STANDARD MONOMIAL THEORY FOR BOTT-SAMELSON VARIETIES OF $GL(n)$

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Abstract. We construct an explicit basis for the coordinate ring of the Bott-Samelson variety $Z_{\mathbf{i}}$ associated to $G = GL(n)$ and an arbitrary sequence of simple reflections $\mathbf{i}$. Our basis is parametrized by certain standard tableaux and generalizes the Standard Monomial basis for Schubert varieties.

In this paper, we prove the results announced in [LkMg] for the case of Type $\mathbf{A}_{n-1}$ (the groups $GL(n)$ and $SL(n)$). That is, we construct an explicit basis for certain "generalized Demazure modules", natural finite-dimensional representations of the group B of upper triangular matrices. These modules can be constructed in an elementary way as flagged Schur modules [Mg1,Mg2,RS1,RS2]. They include as special cases almost all natural examples of B-modules, and their characters include most of the known generalizations of Schur polynomials. We view these representations geometrically via Borel-Weil theory as the space of global sections of a line bundle over a Bott-Samelson variety. Thus, our theory also describes the coordinate ring of this variety.

Notations: $G = GL(n, \mathbb{F})$ or $SL(n, \mathbb{F})$, where $\mathbb{F}$ is an algebraically closed field of arbitrary characteristic or $\mathbb{F} = \mathbb{Z}$; B is the Borel subgroup consisting of upper triangular matrices; T is the maximal torus consisting of diagonal matrices; W is the symmetric group S_n generated by the adjacent transpositions (simple reflections) $s_i = (i, i+1)$; $P_i \supset B$ is the minimal parabolic subgroup of G associated to s_i, namely $P_i = \{\, (x_{ij}) \in G \mid x_{ij} = 0 \text{ if } i > j \text{ and } (i,j) \neq (i+1, i) \,\}$.

For any word $\mathbf{i} = (i_1, \dots, i_l)$, with letters $1 \le i_j \le n-1$, the *Bott-Samelson variety* is the quotient space

$$Z_{\mathbf{i}} = P_{i_1} \times P_{i_2} \times \cdots \times P_{i_l} \ / \ B^l \ ,$$

where B^l acts by

$$(p_1, p_2, \dots, p_l) \cdot (b_1, b_2, \dots, b_l) = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{l-1}^{-1} p_l b_l).$$

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It was originally used [BS, D1] to desingularize the Schubert variety $X_w = \overline{B \cdot wB} \subset G/B$, where $w = s_{i_1} \cdots s_{i_l}$. The desigularization is given by the multiplication map $Z_{\mathbf{i}} \to X_w \subseteq G/B$, $(p_1, \ldots, p_l) \mapsto p_1 \cdots p_l \cdot B$, and $Z_{\mathbf{i}}$ has the structure of an iterated fiber bundle with fiber $\mathbb{P}^1$ in each iteration, so we may loosely think of $Z_{\mathbf{i}}$ as a "factoring" of the Schubert variety into a twisted product of projective lines.

Denote $\mathrm{Gr}(i) = \mathrm{Gr}(i, \mathbb{F}^n)$ the Grassmannian of i-dimensional subspaces of linear n-space, and

$$\mathrm{Gr}(\mathbf{i}) \stackrel{\mathrm{def}}{=} \mathrm{Gr}(i_1) \times \cdots \times \mathrm{Gr}(i_l).$$

We can realize $Z_{\mathbf{i}}$ as a variety of configurations of subspaces of $\mathbb{F}^n$ (a kind of multiple Schubert variety) via the embedding by successive multiplications [Mg2]:

$$\begin{array}{cccc} \mu : & Z_{\mathbf{i}} & \to & \mathrm{Gr}(\mathbf{i}) \\ & (p_1, \ldots, p_l) & \mapsto & (p_1 \mathbb{F}^{i_1},\, p_1 p_2 \mathbb{F}^{i_2}, \ldots,\, p_1 \cdots p_l \mathbb{F}^{i_l}) \end{array}$$

where $0 \subset \mathbb{F}^1 \subset \cdots \subset \mathbb{F}^n$ is the standard flag fixed by B. Although we will not need it here, we note that $\mu(Z_{\mathbf{i}}) \cong Z_{\mathbf{i}}$ can be described explicitly in terms of incidence relations: that is, a configuration of subspaces $(V_1, \ldots, V_l) \in \mathrm{Gr}(\mathbf{i})$ lies in $\mu(Z_{\mathbf{i}})$ exactly if certain inclusions $V_i \subset V_j$ are satisfied, as specified by the combinatorics of wiring diagrams. See [Mg2].

Now, each Grassmannian has a minimal-degree ample line bundle (the Plucker bundle) $\mathcal{O}(1)$, and for any sequence $\mathbf{m} = (m_1, \ldots, m_l)$, $m_j \in \mathbb{Z}_+$, there is an effective line bundle on $\mathrm{Gr}(\mathbf{i})$ given by tensoring the m_jth powers of the Plucker bundles on the factors of $\mathrm{Gr}(\mathbf{i})$: $\mathcal{O}(\mathbf{m}) = \mathcal{O}(1)^{\otimes m_1} \otimes \cdots \otimes \mathcal{O}(1)^{\otimes m_l}$. Denote its restriction to $Z_{\mathbf{i}}$ by $\mathcal{L}_{\mathbf{m}} = \mu^* \mathcal{O}(\mathbf{m})$. We shall be concerned with the B-module

$$H^0(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{m}}),$$

which includes as special cases dual Schur modules (Weyl modules) [FH,F], Demazure modules [D1,LkSh2,LkSb1], skew Schur modules [FH,F], the Schubert modules of Kraskiewicz and Pragacz [KP], and the generalized Schur modules of percent-avoiding diagrams [RS2]. Thus, the characters of these modules include Schur, key, skew Schur, and Schubert polynomials. See [Mg2]. In particular, we obtain a new proof of the classical Standard Monomial Theory for Demazure modules of type A.

