supset t\Lambda_1, \\ \dim(\Lambda_j \backslash \Lambda_{j+1}) = i_{j+1} - i_j \end{array} \right\}.$$

We shall find it convenient to index parabolics by *compositions of* n: that is, sequences of positive integers $\mathbf{d} = (d_1, \ldots, d_h)$ with $d_1 + \cdots + d_h = n$. Given $I = \{0 {=} i_1 < \cdots < i_h\}$, let $i_{h+1} := n$, and define a composition by $d_j := i_{j+1} - i_j$ (so that $i_j = d_1 + \cdots + d_{j-1}$). Then we may rewrite the above more concisely as:

$$G/P_{\widehat{I}} \cong \mathrm{Fl}(\mathbf{d}; V) := \{(\Lambda_1 \overset{d_1}{\supset} \cdots \overset{d_{h-1}}{\supset} \Lambda_h \overset{d_h}{\supset} t\Lambda_1)\}.$$

Denoting $W_{\mathbf{d}} := W_{\widehat{I}}$ and $E_{(j)} := E_{1+i_j}$, $E_{(\bullet)} := (E_{(1)} \supset \cdots \supset E_{(h)})$, we have:

$$\mathrm{Fl}(\mathbf{d}; V) = \coprod_{\pi W_{\mathbf{d}} \in \widetilde{W}/W_{\mathbf{d}}} B \cdot \pi E_{(\bullet)} \ .$$

In particular, for $I = \{0\}$, $\mathbf{d} = (n)$, we have $\widetilde{W}/W_{\mathbf{d}} = \widetilde{W}/S_n \cong \mathbb{Z}^n$, and

$$\mathrm{Gr}(V) = G/P_{\hat{0}} = \coprod_{\mathbf{c} \in \mathbb{Z}^n} X^\circ_{\mathbf{c}} \ ,$$

where $X^\circ_{\mathbf{c}} := B \cdot \tau^{\mathbf{c}} E_1 = \{\Lambda \mid \dim(\Lambda/\Lambda \cap E_j) = \#(\tau^{\mathbf{c}}\mathbb{Z}_{\geq 1} \setminus \mathbb{Z}_{\geq j})\}$.

Next, for any Schubert variety X_π, we define a certain affine open subset, the *opposite cell* $X'_\pi \subset X_\pi$ (meaning the opposite to the cell $X^\circ_\pi \subset X_\pi$, though X'_π itself is generally not a topological cell). Let $E'_k := \mathrm{Span}_{\mathbf{k}}\langle e_i \rangle_{i<k}$ be the complementary space to E_k. Note that E'_k is *not* an A-lattice in V: rather, it is a lattice over the ring $A' = k[t^{-1}] \subset F$. For $\pi \in W$, define $X'_\pi \subset X_\pi \subset \mathrm{Fl}_0(V)$ as the set of $\Lambda_\bullet \in X_\pi$ such that $\Lambda_i \cap E'_i = 0$ for $1 \leq i \leq n$. For example, $E_\bullet \in X'_\pi$ for any $\pi \in W$.

[Note: The condition $\Lambda_i \cap E'_i = 0$ is equivalent to $\Lambda_i \oplus E'_i = V$. Proof: Recall that $\dim_{\mathbf{k}}(\Lambda_i/\Lambda_i \cap E_i) - \dim_{\mathbf{k}}(E_i/E_i \cap \Lambda_i) = 0$, and let $\phi : \Lambda_i/\Lambda_i \cap E_i \subset E'_i \oplus E_i/\Lambda_i \cap E_i \to E_i/\Lambda_i \cap E_i$. Thus $\Lambda_i \cap E'_i = \mathrm{Ker}(\phi) = 0 \Longleftrightarrow \mathrm{Im}(\phi) = E_i/\Lambda_i \cap E_i$ $\Longleftrightarrow E'_i + \Lambda_i = E'_i + E_i = V$.]

More generally, for $\pi \in W_k = \sigma^k W$, $X_\pi \subset \mathrm{Fl}_k(V)$, we let

$$\begin{aligned} X'_\pi &:= \{\Lambda_\bullet \in X_\pi \mid \Lambda_i \cap E'_{i+k} = 0, \ \ 1 \leq i \leq n\} \\ &= \{\Lambda_\bullet \in X_\pi \mid \Lambda_i \oplus E'_{i+k} = V, \ \ 1 \leq i \leq n\}. \end{aligned}$$

Thus $\sigma^k E_\bullet \in X'_\pi$. We define $X'_\pi \subset X_\pi \subset \mathrm{Fl}_k(\mathbf{d}, V)$ similarly: e.g., for $X'_\pi \subset X_\pi \subset \mathrm{Gr}(V)$, we require $\Lambda \cap E'_{k+1} = 0$, so $E_{k+1} \in X'_\pi$.

Now we examine certain affine Grassmannian Schubert varieties which will occur in the following section. Suppose $\mathbf{c} = (c_1, \ldots, c_n)$ satisfies $0 \leq c_1 \leq \cdots \leq c_n \leq n$ and $c_1 + \cdots + c_n = n$. Let $c'_j := \#(\tau^{\mathbf{c}}\mathbb{Z}_{\geq 1} \setminus \tau^j \mathbb{Z}_{\geq 1}) = \#\{i \mid c_i \leq j\}$, the conjugate-complement partition of $\mathbf{c}$. Then we have:

$$X^\circ_{\mathbf{c}} = \left\{\Lambda \in \mathrm{Gr}(V) \mid E_1 \supset \Lambda \supset t^n E_1, \ \dim(\Lambda/\Lambda \cap t^j E_1) = c'_j, \ 1 \leq j \leq n\right\}$$

Since the maximal parabolic $P_{\hat{0}}$ stabilizes $t^j E_1$, we have $X^\circ_{\mathbf{c}} = P_{\hat{0}} \cdot \tau^{\mathbf{c}} E_1$. Furthermore, letting $X^{\circ\prime}_{\mathbf{c}} := X^\circ_{\mathbf{c}} \cap X'_{\mathbf{c}}$, we obtain:

$$X^{\circ\prime}_{\mathbf{c}} = \left\{ \Lambda \in \mathrm{Gr}(V) \;\middle|\; \begin{array}{c} \dim(\Lambda/\Lambda \cap t^j E_1) = c'_j, \ 1 \leq j \leq n \\ E_1 \supset \Lambda \supset t^n E_1, \quad \Lambda \cap t^n E'_1 = 0 \end{array} \right\}.$$

Since $\mathrm{GL}_n(\mathbf{k}) \subset P_{\hat{0}}$ is the joint stabilizer of $t^j E_1$ and $t^j E'_1$, we have $X^{\circ\prime}_{\mathbf{c}} = \mathrm{GL}_n(\mathbf{k}) \cdot \tau^{\mathbf{c}} E_1$.

