

Borel-Weil Theorem for Configuration Varieties and Schur Modules

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Abstract

We present a generalization of the classical Schur modules of $GL(n)$ exhibiting the same interplay among algebra, geometry, and combinatorics. A generalized Young diagram D is an arbitrary finite subset of $\mathbf{N} \times \mathbf{N}$. For each D , we define the Schur module S_D of $GL(n)$. We introduce a projective variety \mathcal{F}_D and a line bundle \mathcal{L}_D , and describe the Schur module in terms of sections of \mathcal{L}_D .

For diagrams with the “northeast” property,

$$(i_1, j_1), (i_2, j_2) \in D \Rightarrow (\min(i_1, i_2), \max(j_1, j_2)) \in D,$$

which includes the skew diagrams, we resolve the singularities of \mathcal{F}_D and show analogs of Bott’s and Kempf’s vanishing theorems. Finally, we apply the Atiyah-Bott Fixed Point Theorem to establish a Weyl-type character formula of the form:

$$\text{char}_{S_D}(x) = \sum_t \frac{x^{\text{wt}(t)}}{\prod_{i,j} (1 - x_i x_j^{-1})^{d_{ij}(t)}},$$

where t runs over certain standard tableaux of D .

Our results are valid over fields of arbitrary characteristic.

Introduction

The two main branches of the representation theory of the general linear groups $G = GL(n, F)$ began with the geometric Borel-Weil-Bott theory and the combinatorial analysis of Schur, Young, and Weyl. In the case when F is of characteristic zero, the geometric theory realizes the irreducible representation of G with highest weight λ as the sections of a line bundle \mathcal{L}_λ over the flag variety $\mathcal{F} = G/B$. By contrast, the Schur-Weyl construction produces this representation inside the tensor powers of the standard representation $V = F^n$, by a

process of symmetrization and anti-symmetrization defined by λ considered as a Young diagram. (See [10] for an accessible reference.)

Combinatorists have examined other symmetrization operations on $V^{\otimes k}$, such as those associated to skew diagrams, and recently more general diagrams D of squares in the plane ([1], [14], [18], [19], [24], [30], [31], [32], [33]). We call the resulting G -representations the Schur modules $S_D \subset V^{\otimes k}$. (In characteristic zero, S_D is irreducible exactly when D is a Young diagram.) Kraskiewicz and Pragacz [19] have shown that the characters of S_D , for D running through the inversion diagrams of the symmetric group on n letters, give an algebraic description of the Schubert calculus for the cohomology of the flag variety \mathcal{F} . (More precisely, the Schubert polynomials are characters of *flagged* Schur modules. We deal with this case in [20].)

In this paper, we attempt to combine the combinatorial and the geometric approaches. We give a geometric definition (valid for all characteristics) for the G -module S_D . That is, for any finite set $D \subset \mathbf{N} \times \mathbf{N}$, we produce S_D as the space of sections of a line bundle over a projective variety \mathcal{F}_D , the configuration variety of D . (This is proved only for diagrams with a “direction” property, but a weaker statement is shown for general diagrams.) Our picture reduces to that of Borel-Weil when D is a Young diagram. See also Bozek and Drechsler [7], [8], where similar varieties are introduced. We prove a conjecture of V. Reiner and M. Shimozono asserting the duality between the Schur modules of two diagrams whose disjoint union is a rectangular diagram.

We can carry out a more detailed analysis for diagrams satisfying a direction condition such as the northeast condition

$$(i_1, j_1), (i_2, j_2) \in D \Rightarrow (\min(i_1, i_2), \max(j_1, j_2)) \in D.$$

To accord with the literature, we will deal exclusively with *northwest* diagrams, but since the modules and varieties with which we are concerned do not change (up to isomorphism) if we switch one row of the diagram with another or one column with another, everything we will say applies with trivial modifications to skew, inversion, Rothe, and column-convex diagrams, and diagrams satisfying any direction condition (NE, NW, SE, SW).

In this case, we find an explicit resolution of singularities of \mathcal{F}_D , and we use Frobenius splitting arguments of Wilberd van der Kallen (based on work of Mathieu, Polo, Ramanathan, et al.) to show the vanishing of certain higher cohomology groups. In particular, the configuration varieties are projectively normal and have rational singularities. This allows us to apply the Atiyah-Bott Fixed Point Theorem to compute the character and dimension of the Schur modules.

For more general diagrams, the above program breaks down because we lack a suitable desingularization of \mathcal{F}_D . It can be carried through, however, for diagrams with at most three rows, since in this case we can use the space of triangles [11] as our desingularization. See [16].

Those interested only in the algebraic and combinatorial side of our results can find the definitions and statements in Sections 1, 2.3, 5.2, and 5.3. Our discussion of geometry begins with Section 2.

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1 $GL(n)$ modules

We introduce the Schur-Weyl construction for arbitrary diagrams over fields of any characteristic.

1.1 Schur modules

A *diagram* is a finite subset of $\mathbf{N} \times \mathbf{N}$. Its elements $(i, j) \in D$ are called *squares*, and we picture (i, j) in the i th row and j th column. We shall often think of D as a sequence (C_1, C_2, \dots, C_r) of columns $C_j \subset \mathbf{N}$. The Young diagram corresponding to $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0)$ is the set $\{(i, j) \mid 1 \leq j \leq n, 1 \leq i \leq \lambda_j\}$. For any diagram D , we let

$$\text{Col}(D) = \{\pi \in \Sigma_D \mid \pi(i, j) = (i', j) \exists i'\}$$

be the group permuting the squares of D within each column, and we define $\text{Row}(D)$ similarly for rows.

Let F be a field. We shall always write $G = GL(n, F)$, B = the subgroup of upper triangular matrices, H = the subgroup of diagonal matrices, and $V = F^n$ the defining representation.

Given a finite set T , we will also use the symbol T to denote the order $|T|$ when appropriate. Thus $GL(T) \stackrel{\text{def}}{=} GL(|T|)$, etc. Let Σ_T be the symmetric group permuting the elements of T . For any left G -space X , Σ_T acts on the right, and G acts on the left, of the cartesian product X^T by:

$$g(x_{t_1}, x_{t_2}, \dots)\pi = (gx_{\pi t_1}, gx_{\pi t_2}, \dots).$$

Now let F have characteristic zero. Define the idempotents α_D, β_D in the group algebra $F[\Sigma_D]$ by

$$\alpha_D = \frac{1}{|\text{Row } D|} \sum_{\pi \in \text{Row } D} \pi, \quad \beta_D = \frac{1}{|\text{Col } D|} \sum_{\pi \in \text{Col } D} \text{sgn}(\pi)\pi,$$

where $\text{sgn}(\pi)$ is the sign of the permutation. Define the *Schur module*

$$S_D \stackrel{\text{def}}{=} V^{\otimes D} \alpha_D \beta_D \subset V^{\otimes D},$$

a representation of G . If $D = \lambda$ is a classical Young diagram and the field F has characteristic zero, $S_D = S_\lambda$ is the Schur-Weyl realization of the irreducible $GL(n)$ -module with highest weight λ .

Note that we get an isomorphic Schur module if we change the diagram by permuting the rows or the columns (i.e., for some permutation $\pi : \mathbf{N} \rightarrow \mathbf{N}$, changing $D = \{(i, j)\}$ to $D' = \{(\pi(i), j) \mid (i, j) \in D\}$, and similarly for columns).

1.2 Weyl modules

Let $U = V^*$, the dual of the defining representation of $G = GL(n, F)$, where F is an infinite field. Given a diagram D , define the alternating product *with respect to the columns*

$$\bigwedge^D U = \{f : V^D \rightarrow F \mid f \text{ multilinear, and } f(v\pi) = \text{sgn}(\pi)f(v) \forall \pi \in \text{Col}(D)\},$$

where *multilinear* means $f(v_1, \dots, v_d)$ is F -linear in each of the $d = |D|$ variables. Consider the multidagonal *with respect to the rows*

$$\Delta^D V = \Delta^{R_1} V \times \Delta^{R_2} V \times \dots \subset V^{R_1} \times V^{R_2} \times \dots = V^D,$$

where R_1, R_2, \dots are the rows of D , and $\Delta^R V = \{(v, v, \dots, v)\} \subset V^R$, the total diagonal in a row. Now define the *Weyl module*

$$W_D \stackrel{\text{def}}{=} \bigwedge^D U |_{\Delta^D V},$$

where $|_{\Delta^D V}$ denotes restriction of functions from V^D to $\Delta^D V$. Since $\Delta^D V$ is stable under the diagonal action of G , W_D is naturally a G -module.

Remark. For F a finite field, we make the following modification. Consider $U = U(F) \hookrightarrow U(\bar{F})$, where \bar{F} is the algebraic closure. That is, identify

$$U = \{f : \bar{F}^n \rightarrow \bar{F} \mid f \text{ is } \bar{F}\text{-linear, and } f(F^n) \subset F\}.$$

Then define

$$W_D \stackrel{\text{def}}{=} \bigwedge^D U \big|_{\Delta^D V(\bar{F})}.$$

This keeps the restriction map from killing nonzero tensors which happen to vanish on the finite set $\Delta^D V(F)$.

With this definition, W_D clearly has the base change property $W_D(L) = W_D(F) \otimes_F L$ for any extension of fields $F \subset L$.

Now consider $W_D(\mathbf{Z})$. This is a free \mathbf{Z} -module, since it is a submodule of the \mathbf{Z} -valued functions on $\Delta^D V$. Suppose D satisfies a direction condition. Then our vanishing results of Proposition 25 (a), along with the appropriate universal coefficient theorems, can be used to show that for any field F ,

$$W_D(F) = W_D(\mathbf{Z}) \otimes_{\mathbf{Z}} F.$$

Proposition 1 *If F has characteristic zero, then $W_D \cong S_D^*$ as G -modules.*

Proof. S_D is the image of the composite mapping

$$V^{\otimes D} \alpha_D \hookrightarrow V^{\otimes D} \xrightarrow{\beta_D} V^{\otimes D} \beta_D.$$

For $U = V^*$, write

$$U^{\otimes D} = \{f : V^D \rightarrow F \mid f \text{ multilinear}\},$$

$$\text{Sym}^D U = \{f : V^D \rightarrow F \mid f \text{ multilinear, and } f(v\pi) = f(v) \forall \pi \in \text{Row}(D)\}.$$

Now, representations of $F[\Sigma_D]$ are completely reducible, so S_D^* is the image of

$$U^{\otimes D} \beta_D \hookrightarrow U^{\otimes D} \xrightarrow{\alpha_D} U^{\otimes D} \alpha_D,$$

and $U^{\otimes D} \beta_D \cong \bigwedge^D U$, $U^{\otimes D} \alpha_D \cong \text{Sym}^D U$.

Now, let

$$\text{Poly}^D U = \{f : V^l \rightarrow F \mid f \text{ homog poly of multidegree } (R_1, \dots, R_l)\},$$

where l is the number of rows of D . Then we have a G -equivariant map

$$\text{rest}_\Delta : \text{Sym}^D U \rightarrow \text{Poly}^D U$$

restricting functions from V^D to the row-multidiagonal $\Delta^D V \cong V^l$. It is well known that rest_Δ is an isomorphism: the symmetric part of a tensor algebra is isomorphic to a polynomial algebra.

Thus we have the commutative diagram

$$\begin{array}{ccc} \bigwedge^D U & \hookrightarrow U^{\otimes D} \xrightarrow{\alpha_D} & \text{Sym}^D U \\ \parallel & & \downarrow \text{rest}_\Delta \\ \bigwedge^D U & \hookrightarrow U^{\otimes D} \xrightarrow{\alpha_D} & \text{Poly}^D U. \end{array}$$

Now, the image in the top row is S_D^* , the image in the bottom row is W_D , and all the vertical maps are isomorphisms, so we have $\text{rest}_\Delta : S_D^* \xrightarrow{\sim} W_D$ an isomorphism. •

If $D = \lambda$ a Young diagram, then W_D is isomorphic to Carter and Lusztig's dual Weyl module for $G = GL(n, F)$. This will follow from Proposition 5 in the following section.

2 Configuration varieties

We define spaces which generalize the flag varieties of $GL(n)$. They are associated to arbitrary diagrams, and reduce to (partial) flag varieties in the case of Young diagrams.

N.B. Although our constructions remain valid over \mathbf{Z} , for simplicity we will assume for the remainder of this paper that F is an algebraically closed field.

2.1 Definitions and examples

Given a finite set C (a column), and $V = F^n$, consider $V^C \cong M_{n \times C}(F)$, the $n \times |C|$ matrices, with a right multiplication of $GL(C)$. Let

$$\text{St}(C) = \{X \in V^C \mid \text{rank } X = |C|\},$$

the Stiefel manifold, and

$$\text{Gr}(C) = \text{St}(C)/GL(C),$$

the Grassmannian.

Also, let

$$\mathcal{L}_C = \text{St}(C) \times^{GL(C)} \det^{-1} \rightarrow \text{Gr}(C)$$

be the Plucker determinant bundle, whose sections are regular functions $f : \text{St}(C) \rightarrow F$ with $f(XA) = \det(A)f(X) \forall A \in GL(C)$. In fact, such global sections can be extended to polynomial functions $f : V^C \rightarrow F$.

