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Littelmann paths for the basic representation of an affine Lie algebra

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Abstract

A highest-weight representation of an affine Lie algebra $\hat{\mathfrak{g}}$ can be modeled combinatorially in several ways, notably by the semi-infinite paths of the Kyoto school and by Littelmann's finite paths. In this paper, we unify these two models in the case of the basic representation of an untwisted affine algebra, provided the underlying finite-dimensional algebra $\mathfrak{g}$ possesses a minuscule representation (i.e., $\mathfrak{g}$ is of classical or E_6, E_7 type).

We apply our "coil model" to prove that the basic representation of $\hat{\mathfrak{g}}$, when restricted to $\mathfrak{g}$, is a semi-infinite tensor product of fundamental representations, and certain of its Demazure modules are finite tensor products.

1. Main results

1.1. Product theorems

Let $\mathfrak{g}$ be a complex simple Lie algebra and $\hat{\mathfrak{g}}$ the corresponding untwisted affine Kac–Moody algebra. The basic representation $\hat{V}(\Lambda_0)$ is the simplest and most important $\hat{\mathfrak{g}}$-module (see Section 2.1 for definitions, as well as [14, Chapter 14], [36, Chapter 10]). One of its remarkable properties is the Factorization Phenomenon. In many cases, the Demazure modules $\hat{V}_z(\Lambda_0) \subset \hat{V}(\Lambda_0)$ are representations of the finite-dimensional algebra $\mathfrak{g}$, and they factor into a tensor product of

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many small $\mathfrak{g}$-modules. Hence the full $\hat{V}(\Lambda_0)$ could be constructed by extending the $\mathfrak{g}$-structure on the semi-infinite tensor power $V \otimes V \otimes \cdots$ of a small $\mathfrak{g}$-module V.

The Kyoto school of Jimbo, Kashiwara, et al. has established this phenomenon in many cases via the theory of perfect crystals for level-zero representations [12,15,18–21], a development of their earlier theory of semi-infinite paths [7]. Although the Kyoto path model is expected to hold in great generality, the proofs have been carried out mostly for the Lie algebras of classical types A, B, C, D, due to the need for case-by-case definitions of perfect crystals. (See [11] for an introduction, and the end of this section for a survey of recent developments.)

In this paper, we extend the Factorization Phenomenon for $\hat{V}(\Lambda_0)$ to the non-classical types E_6 and E_7 by a uniform method which applies whenever $\mathfrak{g}$ possesses a minuscule representation, or more precisely a minuscule coweight. We shall rely on a key property of such coweights which may be taken as the definition. Let $\hat{X}$ be the extended Dynkin diagram associated to $\hat{\mathfrak{g}}$. A coweight $\varpi^\vee$ of $\mathfrak{g}$ is *minuscule* if and only if it is a fundamental coweight $\varpi^\vee = \varpi_i^\vee$ and there exists an automorphism σ of $\hat{X}$ taking the node i to the distinguished node 0. Such automorphisms exist in types A, B, C, D, E_6, E_7.

We let $V(\lambda)$ denote the irreducible $\mathfrak{g}$-module with highest weight λ, and $V(\lambda)^*$ its dual module. Our main representation-theoretic result is:

Theorem 1. *Let $\lambda^\vee$ be an element of the coroot lattice of $\mathfrak{g}$ which is a sum*:

$$\lambda^\vee = \lambda_1^\vee + \cdots + \lambda_m^\vee,$$

where $\lambda_1^\vee, \ldots, \lambda_m^\vee$ are minuscule fundamental coweights (not necessarily distinct), with corresponding fundamental weights $\lambda_1, \ldots, \lambda_m$. Let $\hat{V}_{-\lambda^\vee}(\Lambda_0) \subset \hat{V}(\Lambda_0)$ be the Demazure module corresponding to the anti-dominant translation $t_{-\lambda^\vee}$ in the affine Weyl group.

Then there is an isomorphism of $\mathfrak{g}$-modules:

$$\hat{V}_{-\lambda^\vee}(\Lambda_0) \cong V(\lambda_1)^* \otimes \cdots \otimes V(\lambda_m)^*.$$

Now fix a minuscule coweight $\varpi^\vee$ and its corresponding fundamental weight ϖ. Let N be the smallest positive integer such that $N\varpi^\vee$ lies in the coroot lattice of $\mathfrak{g}$. Then we have the following characterization of the basic irreducible $\hat{\mathfrak{g}}$-module:

Theorem 2. *The tensor power $V_N := (V(\varpi)^*)^{\otimes N}$ possesses non-zero $\mathfrak{g}$-invariant vectors. Fix such a vector v_N, and define the $\mathfrak{g}$-module $V^{\otimes\infty}$ as the direct limit of the sequence*:

$$V_N \hookrightarrow V_N^{\otimes 2} \hookrightarrow V_N^{\otimes 3} \hookrightarrow \cdots,$$

where each inclusion is defined by: $v \mapsto v_N \otimes v$.

Then $\hat{V}(\Lambda_0)$ is isomorphic as a $\mathfrak{g}$-module to $V^{\otimes\infty}$.

We survey some related recent results. After the preliminary version of this paper was distributed, Fourier and Littelmann [8] gave a very general version of the Factorization Phenomenon: for $\mathfrak{g}$ of arbitrary type, arbitrary dominant λ, and arbitrary level ℓ, they prove tensor factorization of $V_{-\lambda}(\ell\Lambda_0)$ as a $\mathfrak{g}$-module. Their argument is analogous to ours, using characters rather than crystals.

Independently of our work, Naito and Sagaki [31–33] gave a Littelmann path model for the so-called Weyl modules $W(\lambda)$ for all λ and $\mathfrak{g}$ of arbitrary type. The Weyl modules were then shown by Chari and Loktev [6] and Fourier and Littelmann [9] to coincide with the level-one Demazure modules $\hat{V}_{-\lambda^\vee}(\Lambda_0)$ provided $\mathfrak{g}$ is simply-laced (types ADE), resulting in a Littelmann path model for these cases.

Fourier and Littelmann [9] also showed that for $\mathfrak{g}$ of arbitrary type, factorization of $V(\ell\Lambda_0)$ holds on the level of $\mathfrak{g} \otimes \mathbb{C}[t]$-modules, provided one replaces tensor products with fusion products. Equivalently, this gives a "path construction" of $V(\ell\Lambda_0)$ of level ℓ, but only as a $\mathfrak{g} \otimes \mathbb{C}[t]$-module, extending the $\mathfrak{g}$-structure on $V^{\otimes\infty}$ by the action of the raising operator E_0. In a parallel development, Benkart, Frenkel, Kang and Lee [2] gave a path construction of $V(\Lambda_0)$ as a $\hat{\mathfrak{g}}$-module, but only for level 1: they defined the action of the raising and lowering operators E_0, F_0 as well as the energy operator d.

Combinatorial definitions of the energy grading for $\mathfrak{g}$ of classical type produce generalizations of the Hall–Littlewood and Kostka–Foulkes polynomials (see the bibliography in [34]), and Sanderson [38] has given a connection with Macdonald polynomials. The recent work of Ram [37] gives another perspective on these topics.

Finally, Beilinson and Drinfeld [1] (see also [3,30]) have given a general geometric version of the Factorization Phenomenon: they construct a deformation of the Schubert varieties of the affine Grassmannian into a product of smaller affine Schubert varieties. Pappas and Rapoport [35] gave an elementary and explicit version of this deformation for $\mathfrak{g} = A_n$, in which the factors are ordinary finite-dimensional Grassmannians. One may also view the classical work of Bott [4] as a topological version of the Factorization Phenomenon. In a work in preparation [29], we elucidate the connections among the many versions of factorization.

1.2. Crystal theorems

Our basic tool to prove our results is Littelmann's combinatorial model [24–26] for representations of Kac–Moody algebras, a vast generalization of Young tableaux. Littelmann's paths and path operators give a flexible construction of the crystal graphs associated to quantum $\mathfrak{g}$-modules by Kashiwara [17] and Lusztig [27] (see also [11,13]). Roughly speaking, we prove Theorem 1 (in Section 2.5) by reducing it to an identity of paths: we construct a path crystal for the affine Demazure module which is at the same time a path crystal for the tensor product.

Theorem 2 follows as a corollary (Section 2.6). To describe the crystal graph of the semi-infinite tensor product, we pass to a semi-infinite limit of Littelmann paths which we call *coils*. We thus recover the Kyoto path model for classical $\mathfrak{g}$, and our results are equally valid for E_6, E_7.