2 Lusztig's isomorphism

In this and the following section, we consider how certain varieties of matrices may be considered as opposite cells in affine Schubert varieties.

2.1 Nilpotent matrices

Let $\mathcal{N} \subset M_{n\times n}(\mathbf{k})$ be the set of nilpotent $n \times n$ complex matrices, on which $GL_n(\mathbf{k})$ acts by conjugation. Lusztig [9] has given an equivariant algebraic isomorphism between $\mathcal{N}$ and the opposite cell of a Schubert variety in $\mathrm{Gr}(V)$.

A matrix in $GL_n(\mathbf{k})$ has a natural A-linear action on V, and for $N \in \mathcal{N}$ we can define $\phi_N : V \to V$,

$$\begin{aligned} \phi_N(v) \;&:=\; \frac{t^{n-1}}{1-t^{-1}N}(v) \\ &=\; t^{n-1}v + t^{n-2}N(v) + t^{n-3}N^2(v) + \cdots + N^{n-1}(v). \end{aligned}$$

Lusztig's isomorphism is given by the map

$$\begin{array}{rccc} \Phi : & \mathcal{N} & \to & \mathrm{Gr}(V) \\ & N & \mapsto & \phi_N(E_1). \end{array}$$

Note that Φ is $GL_n(\mathbf{k})$-equivariant: for $g \in GL_n(\mathbf{k})$, we have $\Phi(gNg^{-1}) = g\,\phi_N(g^{-1}E_1) = g\,\phi_N(E_1) = g\,\Phi(N)$. We also have $E_1 \supset \Phi(N) \supset t^n E_1$ for all $N \in \mathcal{N}$.

We may parametrize the $GL_n(\mathbf{k})$-orbits in $\mathcal{N}$ by n-tuples of integers $\mathbf{b} = (b_1, \ldots, b_n)$, where $n \geq b_1 \geq \cdots \geq b_n \geq 0$ and $b_1 + \cdots + b_n = n$. That is, the orbit $\mathcal{N}_{\mathbf{b}} \subset \mathcal{N}$ consists of those nilpotents whose largest Jordan block has size b_1, the next largest has size b_2, etc. The open orbit of principal nilpotents is $\mathcal{N}_{(n,0,\ldots,0)}$, the closed orbit $\{0\} = \mathcal{N}_{(1,\ldots,1)}$.

Let $\mathbf{c} = (n - b_1, \ldots, n - b_n)$. Applying elementary linear algebra to the description of $X_{\mathbf{c}}^{\circ\prime}$ in the previous section, we may easily show that:

$$\Phi(\mathcal{N}_{\mathbf{b}}) = X_{\mathbf{c}}^{\circ\prime} \qquad \text{and} \qquad \Phi(\overline{\mathcal{N}}_{\mathbf{b}}) = X'_{\mathbf{c}},$$

where $\overline{\mathcal{N}}_{\mathbf{b}}$ denotes the closure. In particular,

$$\Phi(\mathcal{N}) = X'_{(0,n,\cdots,n)} = \{\Lambda \in \mathrm{Gr}(V) \mid \Lambda \overset{n}{\supset} t^n E_1\}.$$

Example. Note that we can renormalize our map by a τ-shift: $\Phi(\overline{\mathcal{N}}_{\mathbf{b}}) = X'_{\mathbf{c}} \cong X'_{\tau^j\mathbf{c}}$ (equivariant isomorphism) for any $j \in \mathbb{Z}$. Thus for the example of §3, §4, we have: $X'_{(-1,0,1)} \cong X'_{(1,2,3)} \cong \Phi(\overline{\mathcal{N}}_{(2,1,0)})$.

Let us write our map in coordinates. We represent $v = \sum_{i=1}^{\infty} a_i e_i \in E_1$ (where $a_i \in \mathbf{k}$) by the semi-infinite column vector with entries a_i; and $\Lambda =$

$\mathrm{Span}_A\langle v_1, \ldots, v_n\rangle \in \mathrm{Gr}(V)$ by the semi-infinite matrix $[v_1, \cdots, v_n]$. Then we may write

$$\Phi(N) \quad = \quad \begin{bmatrix} N^{n-1} \\ \vdots \\ N^2 \\ N \\ I \\ 0 \\ \vdots \end{bmatrix},$$

where I is the identity matrix. From this we see how the *Plucker coordinates* (the $n \times n$ minors of this matrix) restrict to polynomial functions on $\mathcal{N}$. In particular, since the vanishing ideal of the Schubert subvarieties $X_{\mathbf{c}} \subset \mathrm{Gr}(V)$ is generated by the vanishing of certain Plucker coordinates, we obtain generators for the ideal of $\overline{\mathcal{N}}_{\mathbf{b}} \subset \mathcal{N}$. (Cf. Weyman [17]).

2.2 Cyclic quivers

Lusztig [10] has generalized the above isomorphism (and simultaneously another isomorphism of Zelevinsky [18], [8]). The generalization involves a positive integer parameter h, with the case $h = 1$ reducing to our discussion of nilpotent matrices.