For a diagram D with columns C_1, C_2, \dots , we let

$$\text{St}(D) = \text{St}(C_1) \times \text{St}(C_2) \times \dots, \quad \text{Gr}(D) = \text{Gr}(C_1) \times \text{Gr}(C_2) \times \dots, \quad \mathcal{L}_D = \mathcal{L}_{C_1} \boxtimes \mathcal{L}_{C_2} \boxtimes \dots.$$

Recall that $\Delta^D V = \Delta^{R_1} V \times \Delta^{R_2} V \times \dots \subset V^D$ is the *row* multidagonal (as opposed to the column constructions above). Let

$$\mathcal{F}_D^o \stackrel{\text{def}}{=} \text{Im} [\Delta^D V \cap \text{St}(D) \rightarrow \text{Gr}(D)],$$

and define the *configuration variety of D* by

$$\mathcal{F}_D = \overline{\mathcal{F}_D^o} \subset \text{Gr}(D),$$

the Zariski closure of \mathcal{F}_D^o in $\text{Gr}(D)$. We denote the restriction of \mathcal{L}_D from $\text{Gr}(D)$ to \mathcal{F}_D by the same symbol \mathcal{L}_D .

Some properties follow immediately from the definitions. For instance, \mathcal{F}_D is an irreducible variety. Just as for Schur modules and Weyl modules, changing the diagram by permuting the rows or the columns gives an isomorphic configuration variety and line bundle. If we add a column C to D which already appears in D , we get an isomorphic configuration variety, but the line bundle is twisted to have higher degree. Since \mathcal{L}_D gives the Plucker embedding on $\text{Gr}(D)$, it is very ample on \mathcal{F}_D .

Examples. (0) If $D = \lambda$ a Young diagram, \mathcal{F}_D is the variety of partial flags in \mathbf{C}^n containing spaces of dimensions equal to the sizes of the columns of λ . See Proposition 5.

Now set $n = 4$. Identifying $\text{Gr}(k, F^4)$ with $\text{Gr}(k - 1, \mathbf{P}_F^3)$, we may consider the \mathcal{F}_D 's as varieties of configurations in \mathbf{P}^3 . Consider the diagrams:

$$D_1 = \begin{array}{cc} & \square \\ \square & \square \end{array} \quad D_2 = \begin{array}{ccc} & & \square \\ \square & \square & \square \\ & & \square \end{array} \quad D_3 = \begin{array}{ccc} & & \square \quad \square \\ & & \square \\ \square & & \square \quad \square \end{array}$$

(1) \mathcal{F}_{D_1} is the variety of pairs (l, l') , where l, l' are intersecting lines in \mathbf{P}^3 . It is singular at the locus where the two lines coincide.

(2) \mathcal{F}_{D_2} is the variety of triples (l, p, l') of two lines and a point which lies on both of them. The variety is smooth: indeed, it is a fiber bundle over the partial flag variety of a line containing a point. There is an obvious map $\mathcal{F}_{D_2} \rightarrow \mathcal{F}_{D_1}$, which is birational, and is in fact a small resolution of singularities. (C.f. Proposition 14.)

(3) \mathcal{F}_{D_3} is the variety of planes with two marked points (which may coincide). $\mathcal{F}_{D_3}^o$ is the locus where the marked points are distinct, an open, dense G -orbit. The variety is smooth as in the previous example.

$$D_4 = \begin{array}{ccc} & \square & \square \\ \square & & \square \\ & \square & \square \end{array} \quad D_5 = \begin{array}{ccc} & & \square \\ & \square & \\ \square & & \square \\ & \square & \square \end{array} \quad D_6 = \begin{array}{ccc} & & \square \\ & & \square \\ \square & & \square \\ & \square & \square \\ & & \square \end{array}$$

(4) \mathcal{F}_{D_4} is the variety of triples of coplanar lines.

(5) \mathcal{F}_{D_5} is the variety of triples of lines with a common point. This is the

projective dual of the previous variety, since the diagrams are complementary within a 4×3 rectangle. (See Theorem 6.) The variety of triples of lines which intersect pairwise cannot be described by a single diagram, but consists of $\mathcal{F}_{D_4} \cup \mathcal{F}_{D_5}$. (See Section 3.4.)

(6) $\mathcal{F}_{D_6} \cong (\mathbf{P}^3)^4$ contains the $GL(n)$ -invariant subvariety where all four points in \mathbf{P}^3 are colinear. Since the cross-ratio is an invariant of four points on a line, this subvariety contains infinitely many $GL(n)$ orbits.

$$D_7 = \begin{array}{ccccc} \square & \square & \square & & \\ & \square & \square & \square & \\ & & \square & & \\ & & & \square & \end{array} \quad D_8 = \begin{array}{ccccccc} \square & \square & \square & & & & \\ & \square & \square & \square & \square & & \\ & & \square & & & & \\ & & & \square & & & \\ & & & & \square & & \end{array}$$

(7) $\mathcal{F}_{D_7} \cong G \times^B X_\lambda$, the G -orbit version of the Schubert variety $X_\lambda \subset \text{Gr}(2, F^4)$ associated to the partition $\lambda = (1, 2)$. This is the smallest example of a singular Schubert variety.

(8) \mathcal{F}_{D_8} is a smooth variety which maps birationally to \mathcal{F}_{D_7} by forgetting the point associated to the last column. In fact, this is essentially the same resolution as (1) and (2) above. Such resolutions of singularities can be given for arbitrary Schubert varieties of $G = GL(n)$, and generalize Zelevinsky’s resolutions in [34]. C.f. Section 3. •

Theorem 2

$$W_D \cong \text{Im} [\text{rest}_\Delta : H^0(\text{Gr}(D), \mathcal{L}_D) \rightarrow H^0(\mathcal{F}_D, \mathcal{L}_D)],$$

where rest_Δ is the restriction map.

Proof. Note that for $GL(D) = GL(C_1) \times GL(C_2) \times \dots$,

$$H^0(\text{Gr}(D), \mathcal{L}_D) = \{f : V^D \rightarrow F \mid f(XA) = \det(A) f(X) \forall A \in GL(D)\},$$

and recall

$$\bigwedge^D U = \{f : V^D \rightarrow F \mid f \text{ multilinear, and } f(v\pi) = \text{sgn}(\pi) f(v) \forall \pi \in \text{Col}(D)\}.$$

But in fact these sets are equal, because a multilinear, anti-symmetric function $g : V^C \rightarrow F$ always satisfies $g(XA) = \det(A) g(X) \forall A \in GL(C)$. Now W_D and $H^0(\mathcal{F}_D, \mathcal{L}_D)$ are gotten by restricting functions in these identical sets to $\Delta^D V$, so we are done. •

2.2 Diagrams with at most n rows

We say D has $\leq n$ rows if $(i, j) \in D \Rightarrow 1 \leq i \leq n$.

Proposition 3 *If D has $\leq n$ rows, then \mathcal{F}_D has an open dense $GL(n)$ -orbit $\mathcal{F}_D^{\text{gen}}$.*

Proof. Let D have columns C_1, C_2, \dots . Consider a sequence of vectors $X = (v_1, \dots, v_n) \in V^n$. For $C = \{i_1, i_2, \dots\} \subset \{1, \dots, n\}$, define $X(C) \stackrel{\text{def}}{=} \text{Span}_F(v_{i_1}, v_{i_2}, \dots) \in \text{Gr}(C)$ (for X sufficiently general). Consider an element $g \in GL(n)$ as a sequence of column vectors $g = (v_1, \dots, v_n)$. Then

$$g(C) = g \cdot \text{Span}_F(e_{i_1}, e_{i_2}, \dots) = g \cdot I(C),$$

where e_i denotes the i -th coordinate vector and I the identity matrix.

Now define the map

$$\begin{aligned} \Psi : \quad V^n &\rightarrow \Delta^D V \subset V^D \\ (v_1, \dots, v_n) &\mapsto (v_i)_{(i,j) \in D}, \end{aligned}$$

where $(u_{ij})_{(i,j) \in D}$ denotes an element of V^D . Then the composite

$$V^n \xrightarrow{\Psi} \Delta^D V \rightarrow \mathcal{F}_D^o$$

is an onto map taking $g \mapsto (g(C_1), g(C_2), \dots) = g \cdot (I(C_1), I(C_2), \dots)$. Since $GL(n)$ is dense in V^n , its image is dense in \mathcal{F}_D^o , and hence the composite image $\mathcal{F}_D^{\text{gen}} \stackrel{\text{def}}{=} G \cdot (I(C_1), I(C_2), \dots)$ is a dense G -orbit in \mathcal{F}_D . •

Corollary 4

$$\dim \mathcal{F}_D = \# \bigcup_{C \in D} \{ (i, j) \mid 1 \leq i, j \leq n, i \notin C, j \in C \},$$

where the union is over all the columns C of D .

Proof. The dimension of the dense $GL(n)$ -orbit is n^2 minus the dimension of the stabilizer of the point $(I(C_1), I(C_2), \dots)$ above. The stabilizer of each component is the parabolic corresponding to the matrix positions (i, j) such that $i \in C$ or $j \notin C$. Now take complements in the set of positions. •

Proposition 5 *If D is the Young diagram associated to a dominant weight λ of $GL(n)$, then:*

(a) $\mathcal{F}_D \cong G/P$, a quotient of the flag variety $\mathcal{F} = G/B$.

(b) The Borel-Weil line bundle $\mathcal{L}_\lambda \stackrel{\text{def}}{=} G \overset{B}{\times} (\lambda^{-1}) \rightarrow \mathcal{F}$ is the pullback of \mathcal{L}_D under the projection $\mathcal{F} \rightarrow \mathcal{F}_D$.

(c) $\text{rest}_\Delta : H^0(\text{Gr}(D), \mathcal{L}_D) \rightarrow H^0(\mathcal{F}_D, \mathcal{L}_D)$ is surjective, and $W_D \cong H^0(\mathcal{F}_D, \mathcal{L}_D)$.

Proof. (a) Let $\mu = (\mu_1 \geq \mu_2 \geq \dots) = \lambda^t$, the transposed diagram, and let $P = \{(x_{ij}) \in GL(n) \mid x_{ij} = 0 \text{ if } \exists k, i > \mu_k \geq j > \mu_{k+1}\}$, a parabolic subgroup of G . Then G/P is the space of partial flags $V = F^n \supset V_1 \supset V_2 \supset \dots$ consisting of subspaces V_j with $\dim(V_j) = \mu_j$. Clearly $G/P \cong \mathcal{F}_D$.

(b) Let $\Psi : G \rightarrow \Delta^D V \cap \text{St}(D)$ be the map in the proof of the previous proposition. Then the map

$$\begin{aligned} G \times^P F_{\lambda^{-1}} &\rightarrow \mathcal{L}_D = (\text{St}(D) \times^{GL(D)} \det_D^{-1}) |_{\mathcal{F}_D} \\ (g, \alpha) &\mapsto (\Psi(g), \alpha) \end{aligned}$$

is a G -equivariant bundle isomorphism. Then (b) follows by standard arguments.

(c) The surjectivity is a special case of Proposition 25 in Section 4. (See also [13].) The other statement then follows by Prop 2. •

2.3 Complementary diagrams

Theorem 6 *Suppose the rectangular diagram $\text{Rect} = \{1, \dots, n\} \times \{1, \dots, r\}$ is the disjoint union of two diagrams D, D^* . Let W_D, W_{D^*} be the corresponding Weyl modules for $G = GL(n, F)$. Then:*

- (a) *there is an F -linear bijection $\tau : W_D \rightarrow W_{D^*}$ such that $\tau(gw) = \det^r(g') g' \tau(w)$, where g' is the inverse transpose in $GL(n)$ of the matrix g ;*
- (b) *the characters obey the relation $\text{char } W_{D^*}(h) = \det^r(h) \text{char } W_D(h^{-1})$, for diagonal matrices $h \in G$;*
- (c) *if F has characteristic zero, then as G -modules*

$$W_{D^*} \cong \det^{-r} \otimes W_D^* \quad \text{and} \quad S_{D^*} \cong \det^r \otimes S_D^*.$$

Proof. (a) Given $C \subset \{1, \dots, n\}$ (a column set), we considered above the Plucker line bundle

$$\text{St}(C) \times^{GL(C)} \det^{-1} \rightarrow \text{Gr}(D).$$

We may equally well write this as

$$GL(n) \times^{P_C} \det_C^{-1} \rightarrow \text{Gr}(D),$$

where $P_C \stackrel{\text{def}}{=} \{(x_{ij}) \in GL(n) \mid x_{ij} = 0 \text{ if } i \notin C, j \in C\}$ is a maximal parabolic subgroup of $GL(n)$ (not necessarily containing B), and $\det_C : P_C \rightarrow F$ is the multiplicative character $\det_C(x_{ij})_{n \times n} \stackrel{\text{def}}{=} \det_{C \times C}(x_{ij})_{i, j \in C}$.

Hence, if C_1, C_2, \dots, C_r are the columns of D , we may write

$$\text{Gr}(D) \cong G^r / P_D,$$

and the bundle

$$\mathcal{L}_D \cong G^r \times^{P_D} \det_D^{-1},$$

where $P_D \stackrel{\text{def}}{=} P_{C_1} \times \dots \times P_{C_r}$ and $\det_D(X_1, \dots, X_r) \stackrel{\text{def}}{=} \det_{C_1}(X_1) \times \dots \times \det_{C_r}(X_r)$. Under this identification,

$$\mathcal{F}_D \cong \text{closure Im} [\Delta G \hookrightarrow G^r \rightarrow \text{Gr}(D)]$$

(c.f. Proposition 3).