To be more precise, we briefly sketch Littelmann's theory. We define a $\mathfrak{g}$-*crystal*[1] as a *set* $\mathcal{B}$ with a *weight function* $\mathrm{wt}: \mathcal{B} \to \bigoplus_{i=1}^r \mathbb{Z}\varpi_i$, and partially defined *crystal raising and lowering operators* $e_1, \dots, e_r, f_1, \dots, f_r : \mathcal{B} \to \mathcal{B}$ satisfying:

$$\mathrm{wt}\big(f_i(b)\big) = \mathrm{wt}(b) - \alpha_i \quad \text{and} \quad e_i(b) = b' \quad \Leftrightarrow \quad f_i(b') = b.$$

Here $\varpi_1, \dots, \varpi_r$ are the fundamental weights and $\alpha_1, \dots, \alpha_r$ are the simple roots of $\mathfrak{g}$. A *dominant* (or *highest-weight*) element is a $b \in B$ such that $e_i(b)$ is not defined for any i. We say that a

[1] This definition is much weaker than Kashiwara's [16], but adequate for our purposes.

crystal $\mathcal{B}$ is a *model* for a $\mathfrak{g}$-module V if the formal character of $\mathcal{B}$ is equal to the character of V, and the dominant elements of $\mathcal{B}$ correspond to the highest-weight vectors of V. That is:

$$\mathrm{char}(V) = \sum_{b\in B} e^{\mathrm{wt}(b)} \quad \text{and} \quad V \cong \bigoplus_{b \text{ dom}} V\big(\mathrm{wt}(b)\big),$$

where the second sum is over the dominant elements of $\mathcal{B}$. Clearly, a $\mathfrak{g}$-module V is determined up to isomorphism by any model $\mathcal{B}$.

We construct such $\mathfrak{g}$-crystals $\mathcal{B}$ consisting of polygonal paths in the vector space of weights, $\mathfrak{h}^*_{\mathbb{R}} := \bigoplus_{i=1}^r \mathbb{R}\varpi_i$. Specifically:

- The *elements* of $\mathcal{B}$ are certain continuous piecewise-linear mappings $\pi : [0,1] \to \mathfrak{h}^*_{\mathbb{R}}$, up to reparametrization, with initial point $\pi(0) = 0$. We use the notation $\pi = (v_1 \star v_2 \star \cdots \star v_k)$ to denote the polygonal path with vector steps $v_1, \dots, v_k \in \mathfrak{h}^*_{\mathbb{R}}$: that is, the path starting at 0 and moving linearly to the point v_1, then to the point $v_1 + v_2$, etc.
- The *weight* of a path is its endpoint: $\mathrm{wt}(\pi) := \pi(1) = v_1 + \cdots + v_k$.
- The *crystal lowering operator* f_i is defined as follows (and there is a similar definition of the raising operator e_i). Let $\star$ denote the natural associative operation of concatenation of paths, and let any linear map $w : \mathfrak{h}^*_{\mathbb{R}} \to \mathfrak{h}^*_{\mathbb{R}}$ act pointwise on paths: $w(\pi) := (w(v_1) \star \cdots \star w(v_k))$. We will divide a path π into three well-defined sub-paths, $\pi = \pi_1 \star \pi_2 \star \pi_3$, and reflect the middle piece by the simple reflection s_i:

  $$f_i\pi := \pi_1 \star s_i\pi_2 \star \pi_3.$$

  The pieces π_1, π_2, π_3 are determined according to the behavior of the i-height function $h_i(t) = h_i^{\pi}(t) := \langle \pi(t), \alpha_i^{\vee} \rangle$. As the point $\pi(t)$ moves along the path from $\pi(0) = 0$ to $\pi(1) = \mathrm{wt}(\pi)$, this function may attain its minimum value $h_i(t) = M$ several times. If, after the *last* minimum point, $h_i(t)$ never rises to the value $M+1$, then $f_i\pi$ is *undefined*. Otherwise, we define π_2 as the last sub-path of π on which $M \leqslant h_i(t) \leqslant M+1$, and the remaining initial and final pieces of π are π_1 and π_3.

A key advantage of this path model is that the definition of the crystal operators, though complicated, is uniform for all paths. Hence a path crystal is completely specified by giving its set of paths $\mathcal{B}$.

Also, the dominant elements have a neat pictorial characterization, as the paths π which never leave the fundamental Weyl chamber: that is, $h_i^{\pi}(t) \geqslant 0$ for all $t \in [0,1]$ and all $i = 1, \dots, r$. For simplicity we restrict ourselves to *integral* dominant paths, meaning that all the steps are integral weights: $\pi = (v_1 \star v_2 \star \cdots)$, where $v_1, \dots, v_k \in \bigoplus_{i=1}^r \mathbb{Z}\varpi_i$. (For arbitrary dominant paths, see [26].)

Littelmann's Character Theorem [26] states that if π is any integral dominant path with weight λ, then the set of paths $\mathcal{B}(\pi)$ generated from π by $f_1, \dots, f_r$ is a model for the irreducible $\mathfrak{g}$-module $V(\lambda)$. (This $\mathcal{B}(\pi)$ is also closed under $e_1, \dots, e_r$.) Note that we can choose *any* integral path π which stays within the Weyl chamber and ends at λ, and each such choice gives a different (but isomorphic) path crystal modeling $V(\lambda)$. One hopes that any reasonable indexing set for a basis of $V(\lambda)$ is in natural bijection with $\mathcal{B}(\pi)$ for some choice of π. For example, classical Young tableaux for $\mathfrak{g} = \mathfrak{gl}_n\mathbb{C}$ correspond to choosing the steps v_j to be coordinate vectors in $\mathfrak{h}^*_{\mathbb{R}} \cong \mathbb{R}^n$.

Furthermore, we have Littelmann's Product Theorem [26]: if $\pi_1, \ldots, \pi_m$ are dominant integral paths of respective weight $\lambda_1, \ldots, \lambda_m$, then $\mathcal{B}(\pi_1) \star \cdots \star \mathcal{B}(\pi_m)$, the set of all concatenations of paths, is a model for the tensor product $V(\lambda_1) \otimes \cdots \otimes V(\lambda_m)$.

Everything we have said for $\mathfrak{g}$ also holds for the affine algebra $\hat{\mathfrak{g}}$, provided we replace the roots $\alpha_1, \ldots, \alpha_r$ of $\mathfrak{g}$ by the roots $\alpha_0, \alpha_1, \ldots, \alpha_r$ of $\hat{\mathfrak{g}}$; and the weights $\varpi_1, \ldots, \varpi_r$ of $\mathfrak{g}$ by the weights $\Lambda_0, \Lambda_1, \ldots, \Lambda_r$ of $\hat{\mathfrak{g}}$. We also replace the vector space $\mathfrak{h}^*_{\mathbb{R}}$ by $\hat{\mathfrak{h}}^*_{\mathbb{R}} := \bigoplus_{i=0}^{r} \mathbb{R}\Lambda_i \oplus \mathbb{R}\delta$, where δ is the non-divisible positive imaginary root of $\hat{\mathfrak{g}}$. (Indeed, the theory works uniformly for all symmetrizable Kac–Moody algebras.) We denote path crystals for $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ by $\mathcal{B}$ and $\hat{\mathcal{B}}$, respectively.

The theory extends to Demazure modules. For example, to model the affine Demazure module $\hat{V}_z(\Lambda) \subset \hat{V}(\Lambda)$, where z is an affine Weyl group element, we choose a reduced decomposition $z = s_{i_1} \cdots s_{i_m}$ (where s_i are simple reflections) and an integral dominant path π of weight Λ, then define the *Demazure path crystal*:

$$\hat{\mathcal{B}}_z(\pi) := \left\{ f_{i_1}^{k_1} \cdots f_{i_m}^{k_m} \pi \;\middle|\; k_1, \ldots, k_m \geqslant 0 \right\}.$$

Because of the local nilpotence of the lowering operators, this is always a finite set. We may consider $\hat{\mathcal{B}}_z$ as a "crystal Demazure operator" acting on sets of paths.

Littelmann proves [25] that the formal character of $\hat{\mathcal{B}}_z(\pi)$ is equal to the character of $\hat{V}_z(\Lambda)$, and π is the unique dominant path. Now suppose $z = t_{-\lambda^\vee}$, an anti-dominant translation in the affine Weyl group $\hat{W}$, so that $\hat{V}_{-\lambda^\vee}(\Lambda) := \hat{V}_z(\Lambda)$ is a $\mathfrak{g}$-submodule of $\hat{V}(\Lambda)$; and consider $\hat{\mathcal{B}}_{-\lambda^\vee}(\pi) := \hat{\mathcal{B}}_z(\pi)$ as a $\mathfrak{g}$-crystal by forgetting the action of f_0, e_0 and projecting modulo $\mathbb{R}\Lambda_0 \oplus \mathbb{R}\delta$ to $\mathfrak{h}^*_{\mathbb{R}}$. Then Littelmann's Restriction Theorem [26] implies that the $\mathfrak{g}$-crystal $\hat{\mathcal{B}}_{-\lambda^\vee}(\pi)$ is a model for the $\mathfrak{g}$-module $\hat{V}_{-\lambda^\vee}(\Lambda)$.

