From: Peter Magyar (pmagyar@lynx.neu.edu)
To: Nigel Ray (nige@ma.man.ac.uk)
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Re: Bounded flag varieties

Dear Nige,

I have a few comments about your remarkable bounded flag varieties. It turns out that they fit into the machinery of combinatorial algebraic geometry in at least three ways: 1) as Bott-Samelson varieties; 2) as Schubert varieties; and 3) as toric varieties.

Following the notations of your preprint (V. Buchstaber and N. Ray, *Double cobordism, flag manifolds and quantum doubles*, 1996), we let $Z_{n+1} = \mathbf{C}^{n+1}$, Z_i the subspace spanned by the first i coordinate vectors, and $F(Z_{n+1})$ the complete flag variety of Z_{n+1}. Then the *bounded flag variety* $B_n = B(Z_{n+1}) \subset F(Z_{n+1})$ is the n-dimensional complex subvariety

$$B_n = \{0 < U_1 < \cdots < U_n < Z_{n+1} \mid \ \forall\, i,\ Z_{i-1} < U_i\}.$$

1. Given the sequence $\mathbf{i} = (n, n-1, \ldots, 2, 1)$, we associate the Bott-Samelson variety

$$\mathrm{Bott}_{\mathbf{i}} = P_n \times P_{n-1} \times \cdots \times P_2 \times P_1 \,/\, B^n,$$

where $B \subset GL(n+1, \mathbf{C})$ is the subgroup of upper-triangular matrices,

$$P_k = \{\ (x_{ij}) \mid x_{ij} = 0 \text{ unless } i \leq j \text{ or } (i,j) = (k+1,k)\ \}$$

is a parabolic subgroup of almost upper-triangular matrices, and B^n acts freely on the right of the product of the P_k via

$$(p_n, p_{n-1}, \ldots, p_1) \cdot (b_n, b_{n-1}, \ldots, b_1) = (p_n b_n, b_n^{-1} p_{n-1} b_{n-1}, \ldots, b_2^{-1} p_1 b_1).$$

Claim: *The map*

$$\begin{array}{rccc} \tilde{\mu} : & \mathrm{Bott}_{\mathbf{i}} & \rightarrow & \mathrm{Gr}(n, Z_{n+1}) \times \mathrm{Gr}(n-1, Z_{n+1}) \times \cdots \times \mathrm{Gr}(1, Z_{n+1}) \\ & (p_n, p_{n-1}, \ldots, p_1) & \mapsto & (p_n Z_n,\ p_n p_{n-1} Z_{n-1},\ \ldots\ ,\ p_n \cdots p_1 Z_1) \end{array}$$

is an isomorphism from $\mathrm{Bott}_{\mathbf{i}}$ *onto* B_n.

There are two natural coordinate systems on $\mathrm{Bott}_{\mathbf{i}}$ given by

$$(x_n, \ldots, x_1) \in \mathbf{C}^n \ \mapsto\ (p_n, \ldots, p_1) = (I + x_n e_{(n+1,n)}, \ldots, I + x_1 e_{(2,1)})$$

$$(y_n, \ldots, y_1) \in \mathbf{C}^n \ \mapsto\ (p_n, \ldots, p_1) = (\,(I + y_n e_{(n+1,n)}) s_n, \ldots, (I + y_1 e_{(2,1)}) s_1\,),$$

where I is the identity matrix, $e_{(k+1,k)}$ is a subdiagonal coordinate matrix, and s_k is the permutation matrix of the transposition $(k, k+1)$. (That is, s_k is the identity matrix except for a block of the form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ on the diagonal.)

Your subvarieties $X_Q \subset \mathrm{Bott}_{\mathbf{i}}$ for $Q \subset [1, n]$ are given by the equations $x_q = 0$ for $q \notin Q$, and your $Y_Q \subset \mathrm{Bott}_{\mathbf{i}}$ by $y_q = 0$ for $q \notin Q$. It is then clear that the X_Q all intersect transversally, as do the Y_Q. Demazure proves that the collection of the X_Q form a linear basis of the integral cohomology ring $\mathrm{H}^{\cdot}(\mathrm{Bott}_{\mathbf{i}})$, and he computes the self-intersection formula for the $X_k \overset{\text{def}}{=} X_{\{k\}}$:

$$X_k \cdot X_k = -(X_k \cdot X_{k+1} + \cdots + X_k \cdot X_n)$$

with $X_n \cdot X_n = 0$.