Now let $\tau : G^r \rightarrow G^r$, $\tau(g_1, \dots, g_r) = (g'_1, \dots, g'_r)$, where $g' = {}^t g^{-1}$, the inverse transpose of a matrix $g \in G$. Then $\tau(P_D) = P_{D^*}$, and τ induces a map

$$\tau : \text{Gr}(D) \rightarrow \text{Gr}(D^*),$$

as well as a map of line bundles

$$\begin{array}{ccc} \tau : & \mathcal{L}_D & \rightarrow & \mathcal{L}_{D^*} \\ & \parallel & & \parallel \\ & G^r \times_{P_D} \det_D^{-1} & & G^r \times_{P_{D^*}} \det_{D^*}^{-1} \\ & (g_1, \dots, g_r, \alpha) & \mapsto & (g'_1, \dots, g'_r, \det(g'_1, \dots, g'_r)\alpha). \end{array}$$

This map is not G -equivariant. Rather, if we have a section of \mathcal{L}_D , $f : G^r \rightarrow F$ (with $f(gp) = \det_D(p)f(g)$ for $p \in P_D$), then for $g_0 \in G$, we have $\tau(g_0 f) = g'_0 \det(g'_0)^r \tau(f)$ (a section of \mathcal{L}_{D^*}).

Since W_D is the restriction of such functions f to $\Delta G \subset G^r$, and $\tau(\Delta G) \subset \Delta G$, we have an induced map

$$\tau : W_D \rightarrow W_{D^*}$$

(an isomorphism of F vector spaces), satisfying $\tau(g_0 w) = g'_0 \det(g'_0)^r \tau(w)$ for $g_0 \in G$, $w \in W_D$. This is the map required in (a), and now (b), (c) follow trivially. •

3 Resolution of singularities

We define the class of northwest direction diagrams, which includes (up to permutation of rows and of columns) the skew, inversion, Rothe, and column-convex diagrams. We construct an explicit resolution of singularities of the associated configuration varieties by means of “blowup diagrams”. We also find defining equations for these varieties. One should note that the resolutions constructed are not necessarily geometric blowups, and can sometimes be small resolutions, as in Example 8 above.

3.1 Northwest and lexicographic diagrams

We shall, as usual, think of a diagram D either as a subset of $\mathbf{N} \times \mathbf{N}$, or as a list (C_1, C_2, \dots, C_r) of columns $C_j \subset \mathbf{N}$. In this section we examine only configuration varieties, as opposed to line bundles on them, so we shall assume that the columns are without multiplicity: $C_j \neq C_{j'}$ for $j \neq j'$.

A diagram D is *northwest* if it possesses the following property:

$$(i_1, j_1), (i_2, j_2) \in D \Rightarrow (\min(i_1, i_2), \min(j_1, j_2)) \in D.$$

Given two subsets $C = \{i_1 < i_2 < \dots < i_l\}$, $C' = \{i'_1 < i'_2 < \dots < i'_l\} \subset \mathbf{N}$, we say C is *lexicographically less than* C' ($C \stackrel{\text{lex}}{<} C'$) if

$$l < l' \text{ and } i_1 = i'_1, \dots, i_l = i'_l,$$

$$\text{or } \exists m : i_1 = i'_1, \dots, i_{m-1} = i'_{m-1}, i_m < i'_m.$$

In the first case, we say C is an *initial subset* of C' ($C \stackrel{\text{init}}{\subset} C'$).

A diagram $D = (C_1, C_2, \dots)$ is *lexicographic* if $C_1 \stackrel{\text{lex}}{<} C_2 \stackrel{\text{lex}}{<} \dots$. Note that any diagram can be made lexicographic by rearranging the order of columns.

Examples. Of the diagrams considered in the example of the previous section, D_1, D_2, D_3, D_6, D_7 , and D_8 are northwest. However, D_4 and D_5 are not northwest, nor can they be made so by permuting the rows or the columns.

Lemma 7 *If D is northwest, then the lexicographic rearrangement of D is also northwest.*

Proof. (a) I claim that if $j < j'$, then either $C_j \stackrel{\text{lex}}{<} C_{j'}$, or $C_j \stackrel{\text{init}}{\supset} C_{j'}$. Let $C_j = \{i_1 < i_2 < \dots\}$, $C_{j'} = \{i'_1 < i'_2 < \dots\}$. We have assumed $C_j \neq C_{j'}$. Thus $C_j \stackrel{\text{lex}}{<} C_{j'}$ or $C_j \stackrel{\text{lex}}{>} C_{j'}$. In the second case, $C_j \stackrel{\text{init}}{\supset} C_{j'}$, or there is an r such that $i_1 = i'_1, \dots, i_{r-1} = i'_{r-1}, i_r < i'_r$. By the northwest property, this last case would mean $i'_r \in C_j$, with $i_{r-1} = i'_{r-1} < i'_r < i_r$. But this contradicts the definition of C_j . Thus the only possibilities are those of the claim.

(b) It follows immediately from (a) that if $C_1 \stackrel{\text{lex}}{<} C_2 \stackrel{\text{lex}}{<} \dots \stackrel{\text{lex}}{<} C_{s-1} \stackrel{\text{lex}}{>} C_s$, then there is a $t < s$ with $C_{t-1} \stackrel{\text{lex}}{<} C_s$, $C_s \stackrel{\text{init}}{\subset} C_t$, $C_s \stackrel{\text{init}}{\subset} C_{t+1}, \dots, C_s \stackrel{\text{init}}{\subset} C_{s-1}$.

(c) From (b), we see that to rearrange the columns lexicographically requires only the following operation: we start with C_1, C_2, \dots , and when we encounter the first column C_s which violates lexicographic order, we move it as far left as possible, passing over those columns C_i with $C_s \stackrel{\text{init}}{\subset} C_i$. This operation does not destroy the northwest property, as we can easily check on boxes from each pair of columns in the new diagram. By repeating this operation, we get the lexicographic rearrangement, which is thus northwest.

3.2 Blowup diagrams

The combinatorial lemmas of this section will be used to establish geometric properties of configuration varieties.

Given a northwest diagram D and two of its columns $C, C' \subset \mathbf{N}$, the *intersection blowup diagram* $\widehat{D}_{C, C'}$ is the diagram with the same columns as D except that the new column $C \cap C'$ is inserted in the proper lexicographic position (provided $C \cap C' \neq C, C'$).

Lemma 8 Suppose D is lexicographic and northwest, and $C <^{\text{lex}} C'$ are two of its columns. Then: (a) $C \cap C' \stackrel{\text{init}}{\subset} C'$, and (b) if $C \subset C'$, then $C \stackrel{\text{init}}{\subset} C'$.

Proof. (a) If $i \in C_j \cap C_{j'}$ and $i > i' \in C_{j'}$, then $i' \in C_j$ by the northwest property. Similarly for (b). •

Lemma 9 If D is lexicographic and northwest, then $\widehat{D}_{C,C'}$ is also lexicographic and northwest.

Proof. If $C = C_j, C' = C_{j'}$ with $j < j'$, and we insert the column $C \cap C' \stackrel{\text{init}}{\subset} C'$ immediately before C' , then we easily check that the resulting diagram is again northwest. Hence $\widehat{D}_{C,C'}$, which is the lexicographic rearrangement of this, is also northwest by a previous lemma. •

Consider the columns $C_1, C_2, \dots \subset \mathbf{N}$ of a northwest diagram D , and take the smallest collection $\{\widehat{C}_1 <^{\text{lex}} \widehat{C}_2 <^{\text{lex}} \dots\}$ of subsets of \mathbf{N} which contains the C_i and is closed under taking intersections. Then we define a new diagram

$$\widehat{D} = (\widehat{C}_1, \widehat{C}_2, \dots)$$

which we call the *maximal intersection blowup diagram* of D . Blowing up again does not enlarge the diagram: $\widehat{\widehat{D}} = \widehat{D}$. Applying the above lemma repeatedly shows that if D is lexicographic and northwest, then so is \widehat{D} .

Examples. For one of the (non-northwest) diagrams considered previously, we have:

$$D_4 = \begin{array}{cc} \square & \square \\ \square & \square \\ & \square \end{array} \quad \widehat{D}_4 = \begin{array}{ccccc} \square & \square & \square & & \\ & \square & & \square & \square \\ & & & \square & \square & \square \end{array}$$

For the diagrams D_7 and D_8 in the previous examples, $D_8 = \widehat{D}_7$. •

Consider the columns $C \subset \mathbf{N}$ of a diagram D as a partially ordered set under \subset , ordinary inclusion. Given two distinct columns C, C' , we say C' *minimally covers* C (or simply C' *covers* C) if $C \subset C'$ and there is no column of D strictly included between C and C' .

Lemma 10 Let D be a lexicographic northwest diagram, and C_L be the last column of D . Then:

(a) there is a column $C_l \neq C_L$ such that

$$\left(\bigcup_{C \neq C_L} C \right) \cap C_L = C_l \cap C_L;$$

(b) if $\widehat{D} = D$, then C_L covers at most one other column C_l and is covered by at most one other column C_u .

Proof. (a) Now, by Lemma 8, $C \cap C_L \overset{\text{init}}{\subset} C_L$ for any column C . Hence the sets $C \cap C_L$ for $C \neq C_L$ are linearly ordered under inclusion, and there is a largest one $C_l \cap C_L$. Thus

$$\left(\bigcup_{C \neq C_L} C \right) \cap C_L = \bigcup_{C \neq C_L} (C \cap C_L) = C_l \cap C_L.$$

(b) By Lemma 8, the columns with $C \subset C_L$ satisfy $C \overset{\text{init}}{\subset} C_L$ and are linearly ordered, so there is at most one maximal C_u .

Now suppose $C_u, C'_u \overset{\text{lex}}{<} C_L$ are columns of D both covering C_L . Then again by Lemma 8, we have $C_u \cap C'_u \overset{\text{lex}}{\leq} C_u$ or $\overset{\text{lex}}{\leq} C'_u$, so that $C_u \cap C'_u \neq C_L$. But $C_u \cap C'_u$ is between C_L and C_u , and between C_L and C'_u . Hence $C_u = C_u \cap C'_u = C'_u$. •

3.3 Blowup varieties

Let $D = (C_1, C_2, \dots)$ be a lexicographic northwest diagram, and $\widehat{D} = (\widehat{C}_1, \widehat{C}_2, \dots)$ be its maximal intersection blowup. Recall that \widehat{D} is obtained by adding certain columns to D , so there is a natural projection map

$$\begin{aligned} \text{pr} : \text{Gr}(\widehat{D}) &\rightarrow \text{Gr}(D) \\ (V_{\widehat{C}})_{\widehat{C} \in \widehat{D}} &\mapsto (V_C)_{C \in D}, \end{aligned}$$

obtained by forgetting some of the linear subspaces $V_{\widehat{C}} \in \text{Gr}(\widehat{C})$.

Proposition 11 *If D has $\leq n$ rows, then*

$$\text{pr} : \text{Gr}(\widehat{D}) \rightarrow \text{Gr}(D)$$

induces a birational map of algebraic varieties

$$\text{pr} : \mathcal{F}_{\widehat{D}} \rightarrow \mathcal{F}_D.$$

Proof. Consider the dense open sets $\mathcal{F}_{\widehat{D}}^{\text{gen}} \subset \mathcal{F}_{\widehat{D}}$ and $\mathcal{F}_D^{\text{gen}} \subset \mathcal{F}_D$ of Proposition 3, consisting of subspaces in general position. If we consider an element $g \in GL(n)$ as a sequence of column vectors $g = (v_1, \dots, v_n)$, and $C = \{i_1, i_2, \dots\} \subset \{1, \dots, n\}$, recall that we define $g(C) = \text{Span}_F(v_{i_1}, v_{i_2}, \dots) \in \text{Gr}(C)$. By definition, any element of $\mathcal{F}_{\widehat{D}}^{\text{gen}}$ can be written as $(g(\widehat{C}_1), g(\widehat{C}_2), \dots) \in \text{Gr}(D)$ for some $g \in GL(n)$.

Now, any column of \widehat{D} can be written as an intersection of columns of D : $\widehat{C} = C_{j_1} \cap C_{j_2} \cap \dots$. Then we have $g(\widehat{C}) = g(C_{j_1}) \cap g(C_{j_2}) \cap \dots$, so the projection map

$$\begin{aligned} \text{pr} : \mathcal{F}_{\widehat{D}}^{\text{gen}} &\rightarrow \mathcal{F}_D^{\text{gen}} \\ (g(\widehat{C}))_{\widehat{C} \in \widehat{D}} &\mapsto (g(C))_{C \in D} \end{aligned}$$

can be inverted:

$$\begin{aligned} \text{pr}^{-1} : \quad \mathcal{F}_D^{\text{gen}} &\rightarrow \mathcal{F}_{\widehat{D}}^{\text{gen}} \\ (g(C))_{C \in D} &\mapsto (g(\widehat{C}) = g(C_{j_1}) \cap g(C_{j_2}) \cap \cdots)_{\widehat{C} \in \widehat{D}}. \end{aligned}$$

Hence the map is birational on the configuration varieties as claimed. •

3.4 Intersection varieties

Now, given a diagram D , define the *intersection variety* \mathcal{I}_D of D by:

$$\mathcal{I}_D = \{(V_C)_{C \in D} \in \text{Gr}(D) \mid \forall C, C', \dots \in D, \dim(V_C \cap V_{C'} \cap \cdots) \geq |C \cap C' \cap \cdots|\}.$$

Clearly \mathcal{I}_D is a projective subvariety of $\text{Gr}(D)$, and $\mathcal{F}_D \subset \mathcal{I}_D$.