Now we are ready to state our main combinatorial results. For λ a dominant weight, define its dual weight λ^* by the dual $\mathfrak{g}$-module: $V(\lambda^*) = V(\lambda)^*$.

Theorem 3. *Let $\lambda^\vee$ be as in Theorem 1, and let $\mathcal{B}(\lambda)$ denote the $\mathfrak{g}$-crystal generated by the straight-line path (λ). Then the set of concatenated paths $\Lambda_0 \star \mathcal{B}(\lambda_1^*) \star \cdots \star \mathcal{B}(\lambda_m^*)$ is a $\hat{\mathfrak{g}}$-crystal for the Demazure module $\hat{V}_{-\lambda^\vee}(\Lambda_0)$. In fact, there is a unique $\hat{\mathfrak{g}}$-dominant path π with weight Λ_0 such that*:

$$\hat{\mathcal{B}}_{-\lambda^\vee}(\pi) = \Lambda_0 \star \mathcal{B}\big(\lambda_1^*\big) \star \cdots \star \mathcal{B}\big(\lambda_m^*\big) \mod \mathbb{R}\delta.$$

This is to be understood as an equality of sets of paths in $\hat{\mathfrak{h}}^*_{\mathbb{R}}$ mod $\mathbb{R}\delta$, and hence an isomorphism of $\hat{\mathfrak{g}}$-crystals.[2]

Theorem 1 follows immediately from this. Indeed, $s_i \Lambda_0 = \Lambda_0$ for $i = 1, \ldots, r$, so $f_i(\Lambda_0 \star \pi') = \Lambda_0 \star f_i(\pi')$ for any path π'. Thus the right-hand side of the equation in theorem is isomorphic as a $\mathfrak{g}$-crystal to $\mathcal{B}(\lambda_1^*) \star \cdots \star \mathcal{B}(\lambda_m^*)$, which models $V(\lambda_1)^* \otimes \cdots \otimes V(\lambda_r)^*$. See [10] for methods of enumerating the dominant paths in this crystal (and hence computing the isotypic components of the corresponding representation).

Next we give the crystal version of Theorem 2:

[2] The projection modulo $\mathbb{R}\delta$ does not affect the action of the $\hat{\mathfrak{g}}$-crystal operators $f_0, \ldots, f_r$, since $\langle \delta, \alpha_i^\vee \rangle = 0$ for all i. That is, the projection gives an isomorphism of $\hat{\mathfrak{g}}$-crystals.

Theorem 4. *Let $\varpi^\vee$, N be as in Theorem 2. Define the N-fold concatenation $\mathcal{B}_N = \mathcal{B}(\varpi^*) \star \cdots \star \mathcal{B}(\varpi^*)$. Then $\Lambda_0 \star \mathcal{B}_N$ contains a unique $\hat{\mathfrak{g}}$-dominant path $\Lambda_0 \star \hat{\pi}$.*

Define $\hat{\mathcal{B}}_\infty$ as the direct limit of the sequence:

$$\Lambda_0 \star \mathcal{B}_N \hookrightarrow \Lambda_0 \star \mathcal{B}_N \star \mathcal{B}_N \hookrightarrow \Lambda_0 \star \mathcal{B}_N \star \mathcal{B}_N \star \mathcal{B}_N \hookrightarrow \cdots,$$

where the inclusions are given by $\Lambda_0 \star \pi \mapsto \Lambda_0 \star \hat{\pi} \star \pi$. Then $\hat{\mathcal{B}}_\infty$ has a natural $\hat{\mathfrak{g}}$-crystal structure (*defined in the following section*) *which is isomorphic to the $\hat{\mathfrak{g}}$-crystal of $\hat{V}(\Lambda_0)$.*

This theorem is equivalent to the Kyoto path model.

1.3. The coil model

We proceed to define the crystals of semi-infinite paths promised in Theorem 4. Let us introduce a notation for a path π which emphasizes the vector steps going toward the endpoint $\Lambda = \mathrm{wt}(\pi)$ rather than away from the starting point 0. Define:

$$\pi = (\star v_k \star \cdots \star v_1 \vdash \Lambda) := (v' \star v_k \star \cdots \star v_1),$$

the path with endpoint Λ, last step v_1, etc., and first step $v' := \Lambda - (v_k + \cdots + v_1)$, a makeweight to assure that the steps add up to Λ.

A *coil* is an infinite list:

$$\pi = (\cdots \star v_2 \star v_1 \vdash \Lambda),$$

where $\Lambda \in \bigoplus_{i=0}^r \mathbb{Z}\Lambda_i$ and $v_1, v_2, \ldots \in \mathfrak{h}_\mathbb{R}^*$ are level-zero vectors (no Λ_0 component), subject to conditions (i) and (ii) below. For $i = 0, \ldots, r$ and $k > 0$, define:

$$h_i[k] := \langle \Lambda - (v_1 + \cdots + v_k), \alpha_i^\vee \rangle.$$

We require:

(i) For each i and all $k \gg 0$, we have $h_i[k] \geqslant 0$.
(ii) For each i, there are infinitely many k such that $h_i[k] = 0$.

We think of the coil π as a "projective limit" as $k \to \infty$ of the finite paths

$$\pi[k] := (\star v_k \star \cdots \star v_1 \vdash \Lambda).$$

Thus, π stays always at the level $\ell = \langle \Lambda, c \rangle$; only a finite number of steps of π lie outside the fundamental chamber $\hat{C}$ (condition (i)); and π touches each wall of $\hat{C}$ infinitely many times (condition (ii)). We may visualize the coil as jumping from the origin up to level ℓ, winding horizontally around the fundamental chamber infinitely many times, and ending at Λ.

Lemma 5. *For a coil π and $i = 0, \ldots, r$, one of the following is true*:

(i) *$f_i(\pi[k])$ is undefined for all $k \gg 0$;*
(ii) *there is a unique coil π' such that $\pi'[k] = f_i(\pi[k])$ for all $k \gg 0$.*

In the second case, we define $f_i \pi := \pi'$.

Proof. We say that a path π is *i-neutral* if $h_i^\pi(t) \geqslant 0$ for all t and $h_i^\pi(1) = 0$. For a fixed i, divide π into a concatenation: $\pi = (\cdots \star \pi_2 \star \pi_1 \star \pi_0 \vdash \Lambda)$, where each π_j is an i-neutral finite path except for π_0, which is an arbitrary finite path. Now it is clear that if $f_i(\pi_0)$ is undefined, then (i) holds. Otherwise (ii) holds and

$$f_i\pi = \bigl(\cdots \star \pi_2 \star \pi_1 \star f_i(\pi_0) \vdash \Lambda - \alpha_i\bigr). \qquad \square$$

We can immediately carry over the definitions of the path model to coils, including that of (Demazure) path crystals. For example, we say that π is an integral dominant coil if $\pi[k]$ is integral dominant for $k \gg 0$, and hence for all k. There exist integral dominant coils of level $\ell = 1$ only when $\mathfrak{g}$ has a minuscule coweight. We cannot immediately concatenate two coils,[3] but we can concatenate a coil π_1 and a path π_0: that is, $\pi_1 \star \pi_0 := (\pi_1 \star \pi_0 \vdash \mathrm{wt}(\pi_1) + \mathrm{wt}(\pi_0))$.

Proposition 6. *For an integral dominant coil π of weight Λ, the crystal $\hat{\mathcal{B}}(\pi)$ is a model for $\hat{V}(\Lambda)$, and $\hat{\mathcal{B}}_z(\pi)$ is a model for the Demazure module $\hat{V}_z(\Lambda)$.*

Proof. Given an integral dominant coil π and a Weyl group element $z \in \tilde{W}$, we can divide $\pi = \pi_1 \star \pi_0$ in such a way that the crystal Demazure operator $\hat{\mathcal{B}}_z$ acts on π by reflecting intervals in π_0 rather than π_1. This gives an isomorphism between the Demazure crystals generated by the dominant path $\mathrm{wt}(\pi_1) \star \pi_0$ and by the coil π:

$$\begin{aligned} \hat{\mathcal{B}}_z\bigl(\mathrm{wt}(\pi_1) \star \pi_0\bigr) &\xrightarrow{\sim} \hat{\mathcal{B}}_z(\pi_1 \star \pi_0) = \hat{\mathcal{B}}_z(\pi), \\ \mathrm{wt}(\pi_1) \star \pi' &\mapsto \pi_1 \star \pi'. \end{aligned}$$

This proves the assertion about Demazure modules.