The *cyclic quiver* $\widehat{A}_{h-1}$ is the oriented graph:

$$\begin{tikzcd} 1 \arrow[rrr, bend right] & 2 \arrow[l] & \cdots \arrow[l] & h \arrow[l] \end{tikzcd}$$

For a fixed h-tuple of positive integers $\mathbf{d} = (d_1, \cdots, d_h)$, we define the **d**-*dimensional representations* of this quiver to be the affine space

$$M_{d(h)\times d(1)}(\mathbf{k}) \times M_{d(1)\times d(2)}(\mathbf{k}) \times \cdots \times M_{d(h-1)\times d(h)}(\mathbf{k}).$$

(For legibility, we have written $d(j)$ instead of d_j.) That is, a representation $(M_1, \ldots, M_h)$ is a way of replacing each arrow $i \to i-1$ by a linear map $M_i : \mathbf{k}^{d(i)} \to \mathbf{k}^{d(i-1)}$, where we take $d_0 := d_h$. (For all j, k, we write $d_{j+hk} := d_j$.) We have a natural action of the group $\mathrm{GL}_{\mathbf{d}}(\mathbf{k}) := \mathrm{GL}_{d(1)}(\mathbf{k}) \times \cdots \times \mathrm{GL}_{d(h)}(\mathbf{k})$ on the space of representations:

$$(g_1, \ldots, g_h) \cdot (M_1, \ldots, M_h) := (g_h M_1 g_1^{-1}, g_1 M_2 g_2^{-1}, \ldots, g_{h-1} M_h g_h^{-1}).$$

Our main concern is a certain $\mathrm{GL}_{\mathbf{d}}(\mathbf{k})$-stable subvariety $\mathcal{M}$ of the representations, the space of *nilpotent* representations). We define:

$$\mathcal{M} = \mathcal{M}^{\mathbf{d}} := \{(M_1, \ldots, M_h) \mid M_1 M_2 \cdots M_h \in M_{d(h)\times d(h)}(\mathbf{k}) \text{ is nilpotent}\}.$$

The condition is equivalent to $M_{j+1} M_{j+2} \cdots M_h M_1 \cdots M_j \in M_{d(j)\times d(j)}(\mathbf{k})$ being nilpotent for any j. In general $\mathcal{M}$ is a connected but reducible variety. I believe it has at most h components, all of equal dimension.

Examples. (i) For $\mathbf{d}=(1,1,1)$, we have $\mathcal{M}=\{(m_1,m_2,m_3)\in\mathbf{k}^3 \mid m_1m_2m_3=0\}$, the union of the three coordinate planes. Similarly for $\mathbf{d}=(1^n)$.
(ii) For $\mathbf{d}=(2,1,1)$, we have $\mathcal{M}=\{([m_1,m_1'],[m_2,m_2']^T,m_3) \mid (m_2m_2+m_1'm_2')\,m_3=0\}$, with two irreducible components of dimension four.

We define an isomorphism from $\mathcal{M}$ to a union of opposite cells of Schubert varieties in $\mathrm{Fl}(\mathbf{d},V)$. Here we take $n:=d_1+\cdots+d_h$, so that $\mathbf{d}$ is a composition of n, and we consider $V=V_1\oplus\cdots\oplus V_n$, where $V_j=F^{d(j)}$.

First, we embed

$$M_{d(h)\times d(1)}(\mathbf{k})\times\cdots\times M_{d(h-1)\times d(h)}(\mathbf{k}) \hookrightarrow G=GL_n(F),$$

$$M=(M_1,\ldots,M_h)\longmapsto \tilde{M}:=\begin{pmatrix} 0 & M_2 & 0 & \cdots & 0\\ 0 & 0 & M_3 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & M_h\\ t^{-1}M_1 & 0 & 0 & \cdots & 0\end{pmatrix}.$$

That is, $\tilde{M}=\tilde{M}_1+\cdots+\tilde{M}_h$, where $\tilde{M}_i:V\to V$,

$$\tilde{M}_j(v):=\begin{cases} t^{-\delta_{1j}}M_j(v) & \text{for } v\in V_j\\ 0 & \text{for } v\in V_k,\ k\neq j.\end{cases}$$

We adopt the notations $\tilde{M}_{j+hk}:=\tilde{M}_j$, and:

$$\tilde{M}_j^{[k]}:=\underbrace{\tilde{M}_{j-k+1}\cdots\tilde{M}_{j-1}\tilde{M}_j}_{k\text{ factors}}.$$

Note that $\tilde{M}_j\tilde{M}_k=0$ unless $k=j+1$, and for $M\in\mathcal{M}$, we have $\tilde{M}_j^{[hd(j)]}=(\tilde{M}_j^{[h]})^{d(j)}=0$, so that $\tilde{M}^n=0$. Now we can define $\psi_M:V\to V$,

$$\begin{aligned}\psi_M(v) &:= \frac{t^{n-1}}{1-\tilde{M}}(v)\\ &= t^{n-1}\big(\, v_1+\tilde{M}_1(v_1)+\tilde{M}_h\tilde{M}_1(v_1)+\cdots+\tilde{M}_1^{[nh-h]}(v_1)\\ &\qquad + v_2+\tilde{M}_2(v_2)+\tilde{M}_1\tilde{M}_2(v_2)+\cdots+\tilde{M}_2^{[nh-h+1]}(v_2)\\ &\qquad + \cdots\\ &\qquad + v_h+\tilde{M}_h(v_h)+\tilde{M}_{h-1}\tilde{M}_h(v_h)+\cdots+\tilde{M}_h^{[nh-1]}(v_h)\,\big),\end{aligned}$$

where $v=v_1+\cdots+v_h$ with $v_j\in V_j$.

Recall $E_{(\bullet)}=(E_{(1)}\supset\cdots\supset E_{(h)})$, the standard flag in $\mathrm{Fl}(\mathbf{d},V)$, where $E_{(j)}:=E_{1+d(1)+\cdots+d(j-1)}$ and $E^{(j)}\oplus E'_{(j)}=V$. Then Lusztig's isomorphism is given by the map:

$$\begin{array}{rccc}\Psi: & \mathcal{M} & \to & \mathrm{Fl}(\mathbf{d},V)\\ & M & \mapsto & \psi_M(E_{(\bullet)})\end{array}$$

where

$$\psi_M(E_{(\bullet)}):=(\,\psi_M(E_{(1)})\supset\cdots\supset\psi_M(E_{(h)})\,).$$