If $\widehat{D} = D$ (up to rearrangement of column order), then the intersection conditions reduce to inclusions:

$$\mathcal{I}_D = \{(V_C)_{C \in D} \in \text{Gr}(D) \mid C \subset C' \Rightarrow V_C \subset V_{C'}\}.$$

Example. For the diagram D_4 of Section 2.1, and $n = 4$, \mathcal{I}_{D_4} has two irreducible components, \mathcal{F}_{D_4} and \mathcal{F}_{D_5} . That is, as before, if we have three lines in \mathbf{P}^3 with non-empty pairwise intersections, then either they are coplanar, or they all intersect in a point. •

Lemma 12 *Let D be a northwest diagram, and \mathcal{I}_D its intersection variety. Then any configuration $(V_C)_{C \in D} \in \mathcal{I}_D$ satisfies*

$$\dim(V_C + V_{C'} + \cdots) \leq |C \cup C' \cup \cdots|$$

for any columns C, C', \dots of D .

Proof. Without loss of generality, assume D is lexicographic. We use induction on the number of columns in D . Now any list C, C', \dots of columns of D also constitutes a lexicographic northwest diagram, so to carry through the induction we need only prove the statement for *all* the columns C_1, C_2, \dots, C_L of D . Now, by Lemma 10, there is a column $C_l \neq C_L$ such that $(\cup_{C \neq C_L} C) \cap C_L = C_l \cap C_L$. Then we have

$$\begin{aligned} \dim\left(\left(\sum_{C \neq C_L} V_C\right) \cap V_{C_L}\right) &\geq \dim\left(\sum_{C \neq C_L} (V_C \cap V_{C_L})\right) \\ &\geq \dim(V_{C_l} \cap V_{C_L}) \\ &\geq |C_l \cap C_L| \quad \text{since } (V_C) \in \mathcal{I}_D \\ &= \left|\left(\bigcup_{C \neq C_L} C\right) \cap C_L\right|. \end{aligned}$$

Thus we may write

$$\begin{aligned}
 \dim\left(\sum_{C \in D} V_C\right) &= \dim\left(\sum_{C \neq C_L} V_C\right) + \dim(V_{C_L}) - \dim\left(\left(\sum_{C \neq C_L} V_C\right) \cap V_{C_L}\right) \\
 &\leq \left|\bigcup_{C \neq C_L} C\right| + |C_L| - \left|\left(\bigcup_{C \neq C_L} C\right) \cap C_L\right| \quad \text{by induction} \\
 &= \left|\bigcup_{C \in D} C\right| \bullet
 \end{aligned}$$

Lemma 13 *If D is a northwest diagram with $\leq n$ rows and $\widehat{D} = D$ (up to rearrangement of column order), then \mathcal{F}_D is an irreducible component of \mathcal{I}_D .*

Proof. Recall that \mathcal{F}_D is always irreducible. Thus it suffices to show that $\mathcal{F}_D^{\text{gen}}$ is an open subset of \mathcal{I}_D .

Consider the set $\mathcal{I}_D^{\text{gen}}$ of configurations $(V_C)_{C \in D}$ satisfying, for every list C, C', \dots of columns in D ,

$$\dim(V_C + V_{C'} + \dots) = |C \cup C' \cup \dots|$$

and

$$\dim(V_C \cap V_{C'} \cap \dots) = |C \cap C' \cap \dots|.$$

This is an open subset of \mathcal{I}_D by the previous lemma.

I claim that $\mathcal{F}_D^{\text{gen}} = \mathcal{I}_D^{\text{gen}}$. To see this equality, let $(V_C)_{C \in D} \in \mathcal{I}_D$ satisfy the above rank conditions, and we will find a basis $g = (v_1, \dots, v_n)$ of $V = F^n$ such that $V_C = g(C)$ for all C . (C.f. the proof of Proposition 3.)

As before, we consider the columns as a poset under ordinary inclusion. We begin by choosing mutually independent bases for those V_C where C is a minimal element of the poset. This is possible because $\dim \text{Span}(V_C \mid C \text{ minimal}) = \sum_C \text{minim} |C|$.

Now we consider the V_C where C covers a minimal column. We start with the basis vectors already chosen, and add enough vectors, all mutually independent, to span each space. Again, the dimension conditions ensure there will be no conflict in choosing independent vectors, since the V_C can have no intersections with each other except those due to the intersections of columns. The condition $\widehat{D} = D$ ensures that all these intersections are (previously considered) columns.

We continue in this way for the higher layers of the poset. We will not run out of independent basis vectors because all the columns of D are contained in $\{1, \dots, n\}$. •

3.5 Smoothness and defining equations

Proposition 14 *Let D be a northwest diagram with $\leq n$ rows and $\widehat{D} = D$ (up to rearrangement of column order). Then $\mathcal{F}_D = \mathcal{I}_D$, and \mathcal{F}_D is a smooth variety.*

Proof. (a) Let C_L be the last column of D , and let D' be D without the last column. By lemma 10, C_L is covered by at most one other column C_u , and covers at most one other column C_l . If these columns do not exist, take $C_l = \emptyset$, $C_u = \{1, \dots, n\}$.

(b) Now I claim that there is a fiber bundle

$$\begin{array}{ccc} \mathrm{Gr}(C_l, C_L, C_u) & \rightarrow & Z \\ & & \downarrow \\ & & \mathrm{Gr}(D') \end{array}$$

where $\mathrm{Gr}(C_l, C_L, C_u)$ denotes the Grassmannian of $|C_L|$ -dimensional linear spaces which contain a fixed $|C_l|$ -dimensional space and are contained in a fixed $|C_u|$ -dimensional space; and

$$Z = \{(V_{C'})_{C'}, V_L) \in \mathrm{Gr}(D') \times \mathrm{Gr}(C_L) \mid V_{C_l} \subset V_L \subset V_{C_u}\}.$$

This is clear. See also [7].

(c) Note that $\mathcal{I}_D = (\mathcal{I}_{D'} \times \mathrm{Gr}(C_L)) \cap Z$. This is because of the uniqueness of C_l and C_u . Thus the above fiber bundle restricts to

$$\begin{array}{ccc} \mathrm{Gr}(C_l, C_L, C_u) & \rightarrow & \mathcal{I}_D \\ & & \downarrow \\ & & \mathcal{I}_{D'} \end{array} ,$$

which is thus also a fiber bundle.

(d) Now apply the above construction repeatedly, dropping columns of D from the end. Finally we obtain \mathcal{I}_D as an iterated fiber bundle whose fibers at each step are smooth and connected (in fact they are Grassmannians). In particular, \mathcal{I}_D is smooth and connected.

(e) Since \mathcal{I}_D is a smooth, connected, projective algebraic variety, it must be irreducible. But by a previous lemma, \mathcal{F}_D is an irreducible component of \mathcal{I}_D . Therefore $\mathcal{F}_D = \mathcal{I}_D$, a smooth variety. •

Proposition 15 *Let D be a northwest diagram with $\leq n$ rows. Then $\mathcal{F}_D = \mathcal{I}_D$, and the birational projection map $\mathcal{F}_{\widehat{D}} \rightarrow \mathcal{F}_D$ has connected fibers.*

Proof. (a) I claim the following: if \widehat{C} is a column of \widehat{D} such that for all $C \in \widehat{D}$ with $C \not\subset \widehat{C}$ we have $C \in D$, then the projection map $\mathcal{I}_{D \cup \widehat{C}} \rightarrow \mathcal{I}_D$ is onto, with connected fibers.

Suppose $(V_C)_{C \in D}$ is a configuration in \mathcal{I}_D . Let

$$V_u = \bigcap_{\substack{C \in D \\ C \supset \widehat{C}}} V_C \quad \text{and} \quad V_l = \sum_{\substack{C \in D \\ C \subset \widehat{C}}} V_C.$$

Then $\dim(V_u) \geq |\widehat{C}|$ since $(V_C) \in \mathcal{I}_D$, and $\dim(V_l) \leq |\widehat{C}|$ by Lemma 12. Clearly $V_l \subset V_u$. Now choose an arbitrary $V_{\widehat{C}}$ between V_l and V_u with $\dim(V_{\widehat{C}}) = |\widehat{C}|$. Then for any list of columns $C, C', \dots \in D$, we have either:

(i) $\widehat{C} \cap C \cap C' \dots = \widehat{C}$, and

$$V_{\widehat{C} \cap C \cap C' \dots} = V_{\widehat{C}} = V_{\widehat{C}} \cap V_u \subset V_{\widehat{C}} \cap V_C \cap V_{C'} \cap \dots;$$

or (ii) $\widehat{C} \cap C \cap C' \dots \subsetneq \widehat{C}$, so that $\widehat{C} \cap C \cap C' \dots \in D$ by hypothesis, and

$$V_{\widehat{C} \cap C \cap C' \dots} \subset V_l \cap V_C \cap V_{C'} \dots \subset V_{\widehat{C}} \cap V_C \cap V_{C'} \dots.$$

In either case $(V_C)_{C \in D \cup \widehat{C}} \in \mathcal{I}_{D \cup \widehat{C}}$. Thus $\mathcal{I}_{D \cup \widehat{C}} \rightarrow \mathcal{I}_D$ is onto, and the fibers are the Grassmannians $\text{Gr}(V_l, |C|, V_u)$.

(b) We now see that $\mathcal{I}_{\widehat{D}} \rightarrow \mathcal{I}_D$ is onto (with connected fibers) by repeated application of (a), starting with \widehat{C} minimal in the poset of columns of \widehat{D} and proceeding upward.

(c) By the previous proposition, the projection map takes $\mathcal{I}_{\widehat{D}} = \mathcal{F}_{\widehat{D}} \rightarrow \mathcal{F}_D$. But $\mathcal{I}_{\widehat{D}} \rightarrow \mathcal{I}_D$ is onto, so $\mathcal{F}_D = \mathcal{I}_D$, and we are done. •

The above proposition shows that for northwest diagrams, \mathcal{F}_D is defined by the rank conditions of \mathcal{I}_D . In general, we state the

Conjecture 16 *For an arbitrary diagram D , \mathcal{F}_D is the set of configurations satisfying*

$$\begin{aligned} \dim(V_C + V_{C'} + \dots) &\leq |C \cup C' \cup \dots| \\ \dim(V_C \cap V_{C'} \cap \dots) &\geq |C \cap C' \cap \dots| \end{aligned}$$

for every list C, C', \dots of columns of D . Equivalently, the variety defined by these relations is irreducible of the same dimension as \mathcal{F}_D .

4 Cohomology of line bundles

Using the technique of Frobenius splitting, we obtain some surjectivity and vanishing theorems for line bundles on configuration varieties: most importantly, that a (dual) Schur module is the *entire* space of sections of a line bundle over a configuration variety (cf. Theorem 2). We also show that for any northwest diagram D , \mathcal{F}_D is normal, and projectively normal with respect to \mathcal{L}_D (so that global sections of L_D on \mathcal{F}_D extend to $\text{Gr}(D)$); and \mathcal{F}_D has rational singularities.

The material of section 4.1 was shown to me by Wilberd van der Kallen. Also, the statement and proof of Proposition 28 are due to van der Kallen and S.P. Inamdar.

4.1 Frobenius splittings of flag varieties

The technique of Frobenius splitting, introduced by V.B. Mehta, S. Ramanan, and A. Ramanathan [23], [27], [28], [29], is a method for proving certain surjectivity and vanishing results.

Given two algebraic varieties $Y \subset X$ defined over an algebraically closed field F of characteristic $p > 0$, with Y a closed subvariety of X , we say that the pair $Y \subset X$ is *compatibly Frobenius split* if:

- (i) the p^{th} power map $F : \mathcal{O}_X \rightarrow F_*\mathcal{O}_X$ has a splitting, i.e. an \mathcal{O}_X -module morphism $\phi : F_*\mathcal{O}_X \rightarrow \mathcal{O}_X$ such that ϕF is the identity; and
- (ii) we have $\phi(F_*I) = I$, where I is the ideal sheaf of Y .

Mehta and Ramanathan prove the following

Theorem 17 *Let X be a projective variety, Y a closed subvariety, and L an ample line bundle on X . If $Y \subset X$ is compatibly split, then $H^i(Y, L) = 0$ for all $i > 0$, and the restriction map $H^0(X, L) \rightarrow H^0(Y, L)$ is surjective.*

Furthermore, if Y and X are defined and projective over \mathbf{Z} (and hence over any field), and they are compatibly split over any field of positive characteristic, then the above vanishing and surjectivity statements also hold for all fields of characteristic zero. •

Our aim is to show that, for D a northwest diagram, $\mathcal{F}_D \subset \text{Gr}(D)$ is compatibly split. The above theorem and Theorem 2 will then imply that $S_D^* \cong H^0(\mathcal{F}_D, \mathcal{L}_D) = \sum_i (-1)^i H^i(\mathcal{F}_D, \mathcal{L}_D)$, the Euler characteristic of \mathcal{L}_D .

We will also need the following result of Mehta and V. Srinivas [22]:

Proposition 18 *Let Y be a projective variety which is Frobenius split, and suppose there exists a smooth irreducible projective variety Z which is mapped onto Y by an algebraic map with connected fibers. Then Y is normal.*

Furthermore, if Y is defined over \mathbf{Z} , and is normal over any field of positive characteristic, then Y is also normal over all fields of characteristic zero. •

Proposition 19 *Let $f : Z \rightarrow X$ be a separable morphism with connected fibers, where X and Z are projective varieties and X smooth. If $Y \subset Z$ is compatibly split, then so is $f(Y) \subset X$. •*

We will show our varieties are split by using the above proposition to push forward a known splitting due to Ramanathan [29] and O. Mathieu [21].