Now, given $z_1 < z_2 < \cdots$, an infinite chain of Weyl group elements increasing in the Bruhat order, we have the morphisms of $\hat{\mathfrak{g}}$-crystals:

$$\begin{array}{ccccc} \hat{\mathcal{B}}_{z_1}(\Lambda) & \xleftarrow{\sim} & \hat{\mathcal{B}}_{z_1}\bigl(\mathrm{wt}(\pi_1) \star \pi_0\bigr) & \xrightarrow{\sim} & \hat{\mathcal{B}}_{z_1}(\pi) \\ \cap & & & & \cap \\ \hat{\mathcal{B}}_{z_2}(\Lambda) & \xleftarrow{\sim} & \hat{\mathcal{B}}_{z_2}\bigl(\mathrm{wt}(\pi_1') \star \pi_0'\bigr) & \xrightarrow{\sim} & \hat{\mathcal{B}}_{z_2}(\pi) \\ \cap & & & & \cap \\ \vdots & & & & \vdots \\ \hat{\mathcal{B}}(\Lambda) & & & & \hat{\mathcal{B}}(\pi) \end{array}$$

Here $\hat{\mathcal{B}}_z(\Lambda)$ denotes the crystal generated by the straight-line path (Λ). Since the $\hat{\mathfrak{g}}$-crystals at the bottom are the unions of their Demazure crystals, they are isomorphic: $\hat{\mathcal{B}}(\Lambda) \cong \hat{\mathcal{B}}(\pi)$. $\square$

In the situation of Theorem 4, $\hat{\pi}$ is a finite path in $\mathfrak{h}_{\mathbb{R}}^*$ with $\mathrm{wt}(\hat{\pi}) = 0$ such that $(\Lambda \star \hat{\pi})$ is $\hat{\mathfrak{g}}$-dominant and touches all the walls of the Weyl chamber $\hat{C}$. Thus $\hat{\pi}_\infty := (\cdots \star \hat{\pi} \star \hat{\pi} \vdash \Lambda_0)$ is

[3] However, one could define the concatenation to be a "multi-coil" containing several semi-infinite segments. This is closely related to the so-called Mathas tableaux for higher-level $\hat{\mathfrak{g}}$-modules.

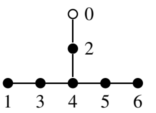
a dominant integral coil. We may view the direct limit $\hat{\mathcal{B}}_\infty$ as a set of coils by identifying each finite path of weight Λ,

$$\Lambda_0 \star \pi_k \star \cdots \star \pi_1 \in \Lambda_0 \star \mathcal{B}_N \star \cdots \star \mathcal{B}_N,$$

with the coil $(\cdots \star \hat{\pi} \star \hat{\pi} \star \pi_k \star \cdots \star \pi_1 \vdash \Lambda)$. Indeed, $\hat{\pi}_\infty$ is the unique dominant coil in this set, and we will prove Theorem 4 in Section 2.6 by showing that $\hat{\mathcal{B}}_\infty$ is identical with the coil crystal $\mathcal{B}(\hat{\pi}_\infty)$.

1.4. Example: E_6

Referring to Bourbaki [5], we write the extended Dynkin diagram $\hat{X} = \hat{E}_6$:



The simple roots are defined inside $\mathbb{R}^6$ with standard basis $\epsilon_1, \ldots, \epsilon_6$. (Our ϵ_6 is $\frac{1}{\sqrt{3}}(-\epsilon_6 - \epsilon_7 + \epsilon_8)$ in Bourbaki's notation.) They are:

$$\alpha_1 = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5) + \frac{\sqrt{3}}{2}\epsilon_6, \qquad \alpha_2 = \epsilon_1 + \epsilon_2,$$
$$\alpha_3 = \epsilon_2 - \epsilon_1, \qquad \alpha_4 = \epsilon_3 - \epsilon_2, \qquad \alpha_5 = \epsilon_4 - \epsilon_3, \qquad \alpha_6 = \epsilon_5 - \epsilon_4.$$

Since E_6 is simply laced, the coroots and coweights may be identified with the roots and weights, with the natural pairing given by the standard dot product on $\mathbb{R}^6$.

We focus on the minuscule coweight $\varpi_1^\vee$ corresponding to the diagram automorphism σ with $\sigma(1) = 0$ and $\sigma(0) = 6$. In this case, the corresponding fundamental representation $V(\varpi_1)$ is also minuscule, meaning that all of its weights are extremal weights $\lambda \in W(E_6) \cdot \varpi_1$. The roots $\alpha_2, \ldots, \alpha_6$ generate the root sub-system $D_5 \subset E_6$, and the reflection subgroup $W(D_5) = \mathrm{Stab}_{W(E_6)}(\varpi_1)$ acts by permuting $\epsilon_1, \ldots, \epsilon_5$ (the subgroup $W(A_4) = S_5$) and by changing an even number of signs $\pm\epsilon_1, \ldots, \pm\epsilon_5$. We have $\dim V(\varpi_1) = |W(E_6)/W(D_5)| = 27$. The weights are:

$$\varpi_1 = \frac{2\sqrt{3}}{3}\epsilon_6,$$
$$S_5 \cdot \frac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5) + \frac{\sqrt{3}}{6}\epsilon_6,$$
$$S_5 \cdot \frac{1}{2}(-\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4 + \epsilon_5) + \frac{\sqrt{3}}{6}\epsilon_6,$$
$$-\frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5) + \frac{\sqrt{3}}{6}\epsilon_6,$$
$$\pm S_5 \cdot \epsilon_1 - \frac{\sqrt{3}}{3}\epsilon_6.$$

The lowest weight is $-\varpi_6 = -\epsilon_5 - \frac{\sqrt{3}}{3}\epsilon_6$, so that $V(\varpi_1)^* = V(\varpi_6)$ and $\varpi_1^* = \varpi_6$.

The simplest path crystal for $V(\varpi_1^*)$ is the set of 27 straight-line paths from 0 to the negatives of the above extremal weights:

$$\mathcal{B}(\varpi_1^*) = \{(v) \mid v \in -W(E_6) \cdot \varpi_1\}.$$

We have $3\varpi_1^\vee \in \bigoplus_{i=1}^6 \mathbb{R}\alpha_i^\vee$ the coroot lattice, so that $N = 3$ in Theorem 2, and this N is always the order of the automorphism σ. The path crystal $\mathcal{B}_3 := \mathcal{B}(\varpi_1^*) \star \mathcal{B}(\varpi_1^*) \star \mathcal{B}(\varpi_1^*)$, the set of all 3-step walks with steps chosen from the 27 weights of $V(\varpi_1^*)$, is a model for $V(\varpi_1^*)^{\otimes 3}$. In this case there is a unique $\mathfrak{g}$-dominant path of weight 0 (not merely a unique $\hat{\mathfrak{g}}$-dominant path),

$$\hat{\pi} := (\varpi_6) \star (\varpi_1 - \varpi_6) \star (-\varpi_1),$$

which corresponds to the one-dimensional space of $\mathfrak{g}$-invariant vectors in $V(\varpi_1^*)^{\otimes 3}$.

Now Theorem 3 states that the affine Demazure module $\hat{V}_{-3m\varpi_1^\vee}(\Lambda_0)$ is modeled by the $\hat{\mathfrak{g}}$-path crystal:

$$\mathcal{B}_{3m} = \{(\Lambda_0 \star v_1 \star \cdots \star v_{3m}) \mid v_j \in -W(E_6) \cdot \varpi_1\},$$

the set of all $3m$-step walks in $\mathbb{R}^6 + \Lambda_0$ starting at Λ_0, with steps chosen from the 27 weights of $V(\varpi_1^*)$. This path crystal is generated from its unique $\hat{\mathfrak{g}}$-dominant path $\Lambda_0 \star \hat{\pi} \star \cdots \star \hat{\pi}$.