We give three coordinate descriptions of Ψ. First, consider the decomposition, $E_1 = \mathbf{k}^{d(1)} \oplus \cdots \oplus \mathbf{k}^{d(h)} \oplus t\mathbf{k}^{d(1)} \oplus \cdots$, so that we can write $V \ni v = u_1 + \cdots + u_h + tu_{h+1} + \cdots$ with $u_i \in \mathbf{k}^{d(i) \mod n}$. Then we may write $\Psi(M) = (\Lambda_1 \supset \cdots \supset \Lambda_h)$ with

$$\Lambda_j = \{u_1 + \cdots + u_h + tu_{h+1} + \cdots \mid u_{i-1} = M_i(u_i) \ \ \forall\, i \le nh - h + j\}\,.$$

Second, we write a partial flag $(\Lambda_1 \supset \cdots \supset \Lambda_h)$ by a semi-infinite matrix of n column-vectors $[v_1, \cdots, v_n]$ which are compatible with all the lattices in the flag: that is, $\Lambda_j = \mathrm{Span}_{\mathbf{k}} \langle v_i \rangle_{i \ge 1 + d(1) + \cdots + d(j-1)}$. We will write $[v_1, \cdots, v_n]$ as a block matrix with blocks of sizes $d_1, \cdots, d_h$. Let I_j be an identity matrix of size d_j, and denote $M_{j+hk} := M_j$, $M_j^{[k]} := M_{j-k+1} \cdots M_{j-1} M_j$. Then:

$$\Psi(M) \quad = \quad \begin{bmatrix} M_1^{[nh-h]} & M_2^{[nh-h+1]} & \cdots & M_h^{[nh-1]} \\ \vdots & \vdots & & \vdots \\ M_1 & M_1 M_2 & \cdots & M_h^{[h]} \\ I_1 & M_2 & \cdots & M_h^{[h-1]} \\ 0 & I_2 & \cdots & M_h^{[h-2]} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I_h \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

Third, taking bases adapted to each lattice Λ_j (i.e. performing column reduction on the above matrix), we obtain:

$$\Lambda_1 = \begin{bmatrix} M_1^{[nh-h]} & 0 \cdots & 0 \\ \vdots & \vdots & \vdots \\ M_h M_1 & 0 \cdots & 0 \\ M_1 & 0 \cdots & 0 \\ I_1 & 0 \cdots & 0 \\ 0 & I_2 \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 \cdots & I_h \\ 0 & 0 \cdots & 0 \\ \vdots & \vdots & \vdots \end{bmatrix}, \qquad \Lambda_2 = \begin{bmatrix} M_2^{[nh-h+1]} & 0 \cdots & 0 \\ \vdots & \vdots & \vdots \\ M_h M_1 M_2 & 0 \cdots & 0 \\ M_1 M_2 & 0 \cdots & 0 \\ M_2 & 0 \cdots & 0 \\ I_2 & 0 \cdots & 0 \\ 0 & I_3 \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 \cdots & I_1 \\ 0 & 0 \cdots & 0 \\ \vdots & \vdots & \vdots \end{bmatrix}, \qquad \text{etc.}$$

From all these descriptions, we see that Ψ is $\mathrm{GL}_{\mathbf{d}}(\mathbf{k})$-equivariant (provided we diagonally embed $\mathrm{GL}_{\mathbf{d}}(\mathbf{k}) \subset \mathrm{GL}_n(\mathbf{k})$). Also $E_1 \supset \psi_M(E_{(1)}) \supset \cdots \supset \psi_M(E_{(h)}) \supset t^n E_1$ for all $M \in \mathcal{M}$.

2.3 Image of Lusztig's isomorphism

The map Ψ embeds $\mathcal{M}$ in $\mathrm{Fl}(\mathbf{d},V)$. We give several descriptions of the image. From the last coordinate description of Ψ, we may easily show:

$$\Psi(\mathcal{M}) = \left\{ \Lambda_\bullet \in \mathrm{Fl}(\mathbf{d},V) \;\middle|\; \begin{array}{c} \begin{array}{ccccccccc} \Lambda_1 & \overset{d(1)}{\supset} & \Lambda_2 & \overset{d(2)}{\supset} & \cdots & \overset{d(h-1)}{\supset} & \Lambda_h & \overset{d(h)}{\supset} & t\Lambda_1 \\ {\scriptstyle d(1)}\cup & & {\scriptstyle d(2)}\cup & & & & {\scriptstyle d(h)}\cup & & \\ t^{n-1}E_{(2)} & \supset & t^{n-1}E_{(3)} & \supset & \cdots & \supset & t^nE_{(1)} & = & t^{n-1}E_{(h+1)} \end{array} \\ \\ \Lambda_1 \cap t^{n-1}E'_{(2)} = \cdots = \Lambda_h \cap t^{n-1}E'_{(h+1)} = 0 \end{array} \right\}.$$

This suggests how to describe the image as a union of opposite cells of Schubert varieties: $\Psi(\mathcal{M}) = \cup_\pi X'_\pi$ for certain $\pi \in \widetilde{W}/W_{\mathbf{d}}$. Let $\mathbb{Z}_{(j)} := \mathbb{Z}_{\geq 1+d(1)+\cdots+d(j-1)}$, and consider the sets $\pi\mathbb{Z}_{(j)}$, $1 \leq j \leq h$, which determine π modulo $W_{\mathbf{d}}$. These sets should contain numbers as small as possible subject to the conditions:

$$\begin{array}{ccccccccc} \pi\mathbb{Z}_{(1)} & \overset{d(1)}{\supset} & \pi\mathbb{Z}_{(2)} & \overset{d(2)}{\supset} & \cdots & \overset{d(h-1)}{\supset} & \pi\mathbb{Z}_{(h)} & \overset{d(h)}{\supset} & \tau\pi\mathbb{Z}_{(1)} \\ {\scriptstyle d(1)}\cup & & {\scriptstyle d(2)}\cup & & & & {\scriptstyle d(h)}\cup & & \\ \tau^{n-1}\mathbb{Z}_{(2)} & \supset & \tau^{n-1}\mathbb{Z}_{(3)} & \supset & \cdots & \supset & \tau^n\mathbb{Z}_{(1)} & & \end{array},$$

where $A \overset{d}{\supset} B$ means $A \supset B$ and $\#(A \setminus B) = d$. There should be at most h such permutations π which are Bruhat-maximal. One can construct them by first maximizing a particular $\pi\mathbb{Z}_{(j)}$, then constructing the rest of the sets, which might not always be possible.