For an integer l , define

$$X_l = G \underbrace{\times^B G \times^B \cdots \times^B G}_{l \text{ factors}} / B,$$

and note that we have an isomorphism

$$\begin{aligned} X_l &\rightarrow (G/B)^l \\ (g, g', g'', \dots) &\mapsto (g, gg', gg'g'', \dots). \end{aligned}$$

Given l permutations w, w', \dots , define also the twisted multiple Schubert variety

$$Y_{w, w', \dots} = \overline{BwB} \overset{B}{\times} \overline{Bw'B} \overset{B}{\times} \dots \subset X_l$$

Proposition 20 (*Ramanathan-Mathieu*) *Let G be a reductive algebraic group over a field of positive characteristic with Weyl group W and Borel subgroup B , and let $w_0, w_1, \dots, w_r \in W$. Then $Y_{w_0, w_1, \dots} \subset X_{r+1}$ is compatibly split. •*

Now, for Weyl group elements u_0, u_1, \dots, u_r , define a variety $\mathcal{F}_{u_0; u_1, \dots, u_r} \subset (G/B)^r$ by

$$\mathcal{F}_{u_0; u_1, \dots, u_r} = \overline{Bu_0B \cdot (u_1B, \dots, u_rB)}.$$

Note that if we take u_0 to be the longest element of W , then

$$\mathcal{F}_{u_0; u_1, \dots, u_r} = \overline{G \cdot (u_1B, \dots, u_rB)}.$$

Proposition 21 (*van der Kallen*) *Let u_0 and w_1, \dots, w_r be Weyl group elements, and define $u_1 = w_1, u_2 = w_1w_2, \dots, u_r = w_1 \dots w_r$. Suppose w_1, \dots, w_r satisfy $\ell(w_1w_2 \dots w_r) = \ell(w_1) + \ell(w_2) + \dots + \ell(w_r)$, or equivalently $\ell(u_j) = \ell(u_{j-1}) + \ell(u_{j-1}^{-1}u_j)$ for all j . Then the pair $\mathcal{F}_{u_0; u_1, \dots, u_r} \subset (G/B)^r$ is compatibly split.*

Proof. Define

$$\begin{aligned} f: X_{r+1} &\rightarrow (G/B)^r \\ (g_0, g_1, \dots, g_r) &\mapsto (g_0g_1, g_0g_1g_2, \dots, g_0g_1 \dots g_r). \end{aligned}$$

We will examine the image under this map of

$$Y \stackrel{\text{def}}{=} Y_{u_0, w_1, \dots, w_r} = \overline{Bu_0B} \overset{B}{\times} \overline{Bw_1B} \overset{B}{\times} \dots \overset{B}{\times} \overline{Bw_rB} \subset X_{r+1}.$$

It is well known that, under the given hypotheses, we have $(Bw_1B) \dots (Bw_rB) = Bw_1 \dots w_r B$, and that the multiplication map

$$Bw_1B \overset{B}{\times} \dots \overset{B}{\times} Bw_rB \rightarrow Bw_1 \dots w_r B$$

is bijective. Thus any element $(g, b_1w_1b'_1, \dots, b_rw_rb'_rB)$ (for $b_i, b'_i \in B$) can be written as $(g, bw_1, w_2, \dots, w_rB) = (gb, w_1, w_2, \dots, w_rB)$ for some $b \in B$, and

$$\begin{aligned} f(Bu_0B \overset{B}{\times} Bw_1B \overset{B}{\times} \dots \overset{B}{\times} Bw_rB) &= f(Bu_0B \overset{B}{\times} w_1 \overset{B}{\times} \dots \overset{B}{\times} w_rB) \\ &= Bu_0B \cdot (u_1B, \dots, u_rB). \end{aligned}$$

Hence $f(Y) = \mathcal{F}_{u_0; u_1, \dots, u_r}$, since our varieties are projective.

Now, f is a separable map with connected fibers between smooth projective varieties, so the compatible splitting of the previous proposition pushes forward by Proposition 19. •

We will need the following lemmas to show that our configuration varieties have rational singularities.

Lemma 22 (Kempf [17]) *Suppose $f : Z \rightarrow X$ is a separable morphism with generically connected fibers between projective algebraic varieties Z and X , with X normal. Let L be an ample line bundle on X , and suppose that $H^i(Z, f^*L^{\otimes n}) = 0$ for all $i > 0$ and all $n \gg 0$.*

Then $R^i f_ \mathcal{O}_Z = 0$ for all $i > 0$. •*

Resuming the notation of Prop 20, let w_0, w_1, \dots, w_r be arbitrary Weyl group elements, and let $\lambda_0, \dots, \lambda_r$ be arbitrary weights of G . Let X_{r+1} be as before, and define the line bundle $\mathcal{L}_{\lambda_0, \dots, \lambda_r}$ on X_{r+1} and on $Y_{w_0, \dots, w_r} \subset X_{r+1}$ as the quotient of $G^{r+1} \times F$ by the B^{r+1} -action

$$(b_0, b_1, \dots, b_r) \cdot (g_0, g_1, \dots, g_r, a) \stackrel{\text{def}}{=} (g_0 b_0, b_0^{-1} g_1 b_1, \dots, b_{r-1}^{-1} g_r b_r, \lambda_1(b_0) \cdots \lambda_r(b_r) a).$$

Note that under the identification $X_{r+1} \cong (G/B)^{r+1}$, $\mathcal{L}_{\lambda_0, \dots, \lambda_r}$ is isomorphic to the Borel-Weil line bundle $G^{r+1} \times^{B^{r+1}} (\lambda_0^{-1}, \dots, \lambda_r^{-1})$.

Lemma 23 (van der Kallen) *Assume $\lambda_0, \dots, \lambda_r$ are dominant weights (possibly on the wall of the Weyl chamber). Then $H^i(Y_{w_0, \dots, w_r}, \mathcal{L}_{\lambda_0, \dots, \lambda_r}) = 0$ for all $i > 0$.*

Proof. Note that $\mathcal{L}_{\lambda_0, \dots, \lambda_r}$ is effective, but not necessarily ample, so we cannot deduce the conclusion directly from Theorem 17.

Recall the following facts from B -module theory [26], [15]:

(a) An excellent filtration of a B -module is one whose quotients are isomorphic to Demazure modules $H^0(\overline{BwB}, \mathcal{L}_\lambda)$, for Weyl group elements w and dominant weights λ .

(b) If M has an excellent filtration, and $\mathcal{E}(M) \stackrel{\text{def}}{=} \overline{BwB} \times^B M$ is the corresponding vector bundle on the Schubert variety $\overline{BwB} \subset G/B$, then $H^i(\overline{BwB}, \mathcal{E}(M)) = 0$ for all $i > 0$, and $H^0(\overline{BwB}, \mathcal{E}(M))$ has an excellent filtration.

(c) Polo's Theorem: If M has an excellent filtration, then so does $(\lambda^{-1}) \otimes M$ for any dominant weight λ .

Now consider the fiber bundle

$$\begin{array}{ccc} Y_{w_1, \dots, w_r} & \rightarrow & Y_{w_0, w_1, \dots, w_r} \\ & & \downarrow \\ & & \overline{Bw_0B} \end{array}$$

which leads to the spectral sequence

$$H^i(\overline{Bw_0B}, \mathcal{E}((\lambda_0^{-1}) \otimes H^j(Y_{w_1, \dots, w_r}, \mathcal{L}_{\lambda_1, \dots, \lambda_r}))) \Rightarrow H^{i+j}(Y_{w_0, w_1, \dots, w_r}, \mathcal{L}_{\lambda_0, \lambda_1, \dots, \lambda_r}).$$

By induction, assume that $H^j(Y_{w_1, \dots, w_r}, \mathcal{L}_{\lambda_1, \dots, \lambda_r}) = 0$ for $j > 0$, and that $H^0(Y_{w_1, \dots, w_r}, \mathcal{L}_{\lambda_1, \dots, \lambda_r})$ has an excellent filtration. Then applying (b) and (c), we find

$$H^i(Y_{w_0, w_1, \dots, w_r}, \mathcal{L}_{\lambda_0, \lambda_1, \dots, \lambda_r}) = H^i(\overline{Bw_0B}, \mathcal{E}((\lambda_0^{-1}) \otimes H^0(Y_{w_1, \dots, w_r}, \mathcal{L}_{\lambda_1, \dots, \lambda_r}))) = 0$$

for $i > 0$, and that $H^0(Y_{w_0, w_1, \dots, w_r}, \mathcal{L}_{\lambda_0, \lambda_1, \dots, \lambda_r})$ has an excellent filtration. •

Corollary 24 (of the proof) *With the above notation, $H^0(Y_{w_0, \dots, w_r}, \mathcal{L}_{\lambda_0, \dots, \lambda_r})$ has an excellent filtration as a B -module. •*

4.2 Frobenius splitting of Grassmannians

We would now like to push forward the Frobenius splittings found above for flag varieties to get splittings of configuration varieties. For this we need a combinatorial prerequisite.

Given a diagram $D = (C_1, C_2, \dots, C_r)$ with $\leq n$ rows, consider a sequence of permutations (Weyl group elements) $u_1, u_2, \dots \in \Sigma_n$ such that, for all j :

- (α) $\ell(u_j) = \ell(u_{j-1}) + \ell(u_{j-1}^{-1}u_j)$, and
- (β) $u_j(\{1, 2, \dots, |C_j|\}) = C_j$.

The first condition says that the sequence is increasing in the weak order on the Weyl group. In the next section, we will give an algorithm which produces such a sequence for any northwest diagram, so that the following theorem will apply:

Proposition 25 *If D a diagram which admits a sequence of permutations u_1, u_2, \dots satisfying (α) and (β) above, then the pair $\mathcal{F}_D \subset \text{Gr}(D)$ is compatibly split for any field F of positive characteristic.*

Hence over an algebraically closed field F of arbitrary characteristic,

- (a) *the cohomology groups $H^i(\mathcal{F}_D, \mathcal{L}_D) = 0$ for $i > 0$;*
- (b) *the restriction map $\text{rest}_\Delta : H^0(\text{Gr}(D), \mathcal{L}_D) \rightarrow H^0(\mathcal{F}_D, \mathcal{L}_D)$ is surjective;*
- (c) *\mathcal{F}_D is a normal variety.*

Proof. By (β), the maximal parabolic subgroups $P_C = \{(x_{ij}) \in GL(n) \mid x_{ij} = 0 \text{ if } i \notin C, j \in C\}$ satisfy $u_i B u_i^{-1} \subset P_{C_i}$. Write

$$\text{Gr}(D) = \text{Gr}(C_1) \times \dots \times \text{Gr}(C_r) \cong G/P_{C_1} \times \dots \times G/P_{C_r},$$

and consider the G -equivariant projection

$$\begin{aligned} \phi : (G/B)^r &\rightarrow \text{Gr}(D) \\ (g_1 B, \dots, g_r B) &\mapsto (g_1 u_1^{-1} P_{C_1}, \dots, g_r u_r^{-1} P_{C_r}) \end{aligned}$$

Then we have $\phi(u_1 B, \dots, u_r B) = (I P_{C_1}, \dots, I P_{C_r})$ and $\phi(\mathcal{F}_{w_0; u_1, \dots, u_r}) = \mathcal{F}_D$, where w_0 is the longest permutation. Since ϕ is a map with connected fibers between smooth projective varieties, we can push forward the compatible splitting for $\mathcal{F}_{w_0; u_1, \dots, u_r} \subset (G/B)^r$ found in the previous section. Applying Theorem 17 and Propositions 18 and 15, we have the assertions of the theorem. •

Note that (b) and (c) of the Proposition are equivalent to the projective normality of \mathcal{F}_D with respect to \mathcal{L}_D .

Conjecture 26 *For any diagram D , and any Weyl group elements u_0, u_1, \dots, u_r , the pairs $\mathcal{F}_D \subset \text{Gr}(D)$ and $\mathcal{F}_{u_0; u_1, \dots, u_r} \subset (G/B)^r$ are compatibly split, and the subvarieties have rational singularities.*

In order to prove the character formula in the last section of this paper, we will need stronger relations between the singular configuration varieties and their desingularizations. In particular, we will show that our varieties have rational singularities.

Lemma 27 *Let X, Y be algebraic varieties with an action of an algebraic group G , and $f : X \rightarrow Y$ an equivariant morphism. Assume that X has an open dense G -orbit $G \cdot x_0$, and take $y_0 = f(x_0)$, $G_0 = \text{Stab}_G y_0$.*

Then $f^{-1}(y_0) = \overline{G_0 \cdot x_0}$. In particular, if G_0 is connected, then $f^{-1}(y_0)$ is connected and irreducible.

Proof. For $F = \mathbf{C}$, this is trivial. Take $x_1 \in f^{-1}(y_0)$, and consider a path $x(t) \in X$ such that $x(0) = x_1$ and $x(t) \in G \cdot x_0$ for small $t > 0$. Then the path $f(x(t))$ lies in $G \cdot y_0$ for small $t \geq 0$, and we can lift it to a path $g(t) \in G$ such that $g(0) = \text{id}$ and $f(x(t)) = g(t) \cdot y_0$ for small $t \geq 0$. Then $\tilde{x}(t) \stackrel{\text{def}}{=} g(t)^{-1} \cdot x(t)$ satisfies $\tilde{x}(0) = x_1$, $\tilde{x}(t) \in G_0 \cdot x_0$ for small $t > 0$.