Taking the direct limit as $m \to \infty$ produces the coil crystal $\hat{\mathcal{B}}_\infty = \hat{\mathcal{B}}(\hat{\pi}_\infty)$ for the basic $\hat{\mathfrak{g}}$-module $\hat{V}(\Lambda_0)$: the set of all semi-infinite walks of the form

$$\begin{aligned}\pi &= \Lambda_0 \star \overbrace{\hat{\pi} \star \cdots \star \hat{\pi}}^{\text{infinite}} \star v_1 \star \cdots \star v_{3m} \\ &= (\cdots \star \hat{\pi} \star \hat{\pi} \star v_1 \star \cdots \star v_{3m} \vdash \Lambda),\end{aligned}$$

with $m > 0$ and $v_j \in -W(E_6) \cdot \varpi_1$. Here the endpoint Λ is $\mathrm{wt}(\pi) := \Lambda_0 + v_1 + \cdots + v_{3m}$. The crystal operators f_i act near the end of the coil, unwinding the coils $\hat{\pi}$ one at a time, right-to-left.

2. Demazure crystals

2.1. Notations

We will work with a complex simple Lie algebra $\mathfrak{g}$ of rank r, a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, the set of roots $\Delta \subset \mathfrak{h}^*$, and the set of coroots $\Delta^\vee \subset \mathfrak{h}$. We write the highest root of Δ as $\theta = a_1\alpha_1 + \cdots + a_r\alpha_r$, and its coroot as $\theta^\vee = a_1^\vee\alpha_1^\vee + \cdots + a_r^\vee\alpha_r^\vee$. Warning: If $\mathfrak{g}$ is not simply laced, $\theta^\vee$ is not the highest root of the dual root system $\Delta^\vee$.

The Weyl group W of $\mathfrak{g}$ is generated by reflections $s_1, \ldots, s_r$ defined by $s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$ for $\lambda \in \mathfrak{h}_\mathbb{R}^* := \bigoplus_{i=1}^r \mathbb{R}\varpi_i$. We have the fundamental Weyl chamber $C = \{\lambda \in \mathfrak{h}_\mathbb{R}^* \mid \langle \lambda, \alpha_i^\vee \rangle \geqslant 0,\ i = 1, \ldots, r\}$. The Weyl group also acts naturally on $\mathfrak{h}_\mathbb{R}$. If we choose a W-invariant bilinear form $(\cdot \mid \cdot)$ on $\mathfrak{h}_\mathbb{R}$, we have the isomorphism $\nu : \mathfrak{h}_\mathbb{R} \to \mathfrak{h}_\mathbb{R}^*$ defined by $\langle \nu(h), h' \rangle = (h|h')$ for $h, h' \in \mathfrak{h}_\mathbb{R}$. We normalize so that $\nu(\theta^\vee) = \theta$ and $\nu(\varpi_i^\vee) = \frac{a_i}{a_i^\vee}\varpi_i$.

Now let $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the untwisted affine Lie algebra of $\mathfrak{g}$, where c is a central element and $d = t\frac{d}{dt}$ is a derivation. (Cf. Kac [14, Chapters 6 and 7].) Then $\hat{\mathfrak{g}}$ has Cartan subalgebra $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$, with dual $\hat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$, where $\langle \Lambda_0, \mathfrak{h}\rangle = \langle \delta, \mathfrak{h}\rangle = 0$ and $\langle \Lambda_0, c\rangle = \langle \delta, d\rangle = 1$.

The simple roots of $\hat{\mathfrak{g}}$ are $\alpha_1, \ldots, \alpha_r$ and $\alpha_0 = \delta - \theta$; the simple coroots are $\alpha_1^\vee, \ldots, \alpha_r^\vee$ and $\alpha_0^\vee = c - \theta^\vee$. The fundamental weights are Λ_0 and $\Lambda_i = \varpi_i + a_i^\vee \Lambda_0$ for $i = 1, \ldots, r$. The affine Weyl group $\hat{W}$ is generated by the reflections $s_0, s_1, \ldots, s_r$ acting on $\hat{\mathfrak{h}}^*_{\mathbb{R}}$. The fundamental Weyl chamber of $\hat{\mathfrak{g}}$ is the cone $\hat{C} = \{\Lambda \in \hat{\mathfrak{h}}^*_{\mathbb{R}} \mid \langle \Lambda, \alpha_i^\vee\rangle \geqslant 0,\ i = 0, \ldots, r\}$ with extremal rays $\Lambda_0, \ldots, \Lambda_r$.

For $\Lambda \in \bigoplus_{i=0}^r \mathbb{N}\Lambda_i$, we have the irreducible highest-weight $\hat{\mathfrak{g}}$-module $\hat{V}(\Lambda)$. We will also consider the Demazure module $\hat{V}_z(\Lambda) := U(\hat{\mathfrak{n}}_+) \cdot v_{z\Lambda}$, where $\hat{\mathfrak{n}}_+$ is the algebra spanned by the positive weight-spaces of $\hat{\mathfrak{g}}$, $z \in \hat{W}$ is a Weyl group element, and $v_{z\Lambda}$ is a non-zero vector of extremal weight $z\Lambda$ in $\hat{V}(\Lambda)$.

For a weight $\Lambda = \lambda + \ell\Lambda_0 + m\delta$, we will ignore the energy $m = \langle \Lambda, d\rangle$ and work modulo $\mathbb{R}\delta$, even when we do not indicate this explicitly. Since $\langle \delta, \alpha_i^\vee\rangle = 0$ for $i = 0, \ldots, r$, the energy has no effect on the path operators f_i, e_i.

Consider the lattice $M = \nu(\bigoplus_{i=1}^r \mathbb{Z}\alpha_i^\vee)$ in $\mathfrak{h}^*_{\mathbb{R}}$. For any $\mu \in M$, there is an element $t_\mu \in \hat{W}$ which acts on the weights of level ℓ as translation by $\ell\mu$: that is,

$$t_\mu(\lambda + \ell\Lambda_0) = \lambda + \ell\mu + \ell\Lambda_0 \pmod{\mathbb{R}\delta}.$$

Furthermore, the affine Weyl group is a semi-direct product of the finite Weyl group with the lattice of translations: $\hat{W} = W \ltimes t_M$.

Consider the anti-dominant translation $z = t_{-\lambda}$ corresponding to a dominant weight $\lambda = \nu(\lambda^\vee) \in C \cap M$. We denote the resulting Demazure module as $\hat{V}_{\lambda^\vee}(\Lambda_0) := \hat{V}_z(\Lambda_0)$. Then the $\hat{\mathfrak{n}}_+$-module $\hat{V}_{\lambda^\vee}(\Lambda_0)$ is also a $\mathfrak{g}$-submodule of $\hat{V}(\Lambda_0)$:

$$\mathfrak{g} \cdot \hat{V}_{\lambda^\vee}(\Lambda_0) \subset \hat{V}_{\lambda^\vee}(\Lambda_0),$$

and these are the only $z \in \hat{W}$ for which $\hat{V}_z(\Lambda_0)$ is a $\mathfrak{g}$-module.

2.2. *Minuscule weights and coweights*

We collect needed facts concerning minuscule weights in root systems. The statements below are well known and easily verified from tables [5,14], although direct proofs are also not difficult (cf. [28]).

We say a non-zero coweight $\varpi^\vee \in \mathfrak{h}_{\mathbb{R}}$ is *minuscule for Δ* if $\langle \alpha, \varpi^\vee\rangle = 0$ or 1 for all positive roots $\alpha \in \Delta_+$. Equivalently, $\varpi^\vee = \varpi_i^\vee$ for some $i = 1, \ldots, r$ with $a_i = \langle \theta, \varpi_i^\vee\rangle = 1$. This implies that $a_i^\vee = 1$ as well, so that $\nu(\varpi_i^\vee) = \varpi_i$. The classification of the minuscule $\varpi_i^\vee$ is most concisely described by listing the pairs $(X, X \setminus \{i\})$, where X is the Dynkin diagram of $\mathfrak{g}$. We have $\varpi_i^\vee$ minuscule when:

$$\begin{aligned}\big(X, X \setminus \{i\}\big) \cong{}& (A_r, A_{r-k} \times A_{k-1}), \quad k = 1, \ldots, r,\\ & (B_r, B_{r-1}), \qquad (C_r, A_{r-1}),\\ & (D_r, D_{r-1}), \qquad (D_r, A_r),\\ & (E_6, D_5), \qquad (E_7, E_6).\end{aligned}$$

There are no minuscule $\varpi_i^\vee$ for $X = E_8$, F_4, or G_2.