Example. For $\mathbf{d} = (1,1,1)$, $n = 3$, we have $\mathbb{Z}_{(j)} = \mathbb{Z}_{\geq j}$, and the three irreducible components are $\pi_1 = [5,9,10] = [2,3,1]\tau^{(1,2,3)}$; $\pi_2 = [8,6,10] = [2,3,1]\tau^{(2,3,1)}$; $\pi_3 = [9,8,7] = [2,3,1]\tau^{(2,2,2)}$. Each π_j is obtained by filling $\pi\mathbb{Z}_{(j)}$ with numbers as small as possible subject to $\pi\mathbb{Z}_{(j)} \overset{d(j)}{\supset} t^{n-1}E_{(j+1)}$, then constructing the rest of the $\pi\mathbb{Z}_{(k)}$.

Specifically, to get $\pi = \pi_1$, we start with $\pi\mathbb{Z}_{\geq 1} \overset{1}{\supset} \mathbb{Z}_{\geq 8}$, yielding $\pi\mathbb{Z}_{\geq 1} = \{5,8,9,10,\ldots\}$. Next $\pi\mathbb{Z}_{\geq 1} \overset{1}{\supset} \pi\mathbb{Z}_{\geq 2} \overset{1}{\supset} \mathbb{Z}_{\geq 9}$, yielding $\pi\mathbb{Z}_{\geq 2} = \{8,9,10,11,\ldots\}$. Similarly $\pi\mathbb{Z}_{\geq 3} = \{8,10,11,12,\ldots\}$, and we conclude $\pi = [5,9,10]$.

The other components are obtained similarly. Note that since the d_j are all equal to a constant $d = 1$, the automorphism $\sigma^d = \sigma$ acts on $\Psi(\mathcal{M})$. In fact, $\pi^{(1)} = \sigma\pi^{(2)}\sigma^{-1} = \sigma^2\pi^{(3)}\sigma^{-2}$.

For a given $\mathbf{d}$, the $GL_{\mathbf{d}}(\mathbf{k})$-orbits of $\mathcal{M}$ are distinguished from each other by certain collections of invariants, the *rank numbers* $\mathbf{r} = (r_j^k)$, where

$$r_j^k := \mathrm{rank}(M_j^{[k]} : \mathbf{k}^{d(j)} \to \mathbf{k}^{d(j-k)}) \quad \text{for} \quad 1 \leq j \leq h,\ 1 \leq k \leq (n-1)h\,.$$

(In fact, it suffices to consider $k < h \cdot \min(d_1, \ldots, d_h)$.) We also define $r_j^0 := d_j$ and $r_j^k := 0$ if not otherwise defined.

For a given collection of rank numbers $\mathbf{r} := (r_{jk})_{jk}$, we denote the corresponding nilpotent quiver orbit by $\mathcal{M}^{\mathbf{r}} \subset \mathcal{M}$. Not every collection of rank numbers is realized by an orbit: rather, the nonnegative integer entries in $\mathbf{r}$ must obey the constraints:

$$m_j^k := r_j^k - r_j^{k+1} - r_{j-1}^{k+1} + r_{j-1}^{k+2} \ \geq\ 0\,.$$

[Note: m_j^k is the multiplicity in $(M_1, \dots, M_h)$ of the indecomposable quiver summand I_j^k defined as follows: letting $\bar{i} := i \bmod h$, we define vector spaces $U_1, \dots, U_h$ by $\oplus_{l=1}^h U_l := \mathrm{Span}_{\mathbf{k}}\langle e_{j-i}\rangle_{0\leq i\leq k}$ with $e_i \in U_{\bar{i}}$; and maps $L_{\bar{i}} : U_{\bar{i}} \to U_{\overline{i-1}}$ with $L_{\bar{i}}(e_i) := e_{i-1}$ and $L_{\overline{j-k}}(e_{j-k}) := 0$. Thus for $h = 1$, the representation I_j^k reduces to a nilpotent Jordan block of size $k+1$.]

The image of a quiver orbit under Ψ can be described as:

$$\Psi(\mathcal{M}^{\mathbf{r}}) = \left\{ \Lambda_\bullet \in \mathrm{Fl}(\mathbf{d}, V) \;\middle|\; \begin{array}{c} \begin{array}{ccccccccc} \Lambda_1 & \overset{d(1)}{\supset} & \Lambda_2 & \overset{d(2)}{\supset} & \cdots & \overset{d(h-1)}{\supset} & \Lambda_h & \overset{d(h)}{\supset} & t\Lambda_1 \\ {}_{d(1)}\cup & & {}_{d(2)}\cup & & & & {}_{d(h)}\cup & & \\ t^{n-1}E_{(2)} & \supset & t^{n-1}E_{(3)} & \supset & \cdots & \supset & t^n E_{(1)} & & \end{array} \\[2ex] \Lambda_1 \cap E_1' = \cdots = \Lambda_h \cap E_h' = 0 \\[1ex] \dim(\Lambda_j/\Lambda_j \cap E_{(k)}) = r_j^{(nh-h+j-k+1)} \quad \forall j,k \end{array} \right\}$$

Here we take $E_{(j+hk)} := t^k E_{(j)}$. The image of the orbit closure $\overline{\mathcal{M}^{\mathbf{r}}}$ is obtained by replacing $=$ by $\leq$ in the Schubert conditions. From this, we may deduce that $\Psi(\mathcal{M}^{\mathbf{r}}) = X_\pi^{\circ\prime}$ and $\Psi(\overline{\mathcal{M}^{\mathbf{r}}}) = X_\pi'$ for a certain explicitly constructable $\pi = \pi^{\mathbf{r}} \in \widetilde{W}/W_{\mathbf{d}}$.

3 The Variety of circular complexes

3.1 Circular complexes and Lusztig's isomorphism

We apply the previous constructions to a particularly simple, but interesting case, intensively considered from a different point of view by Mehta and Trivedi [12]. For positive integers $a \leq b$, we consider the variety of *two-step circular complexes* or *loop-complexes*:

$$\mathcal{L} = \mathcal{L}_{a,b} := \{(X, Y) \in M_{b\times a}(\mathbf{k}) \times M_{a\times b}(\mathbf{k}) \mid XY = 0,\ YX = 0\}.$$

Recall that any finite linear chain-complex can be "rolled up" into such a two-step complex by letting $\mathbf{k}^a$ (resp. $\mathbf{k}^b$) be the direct sum of all the odd-numbered (resp. even-numbered) spaces in the linear complex. This gives a natural map from the variety of chain-complexes to $\mathcal{L}$.