2. The Bott-Samelson variety $\mathrm{Bott}_{\mathbf{i}}$ naturally covers a Schubert variety X_w in the flag variety $F(Z_{n+1})$. The indexing permutation in the Weyl group $W = S_{n+1}$ is $w = s_n s_{n-1} \cdots s_1$ where s_k is the transposition $(k, k{+}1)$. That is, $w(1) = n{+}1$, $w(2) = 1$, $w(3) = 2, \ldots, w(n{+}1) = n$, a cycle of length $n{+}1$. This particular w is known as a *Coxeter element* of W. (See Humphreys' *Coxeter Groups.*)

The natural map is

$$\begin{array}{cccc} \mu : & \mathrm{Bott}_{\mathbf{i}} & \to & X_w \\ & (p_n, \cdots, p_1) & \mapsto & p_n \cdots p_1(Z.) \end{array}$$

where $Z. = (Z_1 < \cdots < Z_{n+1})$ is the standard flag. For a general $\mathbf{i}$, this is a resolution of singularities of X_w, but here the Schubert variety is already a smooth manifold and μ is an *isomorphism.* Thus

$$B_n \cong \mathrm{Bott}_{\mathbf{i}} \cong X_w$$

and your X_Q are the Schubert subvarieties of X_w. Hence, your cohomology calculations are indeed strongly analogous to the Schubert calculus: they compute intersections of Schubert subvarieties inside a smooth ambient Schubert variety, instead of inside the whole flag variety $F(Z_{n+1})$.

The Y_Q are intersections of X_w with the Schubert varieties of the opposite standard flag $(Z_{\{n\}} < Z_{[n-1,n]} < Z_{[n-2,n]} < \cdots)$. These also occur in the Schubert calculus. (See Fulton's new book *Young Tableaux.*)

3. It is easily seen that the complex torus of diagonal matrices in $SL(n+1, \mathbf{C})$ has an open dense orbit on B_n. Hence B_n is a toric variety. (See Fulton's *Introduction to Toric Varieties.*) In general, a toric variety is specified by a *fan* Δ, a collection of polyhedral cones in $\mathbf{R}^n$ with vertex at the origin. The cones must cover $\mathbf{R}^n$ and fit together along their faces like a simplicial complex. In fact, Δ is the cone over a simplicial decomposition of the $(n-1)$-sphere.

In our case, the fan $\Delta = \{\sigma_\epsilon\}$ consists of 2^n cones which are "skewed octants" in $\mathbf{R}^n$. They are indexed by the 2^n sequences $\epsilon = (\pm, \cdots, \pm)$ of pluses and minuses, and

$$\sigma_\epsilon = \mathrm{Span}_{\mathbf{R}_+}(v_1^{\pm}, \cdots v_n^{\pm})$$

where $v_1^+, \cdots, v_n^+$ are the coordinate vectors $z_1, \cdots, z_n$, and

$$v_1^- = -z_1,\ v_2^- = z_1 - z_2,\ v_3^- = z_2 - z_3, \cdots,\ v_n^- = z_{n-1} - z_n.$$

Now the varieties $X_k = X_{\{k\}}$ are the toric divisors corresponding to the rays v_k^+, and $Y_k = Y_{\{k\}}$ correspond to v_k^-. The general intersection theory for toric varieties once again recovers the Schubert calculus on B_n:

$$\mathrm{H}^{\cdot}(B_n) \;\cong\; \frac{\mathbb{Z}[X_1, \ldots, X_n]}{X_k(X_k + \cdots + X_n)} \;\cong\; \frac{\mathbb{Z}[Y_1, \ldots, Y_n]}{Y_k(Y_k - Y_{k+1})}$$

as well as giving the change-of-basis formula $X_k = Y_k - Y_{k+1}$.

Remarks. 1. For a general reductive or Kac-Moody group G with Weyl group W, one again has a Coxeter element $w = s_n s_{n-1} \cdots s_1 \in W$, and all the above remains valid. The only difference is in the structure constants of $\mathrm{H}^{\cdot}(B_n)$, which depend on the root system of G. Could this have some bearing on cobordism with G-structure, for G more general than SU?
2. It is an interesting (and as far as I know open) question to compute the cohomology ring $H^{\cdot}(X)$ for an arbitrary smooth Schubert variety X, not just our $X = B_n$.

Yours, Peter

Peter Magyar
Northeastern University
Boston, MA 02115, USA