Now define the extended Weyl group $\tilde{W}$ as a group of linear mappings on $\hat{\mathfrak{h}}_{\mathbb{R}}^*$: namely, $\tilde{W} := W \ltimes t_L$, where $L = \nu(\bigoplus_{i=1}^r \mathbb{Z}\varpi_i^\vee)$. Let

$$\Sigma := \{\sigma \in \tilde{W} \mid \sigma(\hat{C}) = \hat{C}\},$$

the symmetries in $\tilde{W}$ of the fundamental chamber of $\hat{\mathfrak{h}}_{\mathbb{R}}^*$. The set Σ is a system of coset representatives for $\tilde{W}/\hat{W}$, so that $\tilde{W} = \Sigma \ltimes \hat{W}$. We can extend the Bruhat length function to $\tilde{W}$ as: $l(\sigma w) = l(w\sigma) := l(w)$ for $\sigma \in \Sigma$, $w \in \hat{W}$. Each element $\sigma \in \Sigma$ defines an automorphism of the Dynkin diagram of $\hat{\mathfrak{g}}$ which we also write as σ. For $j = 0, \dots, r$, we have:

$$\sigma(\Lambda_j) = \Lambda_{\sigma(j)} \quad \text{and} \quad \sigma(\alpha_j) = \alpha_{\sigma(j)}.$$

There is a natural correspondence between non-trivial elements of Σ and minuscule coweights. Each $\sigma \in \Sigma$ can be written uniquely as:

$$\sigma = \bar{\sigma}\, t_{-\nu(\varpi_i^\vee)} = \bar{\sigma}\, t_{-\varpi_i},$$

for $\bar{\sigma} \in W$ and $\varpi_i^\vee$ a minuscule coweight. In fact, $\bar{\sigma} = w_0 w_i$, where w_0 is the longest element of W and w_i is the longest element of the parabolic subgroup $W_i := \mathrm{Stab}_W(\varpi_i)$. We have $\bar{\sigma}(\alpha_j) = \alpha_{\sigma(j)}$ for $j \neq i$, and $\bar{\sigma}(\alpha_i) = -\theta$.

We have $\sigma(\Lambda_i) = \bar{\sigma} t_{-\varpi_i}(\varpi_i + \Lambda_0) = \Lambda_0$, so that

$$\sigma(i) = 0.$$

Also, $\bar{\sigma}(\varpi_i) = w_0 w_i(\varpi_i) = w_0(\varpi_i) = -\varpi_i^*$, and $\varpi_{\sigma(0)} = \varpi_i^*$.

The number N appearing in Theorems 2 and 4 is the order of σ in the group Σ, and is also the order of $\varpi^\vee$ in the finite group $\bigoplus_{i=1}^r \mathbb{Z}\varpi_i^\vee / \bigoplus_{i=1}^r \mathbb{Z}\alpha_i^\vee$.

The definition of a minuscule weight ϖ for $\Delta^\vee$ is dual to the above: $\langle \varpi, \alpha^\vee \rangle = 0$ or 1 for all positive coroots $\alpha^\vee \in \Delta_+^\vee$; or equivalently $\varpi = \varpi_i$ and $\langle \varpi, \theta^* \rangle = 1$, where θ^* is highest in the root system $\Delta^\vee$. The fundamental representation $V(\varpi)$ corresponding to a minuscule ϖ has a basis consisting of extremal weight vectors $v_{w(\varpi)}$ for $w \in W$. Note, however, that ϖ need not be minuscule even when the corresponding coweight $\varpi^\vee$ is minuscule.[4] In fact, we have:

Lemma 7. *Let $\varpi_i^\vee$ be a minuscule coweight for Δ, and ϖ_i the corresponding weight.*

(i) *If Δ is simply laced (i.e., all root vectors have the same length), then ϖ_i is minuscule for $\Delta^\vee$.*
(ii) *If Δ is not simply laced, then ϖ_i is minuscule for $\Delta_s^\vee$, the simply-laced root system of short vectors in $\Delta^\vee$. Furthermore, $\alpha_i^\vee \in \Delta_s^\vee$.*

Proof. Part (i) follows from $\nu(\varpi_i^\vee) = (a_i/a_i^\vee)\,\varpi_i = \varpi_i$. Part (ii) is immediately verified for the relevant types B_r and C_r. $\square$

[4] For example, in type B_r with $\Delta = \{\pm\epsilon_i,\ \pm\epsilon_i \pm \epsilon_j \mid 1 \leqslant i < j \leqslant r\}$, we have $\varpi_r = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_r)$ as the only minuscule weight, and $\varpi_1^\vee = \epsilon_1^*$ as the only minuscule coweight.

Consider the *parabolic Bruhat order* on the W-orbit $W \cdot \varpi_i$. That is, the partial order generated by the relations: $\tau_1 < \tau_2$ if $\tau_1 = \tau_2 - d\alpha$ for some positive root α and some $d > 0$. The following result says that if $\varpi_i^\vee$ is minuscule, then this "strong" order is identical to the "weak" order:

Lemma 8. (Stembridge [39]) *Suppose the coweight $\varpi_i^\vee$ is minuscule. Then the Bruhat order on $W \cdot \varpi_i$ has covering relations*:

$$\tau_1 \lessdot \tau_2 \quad \textit{whenever } \tau_1 = \tau_2 - d\alpha_j$$

for some simple *root α_j of W and some $d > 0$.*

Proof. This follows from Stembridge's formulation by noting that $W \cdot \varpi_i \cong W/W_i \cong W \cdot \varpi_i^\vee$. □

2.3. Lakshmibai–Seshadri paths

We examine in detail the path crystals of $V(\varpi_i)$ where $\varpi_i^\vee$ is minuscule. For any dominant weight λ, the most canonical choice of dominant path is the straight-line path from 0 to λ, denoted $\pi = (\lambda)$. The corresponding path crystal $\mathcal{B}(\lambda)$ can be described non-recursively by the *Lakshmibai–Seshadri* (*LS*) *chains* [25]. These are saturated chains in the parabolic Bruhat order on $W \cdot \lambda$, weighted with certain rational numbers:

$$(\tau_1 \gtrdot \cdots \gtrdot \tau_m;\ 0 < a_1 \leqslant \cdots \leqslant a_{m-1} < 1)$$

with $\tau_j \in W \cdot \lambda$, $a_j \in \mathbb{Q}$, and $m \geqslant 1$. We require that if $\tau_{j+1} = \tau_j - d_j\alpha$ for $\alpha \in \Delta_+$, $d_j \in \mathbb{N}$, then $a_j = n_j/d_j$ for some $n_j \in \mathbb{N}$. An LS chain corresponds to the *LS path* defined as:

$$\pi = \bigl(a_1\tau_1 \star (a_2 - a_1)\tau_2 \star (a_3 - a_2)\tau_3 \star \cdots \star (1 - a_{m-1})\tau_m\bigr).$$

Notice that if $a_{j+1} = a_j$ then we may omit the step $0\tau_j$; in this case there may be more than one LS chain producing the same LS path. Nevertheless, $B(\lambda)$ is the set of all *distinct* LS paths [25].

Proposition 9. *Let $\varpi_i^\vee$, $\varpi_l^\vee$ be two minuscule coweights* (*possibly identical*), *and let $\bar{\sigma}_l$ be the linear mapping of $\mathfrak{h}_\mathbb{R}^*$ corresponding to $\varpi_l^\vee$. For each $\pi \in \mathcal{B}(\varpi_i)$, we have $\bar{\sigma}_l\pi \in \mathcal{B}(\varpi_i)$. That is, the linear mapping $\bar{\sigma}_l$ permutes the paths in $\mathcal{B}(\varpi_i)$.*

Proof. Consider an LS chain $(\tau_1 \gtrdot \cdots \gtrdot \tau_m;\ 0 < a_1 \leqslant \cdots \leqslant a_{m-1} < 1)$ corresponding to a path $\pi \in \mathcal{B}(\varpi_i)$. If $\mathfrak{g}$ is simply laced, then ϖ_i is minuscule by Lemma 5(i), and the denominator of a_j is:

$$d_j = \langle \tau_j, \alpha^\vee \rangle = \langle w\varpi_i, \alpha^\vee \rangle = \langle \varpi_i, w\alpha^\vee \rangle = 0 \text{ or } 1.$$

Since $0 < a_j = \frac{n_j}{d_j} < 1$, this means that $m = 1$ and $\pi = (\tau_1) = (w\varpi_i)$ for $w \in W$, a straight-line path of extremal weight. Since $\bar{\sigma}_l$ is an automorphism of the root system Δ, it permutes the elements of a W-orbit, and hence $\bar{\sigma}_l\pi$ is another straight-line LS path.