Now, $\mathcal{L}$ is a subvariety of the representations of the affine quiver $\widehat{A}_1$; a subvariety which is invariant under the natural action of the group $GL_{a,b}(\mathbf{k}) := GL_a(\mathbf{k}) \times GL_b(\mathbf{k})$, namely $(g_a, g_b) \cdot (X, Y) := (g_b X g_a^{-1}, g_a Y g_b^{-1})$. We easily see that $\mathcal{L}$ is a finite union of $GL_{a,b}(\mathbf{k})$-orbits. In fact, $\mathcal{L}$ has exactly $a+1$ open orbits $\mathcal{L}_0^\circ, \ldots, \mathcal{L}_a^\circ$, whose closures give the $a+1$ irreducible components of $\mathcal{L}$:

$$\mathcal{L}_c^\circ := \{(X,Y) \in \mathcal{L} \mid \operatorname{rank} X = c,\ \operatorname{rank} Y = a-c\}, \qquad \mathcal{L}_c := \overline{\mathcal{L}_c^\circ}\,.$$

We define an isomorphism from $\mathcal{L}$ to a union of opposite cells of Schubert varieties in the partial affine flag variety $\mathrm{Fl}(a,b;V)$, where $V = F^n$ and $n = a+b$. Our notation will emphasize the block decomposition $V = F^a \oplus F^b$, as well as:

$$E = \mathbf{k}^a \oplus \mathbf{k}^b \oplus t\mathbf{k}^a \oplus t\mathbf{k}^b \oplus \cdots\ .$$

In this case, Lusztig's isomorphism is given by the map $\Psi : \mathcal{L} \to G/P_a \cong \mathrm{Fl}(a,b;V)$ as:

$$\Psi(X,Y) := \begin{pmatrix} tI_a & tY \\ X & tI_b \end{pmatrix} \mod P_{\widehat{0,a}}$$

where I_m is an identity matrix of size m; or in terms of lattices, $\Psi(X,Y) = (\Lambda_1 \supset \Lambda_2)$, where $E \supset \Lambda_i$ and:

$$\Lambda_1 \ = \begin{bmatrix} 0 & 0 \\ X & 0 \\ I_a & Y \\ 0 & I_b \\ 0 & 0 \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ X & 0 \\ I_a & 0 \\ 0 & I_b \\ 0 & 0 \\ \vdots & \vdots \end{bmatrix} \mod P\ , \qquad \Lambda_2 \ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ Y & 0 \\ I_b & X \\ 0 & I_a \\ 0 & 0 \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ Y & 0 \\ I_b & 0 \\ 0 & I_a \\ 0 & 0 \\ \vdots & \vdots \end{bmatrix} \mod P\ .$$

Here the column vectors of each matrix give an A-basis for $\Lambda_i \subset E$, written with respect to $E = \operatorname{Span}_{\mathbf{k}}\langle e_i \rangle_{i \geq 1}$. The blocks have sizes $a, b, a, b, \ldots$. The map Ψ is $GL_{a,b}(\mathbf{k})$-equivariant, provided we embed $GL_{a,b}(\mathbf{k}) \subset GL_n(\mathbf{k}) \subset G$ as block-diagonal matrices with constant coefficients.

We easily deduce:

$$\Psi(\mathcal{L}_c) = \left\{ \Lambda_\bullet \in \mathrm{Fl}(a,b;V) \;\middle|\; \begin{array}{ccccc} E_{(2)} & \supset & E_{(3)} & & \\ {\scriptstyle b}\cup & & {\scriptstyle a}\cup & & \\ \Lambda_1 & \overset{a}{\supset} & \Lambda_2 & \overset{b}{\supset} & t\Lambda_1 \\ {\scriptstyle a}\cup & & {\scriptstyle b}\cup & & \\ E_{(4)} & \supset & E_{(5)} & & \end{array} \quad \begin{array}{c} \Lambda_1 \cap E'_{(3)} = \Lambda_2 \cap E'_{(4)} = 0 \\ \dim(\Lambda_1/\Lambda_1 \cap E_{(3)}) \leq c \\ \dim(\Lambda_2/\Lambda_2 \cap E_{(4)}) \leq a-c \end{array} \right\}.$$

Here

$$E_{(1)} = E_1,\ \ E_{(2)} = E_{a+1},\ \ E_{(3)} = E_{a+b+1},\ \ E_{(4)} = E_{2a+b+1},\ \ E_{(5)} = E_{2a+2b+1}\ .$$

From this, we may also realize $\mathcal{L}_c$ as a subset of an ordinary flag variety $\mathrm{Fl}(b,a,b;\mathbf{k}^{a+2b})$, the variety of partial flags

$$\mathbf{k}^{a+2b} \overset{b}{\supset} U_1 \overset{a}{\supset} U_2 \overset{b}{\supset} 0\ .$$

In fact, let $\mathbf{k}^{a+2b} = E_{(2)}/E_{(5)}$, with $\mathbf{k}$-basis $\{\bar{e}_{a+1}, \ldots, \bar{e}_{2a+2b}\}$, where $\bar{e}_i := e_i$ mod $E_{(5)}$. Thus $\bar{E}_{(3)} \supset \bar{E}_{(4)}$ is the standard flag in $\mathrm{Fl}(b, a, b; \mathbf{k}^{a+2b})$. We also define the nilpotent linear operator $\bar{t} : \mathbf{k}^{a+2b} \to \mathbf{k}^{a+2b}$ by $\bar{t}(\bar{e}_i) = \bar{e}_{i+n}$ mod $E_{(5)}$. Then we have:

$$\mathcal{L}_c \cong \Psi(\mathcal{L}_c) \cong \left\{ U_\bullet \in \mathrm{Fl}(b, a, b; \mathbf{k}^{a+2b}) \;\middle|\; \begin{array}{c} U_1 \supset \bar{E}_{(4)}, \quad U_2 \subset \bar{E}_{(3)}, \\ \dim(U_1 \cap \bar{E}_{(3)}) \geq a+b-c \\ \dim(U_2 \cap \bar{E}_{(4)}) \geq b-a+c \\ U_1 \cap \bar{E}'_{(3)} = U_2 \cap \bar{E}'_{(4)} = 0 \\ U_2 \supset \bar{t}(U_1) \end{array} \right\}$$

This is precisely the opposite cell of a Schubert variety in $\mathrm{Fl}(b, a, b; \mathbf{k}^{a+2b})$, but with the additional algebraic incidence condition $U_2 \supset \bar{t}(U_1)$, which can be written in terms of the Plucker coordinates of U_1, U_2.