For general F , T. Springer has given the following clever argument. Assume without loss of generality that X is irreducible and $G \cdot y_0$ is open dense in Y . Since an algebraic map is generically flat, and $G \cdot y_0$ is open, all the irreducible components C of $f^{-1}(y_0)$ have the same dimension $\dim C = \dim X - \dim Y$. Let $Z = \overline{G \cdot C}$ be the closure of one of these components. Now, the restriction $f : Z \rightarrow Y$ also satisfies our hypotheses, with $C \subset Z$ again a component of the fiber of the restricted f , so we again have $\dim C = \dim Z - \dim Y$, and $\dim Z = \dim X$. Thus $G \cdot C$ is an open subset of X , since X is irreducible.

Now consider the open set $G \cdot C \cap G \cdot x_0 \subset X$. Choose a point z in this set which does not lie in any other component C' of our original $f^{-1}(y_0)$. For any other component C' , choose a similar point z' . But we have $g \cdot z \in C$, $g' \cdot z' = g_0 g \cdot z \in C'$ for some $g, g', g_0 \in G$, and in fact $g_0 \in G_0$. Thus $C' = g_0 \cdot C$, and G_0 permutes the components transitively. Hence, $G_0 \cdot x_0$ has at least as many irreducible components as the whole $f^{-1}(y_0)$, and the lemma follows. •

Proposition 28 (*Inamdar-van der Kallen*) *Suppose D_1, D_2 are diagrams admitting sequences of permutations with (α) and (β) as above, such that D_2 is obtained by removing some of the columns of D_1 . Denote $\mathcal{F}_1 = \mathcal{F}_{D_1}$, $\mathcal{F}_2 = \mathcal{F}_{D_2}$, $\mathcal{L}_2 = \mathcal{L}_{D_2}$, and consider the projection $\text{pr} : \mathcal{F}_1 \rightarrow \mathcal{F}_2$.*

Then:

(a) $H^0(\mathcal{F}_1, \text{pr}^* \mathcal{L}_2) = H^0(\mathcal{F}_2, \mathcal{L}_2)$, and this G -module has a good filtration (one whose quotients are isomorphic to $H^0(G/B, \mathcal{L}_\lambda)$ for dominant weights λ).

(b) $H^i(\mathcal{F}_1, \text{pr}^* \mathcal{L}_2) = H^i(\mathcal{F}_2, \mathcal{L}_2) = 0$ for all $i > 0$.

(c) $R^i \text{pr}_* \mathcal{O}_{\mathcal{F}_1} = 0$ for all $i > 0$.

(d) If F has characteristic zero, then \mathcal{F}_D has rational singularities for any northwest diagram D .

Proof. (i) Consider a sequence of permutations w_1, w_2, \dots, w_r (where r is the number of columns in D_1) such that $u_1 = w_1, u_2 = w_1 w_2, \dots$ satisfies (α) and

(β), and let $Y = Y_{w_0, w_1, \dots, w_r}$, (where w_0 is the longest permutation). Then we have a commutative diagram of surjective morphisms

$$\begin{array}{ccc} Y & \xrightarrow{\Phi_1} & \mathcal{F}_1 \\ & \searrow \Phi_2 & \downarrow \text{pr} \\ & & \mathcal{F}_2 \end{array}$$

where $\Phi_j = \phi \circ f$, where ϕ and f are the maps defined in the proofs of Propositions 21 and 25 in the cases $D = D_j$. All of these spaces have dense G -orbits. Furthermore, the stabilizer of a general point in \mathcal{F}_D is an intersection of parabolic subgroups and is connected. Thus, by the above lemma, the fibers of Φ_1 are generically connected.

(ii) Now (i) and Lemma 23 insure that the hypotheses of Kempf's lemma (Proposition 22) are satisfied. Thus $R^i(\Phi_1)_* \mathcal{O}_Y = 0$ for $i > 0$, and by the Leray spectral sequence we have, for all $i \geq 0$,

$$H^i(Y, \Phi_1^* \text{pr}^* \mathcal{L}_2) = H^i(\mathcal{F}_1, (\Phi_1)_*(\Phi_1)^* \text{pr}^* \mathcal{L}_2).$$

(iii) Furthermore, \mathcal{F}_1 is normal by the previous Proposition, and Φ_1 is separable with connected fibers, so

$$\begin{aligned} (\Phi_1)_*(\Phi_1)^* \text{pr}^* \mathcal{L}_2 &\cong [(\Phi_1)_*(\Phi_1)^* \mathcal{O}_{\mathcal{F}_1}] \otimes \text{pr}^* \mathcal{L}_2 \\ &\cong \text{pr}^* \mathcal{L}_2. \end{aligned}$$

Thus $H^i(Y, \Phi_1^* \text{pr}^* \mathcal{L}_2) = H^i(\mathcal{F}_1, \text{pr}^* \mathcal{L}_2)$ for all $i \geq 0$.

(iv) An exactly similar argument shows that $H^i(Y, \Phi_2^* \mathcal{L}_2) = H^i(\mathcal{F}_2, \mathcal{L}_2)$ for all $i \geq 0$. But $\Phi_2^* = \Phi_1^* \text{pr}^*$, so for all i ,

$$H^i(\mathcal{F}_1, \text{pr}^* \mathcal{L}_2) = H^i(Y, \Phi_2^* \mathcal{L}_2) = H^i(\mathcal{F}_2, \mathcal{L}_2).$$

But we saw in Lemma 23 that $H^i(Y, \Phi_2^* \mathcal{L}_2)$ vanishes for $i > 0$, so (b) of the present Proposition follows.

(v) We also saw in the proof of Lemma 23 that $H^0(Y, \Phi_2^* \mathcal{L}_2)$ has an excellent filtration as a B -module. But this is equivalent to it having a good filtration as a G -module, so (a) follows.

(vi) Now consider the spectral sequence

$$R^i \text{pr}_* R^j (\Phi_1)_* \mathcal{O}_Y \Rightarrow R^{i+j} (\Phi_2)_* \mathcal{O}_Y.$$

For $i > 0$, we have $R^i(\Phi_1)_* \mathcal{O}_Y = 0$ and $R^i(\Phi_2)_* \mathcal{O}_Y = 0$ by (ii) above. Because of this and the normality of \mathcal{F}_1 , we have for all $i > 0$,

$$\begin{aligned} 0 &= R^i(\Phi_2)_* \mathcal{O}_Y \\ &= R^i \text{pr}_* (\Phi_1)_* \mathcal{O}_Y \\ &= R^i \text{pr}_* \mathcal{O}_{\mathcal{F}_1}. \end{aligned}$$

this shows (c).

(vii) Now take $D_2 = D$ an arbitrary northwest diagram, and $D_1 = \widehat{D}$ its maximal blowup. Then pr is a resolution of singularities by Proposition 14. Assume, as we will show in the next section, that D_1 admits a sequence of permutations as required. Then (c) holds, and this is precisely the definition of rational singularities in characteristic zero, so we have (d). •

4.3 Monotone sequences of permutations

Let $D = (C_1, C_2, \dots, C_r)$ be a northwest diagram with $\leq n$ rows. In this section, we will construct by a recursive algorithm a sequence of permutations $u_1, u_2, \dots, u_r \in \Sigma_n$ satisfying the conditions of the previous section: for all j ,

(α) $\ell(u_j) = \ell(u_{j-1}) + \ell(u_{j-1}^{-1}u_j)$, and

(β) $u_j(\{1, 2, \dots, |C_j|\}) = C_j$.

Given a column $C \subset \mathbf{N}$, a *missing tooth* of C is an integer i such that $i \in C$ but $i+1 \notin C$. The only C with no missing teeth are the initial intervals $\{1, 2, 3, \dots, i\}$. Now consider the leftmost column C_J of our diagram D with missing teeth, and let I be the highest missing tooth in C_J . Because D is northwest, I is a missing tooth of C_j for all $j \geq J$. Define the *derived diagram* $D' = (C'_1, C'_2, \dots)$ by

$$C'_j = \{i \mid i \in C, i < I\} \cup \{i-1 \mid i \in C, i \geq I\}.$$

That is, we erase the I th row of D and push all lower squares upward by one row (orthodontia).

Lemma 29 *If D is northwest with $\leq n$ rows, then D' is northwest with $\leq n-1$ rows and no missing teeth in the first $J-1$ columns*

Proof. The only doubtful case in checking the northwest property is that of two squares (i_1, j_1) and (i_2, j_2) in D' with $j_1 < J_n \leq j_2$ and $i_1 > i_2$. Since $j_1 < J_n$, we have $C_{j_1} = \{1, 2, \dots, i_1-1, i_1, \dots\}$, so that $i_2 \in C_{j_1}$ and $i_2 \in C'_{j_1}$. Hence $(i_2, j_1) \in D'$ as required. •

Now, consider the following elements of Σ_n :

$$\kappa_I^{(n)}(i) = \begin{cases} i & \text{if } i < I \\ i+1 & \text{if } I \leq i < n \\ I & \text{if } i = n \end{cases}$$

Then $\kappa_1^{(n)}, \dots, \kappa_n^{(n)}$ are minimum length coset representatives of the quotient Σ_n/Σ_{n-1} , and for any permutation $\pi \in \Sigma_{n-1}$, we have $\ell(\kappa_n\pi) = \ell(\kappa_n) + \ell(\pi)$. Furthermore, we have

$$C_j = \begin{cases} \{1, 2, \dots, |C_j|\} & \text{for } j < J \\ \kappa_I^{(n)}C'_j & \text{for } j \geq J \end{cases}$$

Now, starting with D , a northwest diagram with $\leq n$ rows, we can define a sequence of derived diagrams $D = D^{(n)}, D^{(n-1)}, \dots, D^{(1)}$, where $D^{(i)} = (D^{(i+1)})'$ is a northwest diagram with $\leq i$ rows. We have the associated sequence of (I, J) , which we denote $(I_n, J_n), (I_{n-1}, J_{n-1}), \dots, (I_1, J_1)$. Finally, for each column C_j , let $m = m(j)$ be the largest number with $J_m \leq j$, and define

$$u_j = \kappa_{I_n}^{(n)} \kappa_{I_{n-1}}^{(n-1)} \cdots \kappa_{I_m}^{(m)}$$

(possibly an empty product, in which case $u_j = \text{id}$). This is a reduced decomposition of u_j , in the sense that $\ell(u_j)$ is the sum of the lengths of the factors. Since $\kappa_i^{(i)} = \text{id}$, and $J_n \leq J_{n-1} \leq \dots$, each u_j is an initial string of u_{j+1} . Thus the u_j have the desired monotonicity property (α) . Property (β) follows easily by induction on n .

5 A Weyl character formula

The results of the last two sections allow us to apply the Atiyah-Bott Fixed Point Theorem to compute the characters of the Schur modules for northwest diagrams. To apply this theorem, we must examine the points of \mathcal{F}_D fixed under the action of H , the group of diagonal matrices. We must also understand the action of H on the tangent spaces at the fixed points.

Using other Lefschetz-type theorems, this data also specifies the dimensions of the Schur modules and the Betti numbers of smooth configuration varieties.

5.1 Fixed points and tangent spaces

The following formula is due to Atiyah and Bott [2] in the complex analytic case, and was extended to the algebraic case by Nielsen [25].

Theorem 30 *Let F be an algebraically closed field, and suppose the torus $H = (F^\times)^n$ acts on a smooth projective variety X with isolated fixed points, and acts equivariantly on a line bundle $L \rightarrow X$. Then the character of H acting on the cohomology groups of L is given by:*

$$\sum_i (-1)^i \text{tr}(h | H^i(X, L)) = \sum_{p \text{ fixed}} \frac{\text{tr}(h | L|_p)}{\det(\text{id} - h | T_p^*X)},$$

where p runs over the fixed points of H , $L|_p$ denotes the fiber of L above p , and T_p^*X is the cotangent space. •

We will apply the formula for $X = \mathcal{F}_D$ a smooth configuration variety, where $D = (C_1, C_2, \dots)$ is a lexicographic northwest diagram with $\leq n$ rows and $\widehat{D} = D$.

Fixed points. Assume for now that the columns are all distinct. Let $H = \{h = \text{diag}(x_1, \dots, x_n) \in GL(n)\}$ act on $\text{Gr}(D)$ and \mathcal{F}_D by the restriction of the $GL(n)$ action. Then by Proposition 14, we have $\mathcal{F}_D = \mathcal{I}_D = \{(V_C)_{C \in D} \in \text{Gr}(D) \mid C \subset C' \Rightarrow V_C \subset V_{C'}\}$, a smooth variety.

A point in $\mathcal{F}_D \subset \text{Gr}(D) = \text{Gr}(C_1) \times \text{Gr}(C_2) \times \dots$ is fixed by H if and only if each component is fixed. Now, the fixed points of H in $\text{Gr}(l, F^n)$ are the coordinate planes $E_{k_1, \dots, k_l} = \text{Span}(e_{k_1}, \dots, e_{k_l})$, where the e_k are coordinate vectors in F^n (c.f. [12]). For instance, the fixed points in $\text{Gr}(C)$ are the n coordinate lines Fe_k . We may describe the fixed points in $\text{Gr}(C)$ as $E_S = \text{Span}(e_k \mid k \in S)$, where $S \subset \{1, \dots, n\}$ is any set with $|S| = |C|$.