If $\mathfrak{g}$ is not simply laced, the paths of $\mathcal{B}(\varpi_i)$ are more complicated. By Lemma 8, we may assume that $\tau_{j+1} = \tau_j - d_j\alpha_{k(j)}$, where $k(j) \in \{1, \ldots, r\}$ and $\alpha_{k(j)}$ is a *simple* root. The turned path is:

$$\bar{\sigma}_l\pi = \big(a_1\bar{\sigma}_l\tau_1 \star (a_2 - a_1)\bar{\sigma}_l\tau_2 \star \cdots\big).$$

Suppose $k(j) = l$ for some j. By Lemma 5(ii), $\alpha_l^\vee$ is a short root of $\Delta^\vee$, and so is $w\alpha_l^\vee$, so we have:

$$d_j = \langle\tau_j, \alpha_l^\vee\rangle = \langle w\varpi_i^\vee, \alpha_l^\vee\rangle = \langle\varpi_i, w\alpha_l^\vee\rangle = 0 \text{ or } 1$$

by the same lemma. This again means that $m = 1$ and $\bar{\sigma}_l\pi$ is a straight-line LS path.

Finally, suppose $k(j) \neq l$ for all $j = 1, \ldots, m$. Then:

$$\bar{\sigma}_l\tau_{j+1} = \bar{\sigma}_l\tau_j - d_j\bar{\sigma}_l\alpha_{k(j)} = \bar{\sigma}_l\tau_j - d_j\alpha_p,$$

where $p := \sigma_l(k(j)) \in \{1, \ldots, r\}$. Hence $\bar{\sigma}_l\tau_j > s_p\bar{\sigma}_l\tau_j = \bar{\sigma}_l\tau_{j+1}$, and $\bar{\sigma}_l\pi$ is an LS path. □

Although we do not need it here, we note that for any minuscule $\varpi_i^\vee$, the LS paths of $B(\varpi_i)$ have at most two linear pieces (cf. [22]).

2.4. Twisted Demazure operators

For a Weyl group element with reduced decomposition $z = s_{i_1} \cdots s_{i_m} \in \hat{W}$ and any path π (not necessarily dominant), we define the *crystal Demazure operator*:

$$\hat{\mathcal{B}}_z(\pi) := \big\{f_{i_1}^{k_1} \cdots f_{i_m}^{k_m}\pi_\lambda \;\big|\; k_1, \ldots, k_m \geqslant 0\big\}.$$

We can extend $\hat{\mathcal{B}}_z$ to an operator taking any set of paths Π to a larger set of paths: $\hat{\mathcal{B}}_z(\Pi) := \bigcup_{\pi\in\Pi}\hat{\mathcal{B}}_z(\pi)$. We have $\hat{\mathcal{B}}_y(\hat{\mathcal{B}}_z(\Pi)) = \hat{\mathcal{B}}_{yz}(\Pi)$ whenever $l(yz) = l(y) + l(z)$. Similarly, we let $\mathcal{B}(\Pi)$ be the set of all paths generated from Π by $f_1, \ldots, f_r, e_1, \ldots, e_r$.

It will be convenient to define a Demazure module $\hat{V}_z(\Lambda)$ for any $z \in \tilde{W} = \Sigma \ltimes \hat{W}$. Now, $\sigma \in \Sigma$ induces an automorphism of $\hat{\mathfrak{g}}$, so for a module $\hat{V}$ we have the twisted module $\sigma\hat{V}$ defined by the action $g \odot v := \sigma^{-1}(g)v$ for $g \in \hat{\mathfrak{g}}$, $v \in \hat{V}$. That is, $\sigma\hat{V}(\Lambda) \cong \hat{V}(\sigma\Lambda)$, and in particular $\sigma\hat{V}(\Lambda_i) \cong \hat{V}(\Lambda_{\sigma(i)})$. Now, for $z = \sigma y$ with $y \in \hat{W}$ define: $\hat{V}_z(\Lambda) := \sigma(\hat{V}_y(\Lambda)) \subset \sigma\hat{V}(\Lambda)$, a twist of an ordinary Demazure module. Thus, $\hat{V}_{\sigma y}(\Lambda) \cong \hat{V}_{\sigma y\sigma^{-1}}(\sigma\Lambda)$ and

$$\hat{V}_{y\sigma}(\Lambda) \cong \hat{V}_y(\sigma\Lambda).$$

Furthermore, $\hat{V}_{\lambda^\vee}(\Lambda_0) := \hat{V}_z(\Lambda_0)$ for $z = t_{-\nu(\lambda^\vee)}$ is a $\mathfrak{g}$-module for any dominant integral coweight $\lambda^\vee \in \bigoplus_{i=1}^r \mathbb{N}\varpi_i^\vee$.

The combinatorial counterpart of this construction is:

$$\hat{\mathcal{B}}_{\sigma y}(\pi) := \sigma\hat{\mathcal{B}}_y(\pi)$$

for $\sigma \in \Sigma$ and $y \in \hat{W}$. All of our statements regarding $\hat{V}_z(\Lambda)$ and $\hat{\mathcal{B}}_z$ for $z \in \hat{W}$ remain valid for $z \in \tilde{W}$.

2.5. Proof of Theorems 1 and 3

Let $\lambda^\vee$ be a dominant integral coweight (not necessarily in the coroot lattice) which can be written:

$$\lambda^\vee = \lambda_1^\vee + \cdots + \lambda_m^\vee,$$

where $\lambda_j^\vee \in \{\varpi_1, \ldots, \varpi_r\}$ are minuscule fundamental coweights (not necessarily distinct) with corresponding weights λ_j and dual weights $\lambda_j^* = -w_0(\lambda_j)$. Then we claim (extending Theorem 1 to the coweight lattice) that there is an isomorphism of $\mathfrak{g}$-modules:

$$\hat{V}_{\lambda^\vee}(\Lambda_0) \cong V(\lambda_1^*) \otimes \cdots \otimes V(\lambda_m^*).$$

This follows immediately from Littelmann's Character and Restriction Theorems combined with the following extension of Theorem 3.

We will choose a certain $\hat{\mathfrak{g}}$-dominant path π_m of weight Λ_0 and show that:

$$\hat{\mathcal{B}}_{-\lambda^\vee}(\pi_m) = \Lambda_0 \star \mathcal{B}(\lambda_1^*) \star \cdots \star \mathcal{B}(\lambda_m^*).$$

Let $\sigma_j \in \Sigma$ correspond to $\lambda_j^\vee$ for $j = 1, \ldots, m$. We define π_m inductively as the last of a sequence of paths $\pi_0, \pi_1, \ldots, \pi_m$:

$$\pi_0 := \Lambda_0, \qquad \pi_j := \sigma_j^{-1}(\pi_{j-1} \star \lambda_j^*).$$

We may picture π_m as jumping up to level Λ_0, winding horizontally around the fundamental alcove $A = (\mathfrak{h}_{\mathbb{R}}^* + \Lambda_0) \cap \hat{C}$, and ending at Λ_0. Indeed, note that $\mathrm{wt}(\pi_0) = \Lambda_0$. For $j > 0$, write $\Lambda_{(j)} := \lambda_j + \Lambda_0$, so that $\sigma_j \Lambda_{(j)} = \Lambda_0$ and $\sigma_j \lambda_j = -\lambda_j^*$. Then by induction:

$$\begin{aligned}
\mathrm{wt}(\pi_j) &= \sigma_j^{-1}\big(\mathrm{wt}(\pi_{j-1}) + \lambda_j^*\big) \\
&= \sigma_j^{-1}\Lambda_0 + \sigma_j^{-1}\lambda^* \\
&= \Lambda_{(j)} - \lambda_j = \Lambda_0,
\end{aligned}$$

so that each π_j has weight Λ_0. Furthermore, since $\Lambda_0 + \lambda_j^* \in \hat{C}$ and σ_j is a automorphism of $\hat{C}$, it is clear that each π_j is indeed a $\hat{\mathfrak{g}}$-dominant path.