3.2 Affine permutations for circular complexes

We wish to identify the image of $\mathcal{L}_c$ as the opposite cell of an affine Schubert variety,

$$\Psi(\mathcal{L}_c) = X'_\pi\,, \quad \text{for some} \quad \pi = \pi_c \in W_a \backslash \widetilde{W} / W_a\,.$$

First, we construct the sets $\pi\mathbb{Z}_{(1)}, \pi\mathbb{Z}_{(2)}$, which then determine π modulo W_a. These sets should contain numbers as small as possible subject to the conditions:

$$\begin{array}{ccccc} \mathbb{Z}_{(2)} & \supset & \mathbb{Z}_{(3)} & & \\ {}^{b}\cup & & {}^{a}\cup & & \\ \pi\mathbb{Z}_{(1)} & \overset{a}{\supset} & \pi\mathbb{Z}_{(2)} & \overset{b}{\supset} & \tau\pi\mathbb{Z}_{(1)} \\ {}^{a}\cup & & {}^{b}\cup & & \\ \mathbb{Z}_{(4)} & \supset & \mathbb{Z}_{(5)} & & \end{array} \qquad \begin{array}{c} \#(\pi\mathbb{Z}_{(1)} \backslash \mathbb{Z}_{(3)}) = c \\ \#(\pi\mathbb{Z}_{(2)} \backslash \mathbb{Z}_{(4)}) = a-c \end{array} \quad ,$$

To construct π according to these constraints, we will divide $[1, n]$ into intervals (blocks) of the form:

$$\underbrace{i, \ldots, j}_{(k)} := [i,\, i{+}1, \ldots, j],$$

where $k = j{-}i{+}1$ is the number of integers in the interval. We perform two subdivisions as follows:

$$\begin{aligned}[1, \ldots, n] := [\ & \underbrace{1, \ldots, c}_{\mathrm{I}\ (c)}, \underbrace{c{+}1, \ldots, a}_{\mathrm{II}\ (a-c)}, \underbrace{a{+}1, \ldots, 2a{-}c}_{\mathrm{III}\ (a-c)}, \\ & \underbrace{2a{-}c{+}1, \ldots, a{+}b{-}c}_{\mathrm{IV}\ (b-a)}, \underbrace{a{+}b{-}c{+}1, \ldots, a{+}b}_{\mathrm{V}\ (c)}\ ] \end{aligned} \quad ,$$

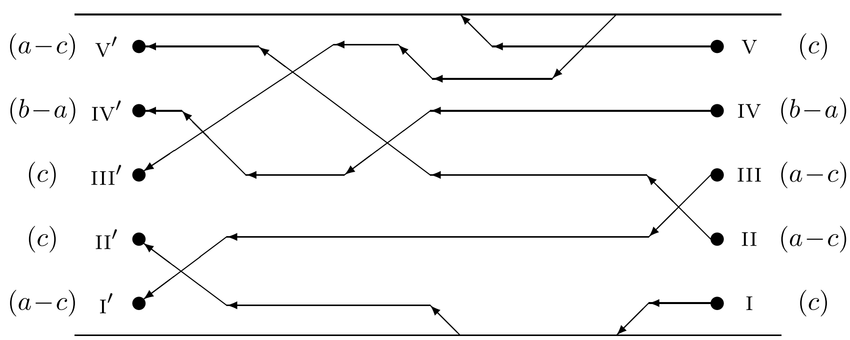
$$\begin{aligned}[1, \ldots, n] := [\ & \underbrace{1, \ldots, a{-}c}_{\mathrm{I}'\ (a-c)}, \underbrace{a{-}c{+}1, \ldots, a}_{\mathrm{II}'\ (c)}, \underbrace{a{+}1, \ldots, a{+}c}_{\mathrm{III}'\ (c)}, \\ & \underbrace{a{+}c{+}1, \ldots, b{+}c}_{\mathrm{IV}'\ (b-a)}, \underbrace{b{+}c{+}1, \ldots, a{+}b}_{\mathrm{V}'\ (a-c)}\ ] \end{aligned} \quad ,$$

where we have numbered the blocks with roman numerals. Now π takes the first set of blocks to the second set, as well as shifting them by powers of τ:

$$\pi := [\ \underbrace{a{+}1,\ldots,a{+}c}_{\mathrm{V}'\ (c)}, \underbrace{a{+}2b{+}c{+}1,\ldots,2a{+}2b}_{\tau(\mathrm{III}')\ (a-c)}, \underbrace{a{+}b{+}1,\ldots,2a{+}b{-}c}_{\tau(\mathrm{II}')\ (a-c)},$$
$$\underbrace{2a{+}b{+}c{+}1,\ldots,a{+}2b{+}c}_{\tau(\mathrm{IV}')\ (b-a)}, \underbrace{3a{+}2b{-}c{+}1,\ldots,3a{+}2b}_{\tau^2(\mathrm{I}')\ (c)}\ ] .$$

Recall that π represents the double coset $W_a\pi W_a$: in fact, π is maximal with respect to the left action of W_a, and minimal with respect to the right action. This is the correct normalization so that $\ell(\pi_c) = \dim_{\mathbf{k}}(\mathcal{L}_c)$.