Hence the fixed points in \mathcal{F}_D are as follows: Take a function τ which assigns to any column C a set $\tau(C) \subset \{1, \dots, n\}$ with $|\tau(C)| = |C|$, and $C \subset C' \Rightarrow \tau(C) \subset \tau(C')$. We will call such a τ a *standard column tabloid* for D . Then the fixed point corresponding to τ is $E_\tau = (E_{\tau(C)})_{C \in D}$.

Tangent spaces at fixed points. We may naturally identify the tangent space $T_{V_0} \text{Gr}(l, F^n) = \text{Hom}_F(V_0, F^n/V_0)$. If V_0 is a fixed point (that is, a space stable under H), then $h \in H$ acts on a tangent vector $\phi \in \text{Hom}_F(V_0, F^n/V_0)$ by $(h \cdot \phi)(v) = h(\phi(h^{-1}v))$. For $(V_C)_{C \in D} \in \text{Gr}(D)$, we have $T_{(V_C)} \text{Gr}(D) = \bigoplus_{C \in D} \text{Hom}(V_C, F^n/V_C)$. Furthermore, if $(V_C)_{C \in D} \in \mathcal{F}_D$, then

$$T_{(V_C)} \mathcal{F}_D = \left\{ \phi = (\phi_C)_{C \in D} \in \bigoplus_{C \in D} \text{Hom}(V_C, F^n/V_C) \mid C \subset C' \Rightarrow \phi_{C'}|_{V_C} \equiv \phi_C \pmod{V_{C'}} \right\}$$

(that is, the values of ϕ_C and $\phi_{C'}$ on V_C agree up to translation by elements of $V_{C'}$). See [12].

For a fixed point $E_\tau = (E_{\tau(C)})$, we will find a basis for T_{E_τ} consisting of eigenvectors of H . Now, the eigenvectors in $T_{E_\tau} \text{Gr}(D) = \bigoplus_{C \in D} \text{Hom}(V_C, F^n/V_C)$ are precisely $\phi^{ijC_0} = (\phi_C^{ijC_0})_{C \in D}$, where $i, j \leq n$, C_0 is a fixed column of D , and $\phi_C^{ijC_0}(e_i) \stackrel{\text{def}}{=} \delta_{C_0, C} \delta_{ij} e_j$ (δ being the Kronecker delta). The eigenvalue is

$$h \cdot \phi^{ijC_0} = \text{diag}(x_1, \dots, x_n) \cdot \phi^{ijC_0} = x_i^{-1} x_j \phi^{ijC_0}.$$

To obtain eigenvectors of $T_{E_\tau} \mathcal{F}_D$, we must impose the compatibility conditions. An eigenvector ϕ with eigenvalue $x_i^{-1} x_j$ must be a linear combination $\phi = \sum_{C \in D} a_C \phi^{ijC}$ with $a_C \in F$. By the compatibility, we have that

$$C \subset C', i \in t(C), j \notin t(C') \Rightarrow a_C = a_{C'}.$$

We wish to find the number d_{ij} of linearly independent solutions of this condition for a_C .

Given a poset with a relation \subset , define its *connected components* as the equivalence classes generated by the elementary relations $x \sim y$ for $x \subset y$. Now for a given i, j consider the poset whose elements are those columns C of D such

that $i \in t(C)$, $j \notin t(C)$, with the relation of ordinary inclusion. Then d_{ij} is the number of components of this poset.

Note that the eigenvectors for all the eigenvalues span the tangent space. Thus

$$\det(\text{id} - h \mid T_{E_\tau}^*) = \prod_{i \neq j} (1 - x_i x_j^{-1})^{d_{ij}(\tau)}.$$

Bundle fibers above fixed points. Finally, let us examine the line bundles L on \mathcal{F}_D obtained by giving each column C a multiplicity $m(C) \geq 0$. If $m(C) > 0$ for all columns C of D , then $L \cong \mathcal{L}_{D'}$ for the diagram D' with the same columns as D , each repeated $m(C)$ times. If some of the $m(C) = 0$, then L is the *pullback* of $\mathcal{L}_{D'}$ for the diagram D' with the same columns as D , each taken $m(C)$ times, where 0 times means deleting the column. In the second case, L is effective, but not ample.

It follows easily from the definition that

$$\text{tr}(h \mid L|_{E_\tau}) = x_1^{-\text{wt}_1(\tau)} \cdots x_n^{-\text{wt}_n(\tau)},$$

where

$$\text{wt}_i(\tau) = \sum_{\substack{C \\ i \in \tau(C)}} m(C).$$

Hence we obtain:

$$\sum_i (-1)^i \text{tr}(h \mid H^i(\mathcal{F}_D, \mathcal{L}_D)) = \sum_\tau \frac{\prod_i x_i^{-\text{wt}_i(\tau)}}{\prod_{i \neq j} (1 - x_i x_j^{-1})^{d_{ij}(\tau)}},$$

where τ runs over the standard column tabloids of D .

5.2 Character formula

We summarize in combinatorial language the implications of the previous section.

We think of a diagram D as a list of columns $C_1, C_2, \dots \subset \mathbf{N}$, possibly with repeated columns. Given a diagram D , the *blowup diagram* \widehat{D} is the diagram whose columns consist of all the columns of D and all possible intersections of these columns. We will call the columns which we add to D to get \widehat{D} the *phantom columns*.

We may define a *standard column tabloid* τ for the diagram \widehat{D} with respect to $GL(n)$, to be a filling (i.e. labeling) of the squares of \widehat{D} by integers in $\{1, \dots, n\}$, such that:

- (i) the integers in each column are strictly increasing, and
- (ii) if there is an inclusion $C \subset C'$ between two columns, then all the numbers in the filling of C also appear in the filling of C' .

Given a tabloid τ for \widehat{D} , define integers $\text{wt}_i(\tau)$ to be the number of times i appears in the filling, but *not counting i 's which appear in the phantom columns*. Also define integers $d_{ij}(\tau)$ to be the number of connected components of the following graph: the vertices are columns C of \widehat{D} such that i appears in the filling of C , but j does not; the edges are (C, C') such that $C \subset C'$ or $C' \subset C$. (An empty graph has zero components.)

Recall that a diagram D is *northwest* if $i \in C_j, i' \in C_{j'} \Rightarrow \min(i, i') \in C_{\min(j, j')}$. The following theorem applies without change to *northeast* diagrams and any other diagrams obtainable from northwest ones by rearranging the order of the rows and the order of the columns. Also, we can combine it with Theorem 6 to compute the character for the complement of a northwest diagram in an $n \times r$ rectangle.

Denote a diagonal matrix by $h = \text{diag}(x_1, \dots, x_n)$.

Theorem 31 *Suppose D is a northwest diagram with $\leq n$ rows, and F an algebraically closed field. Then:*

(a) *The character of the Weyl module W_D (for $GL(n, F)$) is given by*

$$\text{char}_{W_D}(h) = \sum_{\tau} \frac{\prod_i x_i^{-\text{wt}_i(\tau)}}{\prod_{i \neq j} (1 - x_i x_j^{-1})^{d_{ij}(\tau)}}$$

where τ runs over the standard tabloids for \widehat{D} .

(b) *For F of characteristic zero, the character of the Schur module S_D (for $GL(n, F)$) is given by*

$$\text{char}_{S_D}(h) = \sum_{\tau} \frac{\prod_i x_i^{\text{wt}_i(\tau)}}{\prod_{i \neq j} (1 - x_i^{-1} x_j)^{d_{ij}(\tau)}}$$

where τ runs over the standard tabloids for \widehat{D} .

Example. Consider the following diagram and some of its standard tabloids for $n = 3$:

$$\begin{array}{cccc}
 & & \square & \\
 D = & \square & \square & \square \\
 & & \square & \square & \square \\
 & & & & & & \square \\
 \widehat{D} = & \square & \square & \square & \square \\
 & & & & & & \square & \square & \square
 \end{array}$$

$$\begin{array}{cccc}
 & & 1 & \\
 \tau_1 = & 2 & 2 & 2 & 2 \\
 & & 3 & 3 & 3 \\
 & & & & & & 2 & 2 & 1
 \end{array}
 \quad
 \begin{array}{cccc}
 & & 2 & \\
 \tau_2 = & 3 & 3 & 1 & 1 \\
 & & 3 & 3 & 3
 \end{array}$$

Note that since the order of the columns is irrelevant, we may insert the phantom column of \widehat{D} in the middle of D rather than at the end. The tabloid τ_1 has weight monomial $x_1 x_2^3 x_3^3$ (since the entry 2 in the phantom column is not counted in

the weight), and the denominator multiplicities are: $d_{13} = d_{21} = d_{23} = d_{31} = d_{32} = 1$, the rest zero. Thus, its contribution to $\text{char}S_D$ is:

$$\frac{x_1 x_2^3 x_3^3}{(1 - x_1^{-1} x_3)(1 - x_2^{-1} x_1)(1 - x_2^{-1} x_3)(1 - x_3^{-1} x_1)(1 - x_3^{-1} x_2)} .$$

The tabloid τ_2 has weight $x_1^2 x_2 x_3^4$, and $d_{31} = 2$, $d_{12} = d_{21} = d_{32} = 1$, the rest zero. Its contribution is:

$$\frac{x_1^2 x_2 x_3^4}{(1 - x_3^{-1} x_1)^2 (1 - x_1^{-1} x_2)(1 - x_2^{-1} x_1)(1 - x_3^{-1} x_2)} .$$

New standard tabloids can be obtained from any given one by applying a permutation of $\{1, 2, 3\}$ to the entries, and rearranging the new entries in the columns to make them increasing. For instance, $\tau_3 = \pi\tau_2$, where π is the three-cycle (123) . Altogether there are 24 standard tabloids for \widehat{D} grouped in four orbits of Σ_3 . Note that in this case the standard tabloids are standard skew tableaux in the usual sense: they are fillings with the columns strictly increasing top-to-bottom, and the rows weakly increasing right-to-left. This is true in general when \widehat{D} is a skew diagram, though not all the standard tableaux are obtained in this way.

Working out our formula, we find that the 24 rational terms simplify to the polynomial

$$\begin{aligned} \text{char } S_D = & (x_1 x_2 + x_1 x_3 + x_2 x_3) \\ & (x_1^3 x_2^2 + x_1^2 x_2^3 + x_1^3 x_2 x_3 + 2 x_1^2 x_2^2 x_3 \\ & + x_1 x_2^3 x_3 + x_1^3 x_3^2 + 2 x_1^2 x_2 x_3^2 + 2 x_1 x_2^2 x_3^2 \\ & + x_2^3 x_3^2 + x_1^2 x_3^3 + x_1 x_2 x_3^3 + x_2^2 x_3^3) . \end{aligned}$$

Since D is a skew diagram, we can also compute this using the Littlewood-Richardson Rule:

$$\text{char } S_D = s_{(4,3,0)} + s_{(4,2,1)} + s_{(3,3,1)} ,$$

where s_λ is the classical Schur polynomial with highest weight λ . There seems to be a subtle relationship between the column tabloids of \widehat{D} and the LR tableaux of D , and it may be possible to prove the LR rule using the present methods. •

Proof of the Theorem. (i) Consider the map $\text{pr} : \mathcal{F}_{\widehat{D}} \rightarrow \mathcal{F}_D$, and the pullback line bundle $\text{pr}^* \mathcal{L}_D$. This is the bundle on $\mathcal{F}_{\widehat{D}}$ which corresponds to giving multiplicity $m_C = 0$ to the phantom columns C of \widehat{D} . Let RHS denote the right hand side of our formula in (a). Then by the analysis of Section 5.1, RHS is equal to the right hand side of the Atiyah-Bott formula (Theorem 30) for $X = \mathcal{F}_{\widehat{D}}$, $L = \text{pr}^* \mathcal{L}_D$. Thus

$$\text{RHS} = \text{char} \sum_i (-1)^i H^i(\mathcal{F}_{\widehat{D}}, \text{pr}^* \mathcal{L}_D).$$

(ii) By Proposition 28, we have $H^i(\mathcal{F}_{\widehat{D}}, \text{pr}^* \mathcal{L}_D) = 0$ for $i > 0$, and $H^0(\mathcal{F}_{\widehat{D}}, \text{pr}^* \mathcal{L}_D) = H^0(\mathcal{F}_D, \mathcal{L}_D)$. Thus $\text{RHS} = \text{char } H^0(\mathcal{F}_D, \mathcal{L}_D)$.

(iii) By Proposition 25, the restriction of global sections of \mathcal{L}_D from $\text{Gr}(D)$ to \mathcal{F}_D is surjective, and we have

$$\begin{aligned} \text{RHS} &= \text{char } H^0(\mathcal{F}_D, \mathcal{L}_D) \\ &= \text{char } \text{Im}(\text{rest} : H^0(\text{Gr}(D), \mathcal{L}_D) \rightarrow H^0(\mathcal{F}_D, \mathcal{L}_D)) \\ &= \text{char } W_D. \end{aligned}$$

The last equality holds by Proposition 2, and we have proved (a). Then (b) follows because $S_D = (W_D)^*$. •

5.3 Dimension formula

We compute the dimension of the $GL(n)$ -module S_D by means of the Hirzebruch-Riemann-Roch Theorem. Again, each standard column tabloid of \widehat{D} gives a contribution, which is a rational number, possibly negative. The contribution is the value of a certain multivariable polynomial RR at a sequence of integers specific to the tabloid. For practical purposes, it is easier to find the dimension by specializing our character formula above to the identity element. However, the present formula gives quite a striking expression for this dimension as a sum and difference of fractions.