We will now prove Theorem 3 by showing that the Demazure operator $\hat{\mathcal{B}}_{-\lambda^\vee}$ "unwinds" π_m starting from its endpoint. To compute:

$$\hat{\mathcal{B}}_{-\lambda^\vee}(\pi_m) = \hat{\mathcal{B}}_{-\lambda_1^\vee} \cdots \hat{\mathcal{B}}_{-\lambda_m^\vee}(\pi_m),$$

it suffices to prove:

Lemma 10. *For $j = m, m-1, \ldots, 1$, we have*:

$$\hat{\mathcal{B}}_{-\lambda_j^\vee}\big(\pi_j \star \mathcal{B}(\lambda_{j+1}^*) \star \cdots \star \mathcal{B}(\lambda_m^*)\big) = \pi_{j-1} \star \mathcal{B}(\lambda_j^*) \star \mathcal{B}(\lambda_{j+1}^*) \star \cdots \star \mathcal{B}(\lambda_m^*).$$

Proof. For $j = m$, we compute:

$$\begin{aligned}
\hat{\mathcal{B}}_{-\lambda_m^\vee}(\pi_m) &= \hat{\mathcal{B}}_{w_m w_0 \sigma_m}\big(\sigma_m^{-1}\big(\pi_{m-1} \star \lambda_m^*\big)\big) \\
&= \hat{\mathcal{B}}_{w_m w_0}\big(\pi_{m-1} \star \lambda_m^*\big) \\
&\overset{\text{(I)}}{=} \hat{\mathcal{B}}_{w_0 w_m^*}\big(\pi_{m-1} \star \lambda_m^*\big) \\
&\overset{\text{(II)}}{=} \pi_{m-1} \star \hat{\mathcal{B}}_{w_0 w_m^*}\big(\lambda_m^*\big) \\
&\overset{\text{(III)}}{=} \pi_{m-1} \star \hat{\mathcal{B}}_{w_0}\big(\lambda_m^*\big) \\
&= \pi_{m-1} \star \mathcal{B}\big(\lambda_m^*\big).
\end{aligned}$$

The equalities are justified as follows: (I) Here $w_m^* := w_0 w_m w_0$, the longest element in $\mathrm{Stab}_W(\lambda_m^*) = \mathrm{Stab}_W(-w_0\lambda_m)$. (II) If a path π is *i-neutral*, meaning $\langle \pi(t), \alpha_i^\vee \rangle \geqslant 0$ and $\langle \mathrm{wt}\,\pi, \alpha_i^\vee \rangle = 0$, then it is clear from the definition of f_i that $f_i(\pi \star \pi') = \pi \star f_i(\pi')$ for any path π', and the two sides are both defined or both undefined. Now note that the π_j are i-neutral for $i = 1, \ldots, r$. (III) follows from $\hat{\mathcal{B}}_{w_0 w_m^*}(\lambda^*) = \hat{\mathcal{B}}_{w_0 w_m^*}\hat{\mathcal{B}}_{w_m^*}(\lambda^*) = \hat{\mathcal{B}}_{w_0}(\lambda^*)$.

For $j < m$, letting $\mathcal{B}_{j+1} := \mathcal{B}(\lambda_{j+1}^*) \star \cdots \star \mathcal{B}(\lambda_m^*)$, we have:

$$\begin{aligned}
\hat{\mathcal{B}}_{-\lambda_j^\vee}(\pi_j \star \mathcal{B}_{j+1}) &= \hat{\mathcal{B}}_{w_0 w_j^* \sigma_j}\big(\sigma_j^{-1}\big(\pi_{j-1} \star \lambda_j^*\big) \star \mathcal{B}_{j+1}\big) \\
&\overset{\text{(I)}}{=} \hat{\mathcal{B}}_{w_0 w_j^*}\big(\pi_{j-1} \star \lambda_j^* \star \sigma_j \mathcal{B}_{j+1}\big) \\
&\overset{\text{(IV)}}{=} \hat{\mathcal{B}}_{w_0 w_j^*}\big(\pi_{j-1} \star \lambda_j^* \star \mathcal{B}_{j+1}\big) \\
&\overset{\text{(II)}}{=} \pi_{j-1} \star \hat{\mathcal{B}}_{w_0 w_j^*}\big(\lambda_j^* \star \mathcal{B}_{j+1}\big) \\
&\overset{\text{(V)}}{=} \pi_{j-1} \star \mathcal{B}\big(\lambda_j^* \star \mathcal{B}_{j+1}\big) \\
&\overset{\text{(VI)}}{=} \pi_{j-1} \star \mathcal{B}\big(\lambda_j^*\big) \star \mathcal{B}_{j+1}.
\end{aligned}$$

Here (I) and (II) are as above. (IV) follows from Proposition 7. (V) follows from the Combinatorial Excellent Filtration Theorem [23, Proposition 12], which implies that $\lambda_j^* \star \mathcal{B}_{j+1}$ is isomorphic to a union of Demazure crystals $\mathcal{B}_y(\mu)$ with $y \geqslant w_j^*$. (VI) Both sides are stable under the f_i, e_i for $i = 1, \ldots, r$, and they contain the same $\mathfrak{g}$-dominant paths, hence they are identical.

This concludes the proof of the lemma, and hence of Theorem 3. □

2.6. *Proof of Theorems 2 and 4*

Fix a minuscule coweight $\varpi^\vee$ with corresponding weight ϖ, dual weight ϖ^*, and automorphism $\sigma \in \Sigma$ of order N. The N-fold concatenation $\mathcal{B}(\varpi^*) \star \cdots \star \mathcal{B}(\varpi^*)$ contains the path $\hat{\pi} := \pi_N = (\sigma^{-N}(\varpi^*) \star \cdots \star \sigma^{-2}(\varpi^*) \star \sigma^{-1}(\varpi^*))$, which is $\mathfrak{g}$-dominant with weight 0. Thus $V(\varpi^*)^{\otimes N}$ possesses a corresponding invariant vector v_N.

Since $-N\varpi^\vee \in \bigoplus_{i=1}^r \mathbb{Z}\alpha_i$ and the translation $t_{-N\varpi^\vee}$ lies in $\hat{W}$, the twisted Demazure module $V_{mN\varpi^\vee}(\Lambda_0)$ is an ordinary Demazure submodule of $\hat{V}(\Lambda_0)$. In fact, the basic $\hat{\mathfrak{g}}$-module, considered as a $\mathfrak{g}$-module, is a direct limit of these Demazure modules:

$$\hat{V}(\Lambda_0) = \varinjlim V_{N\varpi^\vee} \hookrightarrow V_{2N\varpi^\vee} \hookrightarrow V_{3N\varpi^\vee} \hookrightarrow \cdots.$$

By Theorem 1, the $\mathfrak{g}$-module on the right-hand side is isomorphic to:

$$\varinjlim V(\varpi^*)^{\otimes N} \overset{\phi_1}{\hookrightarrow} V(\varpi^*)^{\otimes 2N} \overset{\phi_2}{\hookrightarrow} V(\varpi^*)^{\otimes 3N} \overset{\phi_3}{\hookrightarrow} \cdots$$

for some injective $\mathfrak{g}$-module morphisms $\phi_1, \phi_2, \phi_3, \ldots.$

Now, if $\phi, \psi : V_1 \hookrightarrow V_2$ are any two injective $\mathfrak{g}$-module morphisms, and ξ_1 an automorphism of V_1, then complete reducibility implies that there exists an automorphism ξ_2 of V_2 which completes the commutative diagram:

$$\begin{tikzcd} V_1 \arrow[r, hook, "\phi"] \arrow[d, "\xi_1", "\wr"'] & V_2 \arrow[d, "\xi_2", "\wr"'] \\ V_1 \arrow[r, hook, "\psi"] & V_2 \end{tikzcd}$$

Thus, the inclusions $\psi_m : V(\varpi^*)^{\otimes Nm} \hookrightarrow V(\varpi^*)^{\otimes N(m+1)}$, $v \mapsto v_N \otimes v$ appearing in Theorem 2 define the same $\mathfrak{g}$-module as the above maps ϕ_m, and we conclude that $V(\Lambda_0)$ is isomorphic as a $\mathfrak{g}$-module to the infinite tensor product as claimed.

Now consider the situation of Theorem 4. Recall the coil $\hat{\pi}_\infty := (\cdots \star \hat{\pi} \star \hat{\pi} \vdash \Lambda_0)$, and consider the commutative diagram:

$$\begin{array}{ccc} \hat{\mathcal{B}}_{-N\varpi^\vee}(\Lambda_0 \star \hat{\pi}) & \overset{\sim}{\to} & \hat{\mathcal{B}}_{-\varpi^\vee}(\hat{\pi}_\infty) \\ \downarrow & & \cap \\ \hat{\mathcal{B}}_{-2N\varpi^\vee}(\Lambda_0 \star \hat{\pi} \star \hat{\pi}) & \overset{\sim}{\to} & \hat{\mathcal{B}}_{-2N\varpi^\vee}(\hat{\pi}_\infty) \\ \downarrow & & \cap \\ \vdots & & \vdots \\ \hat{\mathcal{B}}_\infty & & \hat{\mathcal{B}}(\hat{\pi}_\infty) \end{array}$$

where the vertical maps on the left are those of the direct limit. As in the proof of Proposition 6, the direct limit crystals on the bottom are isomorphic, so Theorem 4 now follows by applying the conclusion of Proposition 6.

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