To analyze the decomposition of π into simple reflections, we construct its loop wiring diagram. As before, the strip below (with top and bottom edges identified) represents a cylinder with $n = a + b$ dots on either end. For each i we write $\pi(i) = \bar{\pi}(i) + nj$, and we draw a wire connecting the dot i on the right to the dot $\bar{\pi}(i)$ on the left, but looping upwards (around the cylinder) j times. We will group the wires into five *cables* corresponding to our blocks I, ... ,V (on the right) and I′, ... , V′ (on the left), so that the cable starting at I represents c non-crossing wires, etc. As a final simplification, instead of drawing the diagram for π, we instead draw the diagram for $\tau^{-1}\pi$ (a harmless normalization, since τ is in the center of $\widetilde{W}$).



Whenever a cable with k wires crosses one with k' wires, we have a total of kk' wire crossings. Thus the six cable-crossings of our picture give a wire-crossing total of:

$$\ell(\pi) = (a-c)^2 + c^2 + (a-c)(b-a) + 2c(a-c) + c(b-a) = ab\,,$$

which we may confirm by checking directly that $\dim(\mathcal{L}_c) = ab$.

Now we may write a reduced decomposition for π as follows. For integers i, k, define the affine permutation $s_i^{[k]} := s_i s_{i-2} \cdots s_{i-2k+2}$, which has k mutually commuting factors. Recall our convention $s_{i+nj} := s_i$. For each cable crossing:

$$
\begin{array}{ccc}
 & \underbrace{i+1,\dots,i+k}_{(k)} & & \underbrace{i+1,\dots,i+j}_{(j)} & \\
 & & \times & & \\
 \underbrace{i+k+1,\dots,i+j+k}_{(j)} & & & & \underbrace{i+j+1,\dots,i+j+k}_{(k)}
\end{array}
$$

we define the associated "totally commutative" permutation:

$$s_{i+1}^{[j,k]} \;:=\; s_{i+k}^{[1]} s_{i+k+1}^{[2]} \cdots s_{i+k+m}^{[\min(m,j,k)]} \cdots s_{i+j+k}^{[\min(j,k)]} \cdots s_{i+j+m'}^{[\min(m',j,k)]} \cdots s_{i+j+1}^{[2]} s_{i+j}^{[1]}.$$

Finally, we can write

$$\pi = \tau\, s_1^{[a-c,c]}\, s_{a+1}^{[b-a,c]}\, s_{b+1}^{[a-c,c]}\, s_{a+1}^{[a-c,b-a]}\, s_{b+a-c+1}^{[c,c]}\, s_{c+1}^{[a-c,a-c]}\ ,$$

where the six factors (other than τ) correspond to the cable crossings, listed left to right.

3.3 Bott-Samelson resolution

We can use the above data to give a Bott-Samelson resolution of singularities for $\mathcal{L}_c$. Although this is clearly far from a minimal resolution, it brings the circular complexes into the framework of Frobenius splittings and other results for Bott-Samelson varieties (cf. Mathieu [11], Kumar [7], Ramanathan [14], …). In particular, we have the following results proved by Mehta-Trivedi [12]

Theorem *The variety of circular complexes $\mathcal{L}_c$ and the closures of all its $GL_n(\mathbf{k})$-orbits are normal, Cohen-Macaulay, and have rational singularities.*

The construction of the affine Bott-Samelson variety $Z_{\mathbf{i}}$ corresponding to a reduced word $\mathbf{i} = (i_1, \dots, i_l)$ is exactly analogous to (and includes as a special case) the construction for $GL_n(\mathbf{k})$. (Cf. §1.4.)

We illustrate with the simplest example in our case: $c = 1$, $a = 2$, $b = 3$, $n = 5$ so that each of the blocks I, … , V has size 1, and our cable diagram in the previous section is a simple wiring diagram. Then $\pi = s_1 s_3 s_4 s_3 s_0 s_2$, and the Bott-Samelson variety is:

$$
Z_{\mathbf{i}} \;:=\; \left\{ \begin{array}{c} (\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_4') \\ \in \mathrm{Gr}(V)^6 \end{array} \;\middle|\; \begin{array}{ccccccccccc}
 & & & & & E_4 & & & & & \\
 & & & & \swarrow & & \nwarrow & & & & \\
 E_1 & \leftarrow & E_2 & \leftarrow & E_3 & \leftarrow & \Lambda_4' & \leftarrow & E_5 & \leftarrow & tE_1 \\
 & \nwarrow & & \swarrow & & \nwarrow & & \nwarrow & & \swarrow & \\
 \Lambda_1 & \leftarrow & \Lambda_2 & \leftarrow & \Lambda_3 & \leftarrow & \Lambda_4 & \leftarrow & \Lambda_5 & \leftarrow & t\Lambda_1
\end{array} \right\}.
$$

Here $E_j := \mathrm{Span}_{\mathbf{k}}\langle e_i \rangle_{i \geq j}$, and each arrow $U \leftarrow V$ indicates the conditions $U \supset V$, $\dim_{\mathbf{k}}(U/V) = 1$. We construct a point of $Z_{\mathbf{i}}$ by starting with the standard flag $E_1 \supset E_3 \supset \cdots$, and successively choosing the spaces Λ_2, Λ_4', Λ_5, Λ_4, Λ_1,

Λ_3, corresponding to the letters 1,3,4,3,0,2. Each such choice corresponds to a fibration with fiber $\mathbb{P}^1$, hence $Z_{\mathbf{i}}$ is smooth of dimension $\ell(\pi) = 6$. As we did for X_π, we can embed $Z_{\mathbf{i}}$ into a finite-dimensional flag variety for the $\mathbf{k}$-vector space $t^{-1}E_3/tE_1$, since $t^{-1}E_3 \supset \Lambda \supset tE_1$ for $\Lambda = \Lambda_1, \ldots, \Lambda_5, \Lambda_4'$.

We can define a regular, birational map of $Z_{\mathbf{i}}$ onto X_π by forgetting all the spaces except (Λ_1, Λ_3). This map is generically one-to-one because generically all the spaces are determined by Λ_1, Λ_3: that is, $\Lambda_2 = \Lambda_1 \cap E_1$, $\Lambda_5 = t\Lambda_1 + tE_1$, etc. To desingularize the opposite cell $X_\pi' \cong \mathcal{L}_c$, we consider the subset of $Z_{\mathbf{i}}$ where Λ_1, Λ_3 are generic with respect to the opposite standard flag E_1', E_3'.

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