Let $M = \dim \mathcal{F}_D$, the dimension of the configuration variety (which may be computed as in Corollary 4). Define an $(M + 1)$ -variable polynomial, homogeneous of degree M , by

$$\text{RR}_M(b; r_1, \dots, r_M) = \text{coeff at } U^M \text{ of } \left(\exp(bU) \prod_{i=1}^M \frac{r_i U}{1 - \exp(-r_i U)} \right),$$

where the right side is understood as a Taylor series in U . For example,

$$\text{RR}_2(b; r_1, r_2) = \frac{1}{2}b^2 + \frac{1}{2}r_1 b + \frac{1}{2}r_2 b + \frac{1}{12}r_1^2 + \frac{1}{4}r_1 r_2 + \frac{1}{12}r_2^2.$$

For general M the polynomial is factorable.

Now, for each standard tabloid τ , define a multiset of integers $r(\tau) = \{r_1, r_2, \dots, r_M\}$ by inserting the entry $i - j$ with multiplicity $d_{ij}(\tau)$ for each ordered pair (i, j) :

$$r(\tau) = \bigsqcup_{1 \leq i, j \leq n} d_{ij}(\tau) \cdot \{i - j\}.$$

That is, the total multiplicity of the integer k in the multiset $r(\tau)$ is

$$\sum_{\substack{1 \leq i, j \leq n \\ i - j = k}} d_{ij}(\tau).$$

Let $b(\tau)$ be the sum of the entries of the tabloid τ of \widehat{D} , *not counting entries in the phantom columns*:

$$b(\tau) = \sum_{(i,j) \in D} \tau(i,j) .$$

Theorem 32 *The dimension of the Weyl module W_D for $GL(n)$ (and of the Schur module S_D in characteristic zero) is*

$$\begin{aligned} \dim W_D = \dim S_D &= \sum_{\tau} \frac{RR_M(b(\tau); r(\tau))}{\prod_{k=1}^M r(\tau)_i} \\ &= \sum_{\tau} \frac{RR_M(b(\tau); r(\tau))}{\prod_{i \neq j} (i-j)^{d_{ij}(\tau)}} \end{aligned}$$

where τ runs over the standard column tabloids for \widehat{D} .

Examples. Consider the diagram

$$D = \square \square \dots \square$$

consisting of a single row of k squares, and take $n = 2$. Then $\mathcal{F}_D = \mathbf{P}^1$ and $RR_1(b; r_1) = b + \frac{1}{2}r_1$. Also $\mathcal{L}_D \cong \mathcal{O}(k)$, and our formula becomes

$$\begin{aligned} \dim S_D &= \dim H^0(\mathbf{P}^1, \mathcal{O}(k)) \\ &= \frac{2k + \frac{1}{2}(1)}{2-1} + \frac{k + \frac{1}{2}(-1)}{1-2} \\ &= k + 1 . \end{aligned}$$

Now let D be the diagram of the previous section, whose list of columns is $D = \{\{1, 2\}, \{2, 3\}, \{2, 3\}, \{3\}\}$, and take $n = 3$. Then the variety \mathcal{F}_D is 5 dimensional, and the Riemann-Roch polynomial $RR_5(b; r_1, \dots, r_5)$ is a 6-variable polynomial with 172 terms homogeneous of degree 5. However, the formula calls only for evaluating this polynomial at small integer arguments, which is well within the range of easy computer calculations. The result is:

$$\begin{aligned} \dim S_D &= \frac{87437}{1440} - \frac{2795}{288} - \frac{1431}{640} + \frac{71}{5760} - \frac{110123}{5760} + \frac{1431}{640} - \frac{71}{5760} - \frac{25}{1152} \\ &\quad - \frac{2795}{288} + \frac{160}{57} - \frac{1152}{25} - \frac{640}{3} - \frac{640}{3} - \frac{25}{1152} + \frac{57}{160} - \frac{2795}{288} \\ &\quad - \frac{25}{1152} - \frac{71}{5760} + \frac{1431}{640} - \frac{110123}{5760} + \frac{71}{5760} - \frac{1431}{640} - \frac{2795}{288} + \frac{87437}{1440} \\ &= 45 , \end{aligned}$$

which agrees with the character formula derived above. •

The Theorem follows immediately from the Hirzebruch-Riemann-Roch Theorem [4], combined with Bott's Residue Formula [5], [3], according to the method of Ellingsrud and Stromme [9].

Proposition 33 *Suppose the torus $T = \mathbf{C}^*$ is one-dimensional, and acts with isolated fixed points.*

Let $\mathbf{v} = 1$ in the Lie algebra $\mathfrak{t} = \mathbf{C}$, and at each T -fixed point p , let $b(p) = \text{tr}(\mathbf{v} | L|_p)$. Denote the \mathbf{v} -eigenspace decomposition of the tangent space by $T_p X = \bigoplus_{i=1}^M \mathbf{C}_{r_i(p)}$, where $r_i(p)$ are the integer eigenvalues. Also, define the polynomial $RR_M(b; r_1, \dots, r_M)$ as above.

Then the algebraic Euler characteristic of L is given by:

$$\sum_i (-1)^i \dim H^i(X, L) = \sum_{p \text{ fixed}} \frac{RR_M(b(p); r_1(p), \dots, r_M(p))}{\det(\mathbf{v} | T_p X)} .$$

In our case, we consider $X = \mathcal{F}_{\widehat{D}}$ over the field $F = \mathbf{C}$ and take the T in the Proposition to be $\mathbf{C}^* \subset GL(n)$, $q \rightarrow \text{diag}(q^{-1}, q^{-2}, \dots, q^{-n})$. (This is the principal one-dimensional subtorus corresponding to the half-sum of positive roots.) Then the tangent eigenvalues at the fixed points E_τ specialize to the subtorus, and give us the information required to compute the dimension of $W_D = H^0(\mathcal{F}_{\widehat{D}}, \pi^* \mathcal{L}_D)$. (We may check directly that the fixed points of the subtorus are identical to those of the large torus of all diagonal matrices.) Now, the dimension of the Weyl module is independent of the field F by the Remark in section 1.2, so our Theorem holds in general. •

5.4 Betti numbers

In this section, we compute the betti numbers of the smooth configuration varieties of Section 5.1.

Proposition 34 *(Bialynicki-Birula [6]) Let X be a smooth projective variety over an algebraically closed field F , acted on by the one-dimensional torus F^\times with isolated fixed points. Then there is a decomposition*

$$X = \coprod_{p \text{ fixed}} X_p,$$

where the X_p are disjoint, locally closed, F^\times -invariant subvarieties, each isomorphic to an affine space $X_p \cong \mathbf{A}^{d^+(p)}$.

The dimensions $d^+(p)$ are given as follows. Let the tangent space $T_p X \cong \bigoplus_{n \in \mathbf{Z}} a_n(p) F_n$, where $a_n(p) \in \mathbf{N}$ and F_n is the one-dimensional representation of F^\times for which the group element $t \in F^\times$ acts as the scalar t^n . Then

$$d^+(p) = \sum_{n>0} a_n(p).$$

•

Over \mathbf{C} , the above proposition does not quite give a CW decomposition for X , since the boundaries of the cells need not lie in cells of lower dimension.

Nevertheless, $\dim_{\mathbf{R}} \partial X_p \leq \dim_{\mathbf{R}} X_p - 2$, and this is enough to fix the betti numbers $\beta_i = \dim_{\mathbf{R}} H^i(X, \mathbf{R})$. Namely, $\beta_{2i} = \#\{p \mid d^+(p) = i\}$, and $\beta_{2i+1} = 0$.

Now apply this to $X = \mathcal{F}_D$ of Section 5.1, acted on by the n -dimensional torus H . Again consider the principal embedding

$$\begin{aligned} F^\times &\rightarrow H \\ q &\mapsto \text{diag}(q^{-1}, q^{-2}, \dots, q^{-n}). \end{aligned}$$

An eigenvector of weight $x_i x_j^{-1}$ is of positive F^\times -weight exactly when $i < j$. Thus, for a fixed point (standard tabloid) τ of D , define

$$d^+(\tau) = \sum_{i < j} d_{ij}(\tau).$$

We then have the

Proposition 35 *Suppose $F = \mathbf{C}$, and D is a northwest diagram with $\leq n$ rows and $\widehat{D} = D$. Then the betti numbers of \mathcal{F}_D are*

$$\beta_{2i} = \#\{\tau \mid d^+(\tau) = i\}, \quad \beta_{2i+1} = 0,$$

and the Poincare polynomial

$$P(x, \mathcal{F}_D) \stackrel{\text{def}}{=} \sum_i \beta_i x^i = \sum_{\tau} x^{2d^+(\tau)},$$

where τ runs over the standard tabloids of D . •

In fact, our proof shows the above proposition for a broader class of spaces. Suppose $D = (C_1, C_2, \dots)$ is an arbitrary diagram such that the variety

$$\text{Inc}_D \stackrel{\text{def}}{=} \{(V_C)_{C \in D} \in \text{Gr}(D) \mid C \subset C' \Rightarrow V_C \subset V_{C'}\}$$

is smooth. Then the proposition holds with \mathcal{F}_D replaced by Inc_D .

References

- [1] K. Akin, D. Buchsbaum, and J. Weyman, Schur functors and Schur complexes, *Adv. Math.* **44** (1982), 207-277.
- [2] M.F. Atiyah and R. Bott, A Lefschetz fixed point formula for elliptic complexes, *Ann. Math.* **88** (1968), 451-491.
- [3] M.F. Atiyah and R. Bott, The Moment map and equivariant cohomology, *Topology* **23** (1985), 1-28.

- [4] M.F. Atiyah and I.M. Singer, Index of elliptic operators *III*, *Ann. Math.* **87** (1968), 546–604.
- [5] R. Bott, A Residue formula for holomorphic vector fields, *J. Diff. Geom.* **1** (1967), 311–330.
- [6] A. Bialynicki-Birula, Some theorems on actions of algebraic groups, *Ann. Math.* **98** (1973), 480–497.
- [7] M. Bozek and K. Drechsler, *Math. Nachr.* **99** (1980), 251.
- [8] M. Bozek and K. Drechsler, *Beitr. Alg. Geom.* **20** (1985), 203.
- [9] G. Ellingsrud and S.A. Stromme, Bott’s formula and enumerative geometry, preprint 1994.
- [10] W. Fulton and J. Harris, “Representation Theory, A First Course”, Springer Verlag, New York, 1991.
- [11] W. Fulton and R. MacPherson, A compactification of configuration spaces, *Ann. Math.* **139** (1994), 183–225.
- [12] J. Harris, “Algebraic Geometry, a First Course”, Springer Verlag, New York, 1991.
- [13] S.P. Inamdar, Ph.D. Thesis, Tata Institute of Fundamental Research, Bombay, 1993.
- [14] G.D. James and M.H. Peel, Specht series for skew representations of symmetric groups, *J. Alg.* **56** (1979), 343–364.
- [15] W. van der Kallen, “Frobenius splittings and B -modules”, Tata Inst. Lect. Math. & Phys. **84**, Springer Verlag, 1993.
- [16] W. van der Kallen and P. Magyar, The Space of triangles, vanishing theorems, and combinatorics, preprint alg-geom/9601022.
- [17] G. Kempf, Linear systems on homogeneous spaces, *Ann. Math.* **103** (1976), 557–591.
- [18] W. Kraskiewicz, Reduced decompositions in Weyl groups, preprint.
- [19] W. Kraskiewicz and P. Pragacz, *Foncteurs de Schubert*, C.R. Acad. Sci. Paris **304** Ser I No 9 (1987), 207–211.
- [20] P. Magyar, Bott-Samelson varieties and configuration spaces, preprint alg-geom/9611019.
- [21] O. Mathieu, Filtrations of B -modules, *Duke Math. J.* **59** (1989), 421–442.

- [22] V.B. Mehta and V. Srinivas, Normality of Schubert varieties in G/B , *Amer. J. Math.* **109** (1987), 987-989.
- [23] V.B. Mehta and A. Ramanathan, Frobenius splitting and cohomology vanishing for Schubert varieties, *Ann. Math.* **122** (1985), 27-40.
- [24] G. Murphy and M.H. Peel, Representation of Symmetric Groups by Bad Shapes, *J. Algebra* **116** (1988), 143-154.
- [25] H.A. Nielsen, Diagonalizably linearized coherent sheaves, *Bull. Soc. Math. France* **102** (1974), 85-97.
- [26] P. Polo, Varietes de Schubert et excellentes filtrations, *Asterisque* **173-174** (1989), 281-311.
- [27] S. Ramanan and A. Ramanathan, Projective normality of flag varieties and Schubert varieties, *Publ. Math. I.H.E.S.* **65** (1987), 61-90.
- [28] A. Ramanathan, Schubert varieties are arithmetically Cohen-Macaulay, *Invent. Math.* **80** (1985), 283-294.
- [29] A. Ramanathan, Equations defining Schubert varieties and Frobenius splitting, *Publ. Math. I.H.E.S.* **65** (1987), 61-90.
- [30] V. Reiner and M. Shimozono, Key polynomials and a flagged Littlewood-Richardson rule, to appear in *J. Combin. Theory*, Ser. A.
- [31] V. Reiner and M. Shimozono, On flagged Schur modules of general shape, preprint 1993.
- [32] V. Reiner and M. Shimozono, Specht series for column-convex diagrams, to appear in *J. Alg.*.
- [33] David J. Woodcock, A Vanishing Theorem for Schur Modules, *J. Alg.* **165** (1994), 483-506.
- [34] A. Zelevinsky, Small resolutions of singularities of Schubert varieties, *Funct. Anal. App.* **17** (1983), 142-144.