An example will give the flavor of our results. Let $G = GL(3)$, $\mathbf{i} = (1, 2, 1)$, $\mathbf{m} = (1, 1, 1)$. We may write

$$\mathrm{Gr}(\mathbf{i}) = \mathrm{Gr}(1) \times \mathrm{Gr}(2) \times \mathrm{Gr}(1)$$

$$Z_{\mathbf{i}} \cong \{\ (V_1, V_2, V_1') \in \mathrm{Gr}(\mathbf{i}) \ \mid\ \mathbb{F}^2 \supset V_1 \subset V_2 \supset V_1'\ \}.$$

Then $H^0(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{m}})$ is spanned by all products of the form

$$\Delta_{abcd} = \Delta_a(x) \Delta_{bc}(y) \Delta_d(z),$$

where $1 \leq a, b, c, d \leq 3$ and Δ_a, Δ_{bc}, Δ_d mean minors on the corresponding rows of the homogeneous coordinates on $\mathrm{Gr}(\mathbf{i})$:

$$(x, y, z) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \\ y_{31} & y_{32} \end{pmatrix} \times \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathrm{Gr}(\mathbf{i}).$$

For example, $\Delta_{2132} = x_1(y_{11}y_{32} - y_{31}y_{12})z_2$. The sequence of indices $\tau = abcd$ indexing a spanning vector Δ_{abcd} is called a *tableau*. Theorems 1 and 2 below allow us to select a basis of $H^0(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{m}})$ from the spanning set, corresponding to the set of *standard tableaux*:

$$\tau \in \mathcal{T}(\mathbf{i}, \mathbf{m}) = \{1121, 2121, 2122, 1131, 2131, 2231, 2232,\\ 1122, 1132, 2132, 1133, 2133, 2233\}$$

Since each Δ_{abcd} is an eigenvector of the diagonal matrices, this allows us to compute the character of the B-module $H^0(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{m}})$.

In the general case, we give two descriptions of the standard tableaux $\mathcal{T}(\mathbf{i}, \mathbf{m})$. The first (§1.2) is in the spirit of the monotone lifting property of classical Standard Monomial Theory [LkSd1, LkSd2] (which in turn generalizes Young's increasing-in-rows-and-columns definition). The second (§1.4) is in terms of the refined Demazure character formula and crystal lowering operators of Lascoux and Schutzenberger [LcSb1] and Littelmann [Lt1, Lt2, Lt3]. This description is more suited to computations, and it gives an efficient algorithm for generating the standard tableaux. The above list of 13 tableaux, for example, can be computed by hand in less than a minute.

The paper is organized as follows. In Section 1 we define the standard tableaux and state the main theorems. In Section 2 we prove the equivalence of our two definitions of standard tableaux by an elementary argument. In Section 3 we show that our standard monomials form a basis: first we show independence, then use the Demazure character formula to argue that our modules have the same dimension as the number of standard tableaux. Essential to the proof are the vanishing theorems of Mathieu and Kumar for Bott-Samelson varieties [Mt1,Mt2,Ku].

Reiner and Shimozono [RS2] give another combinatorial interpretation of our tableaux. There are also intriguing connections between our basis and that of Brian Taylor [T] for a special case of our modules.

1. Definitions and main results

1.1. Tableaux.

We will use integer sequences to index several types of objects. Hence we will call an integer sequence a "word", a "tableau", etc., depending on what it indexes in a given context.

A *word* is a sequence $\mathbf{i} = (i_1, \dots, i_l)$ with $i_j \in \{1, \dots, n-1\}$, and $\mathbf{i}$ is *reduced* if $s_{i_1} \cdots s_{i_l} = w \in W$ is a minimal-length decomposition of w into simple reflections.

A *tableau* is a sequence $\tau = (r_1, \dots, r_N)$ with $r_j \in \{1, 2, \dots, n\}$. For $\tau = (r_1, \dots, r_N)$, $\tau' = (r'_1, \dots, r'_{N'})$, we define the *concatenation*

$$\tau \star \tau' = (r_1, \dots, r_N, r'_1, \dots, r'_{N'}).$$

Let $\emptyset$ denote the empty tableau and define $\emptyset \star \tau = \tau \star \emptyset = \tau$ for any tableau τ.

A *column* of size i is a tableau $\kappa = (r_1, \dots, r_i)$ with $1 \le r_1 < \cdots < r_i \le n$. The symmetric group W acts on columns as follows: for a permutation w on n letters and a column $\kappa = (r_1, \dots, r_i)$, the column $w \cdot \kappa$ is the increasing rearrangement of $(w(r_1), \dots, w(r_i))$. The *fundamental weight columns* are the initial sequences:

$$\varpi_i = (1, 2, \dots, i).$$

The *Bruhat order* on columns is defined by elementwise comparison: $\kappa = (r_1, \dots, r_i) \leq \kappa' = (r'_1, \dots, r'_i)$ if and only if $r_1 \leq r'_1, \dots, r_i \leq r'_i$.

For a word $\mathbf{i} = (i_1, \dots, i_l)$, and a sequence $\mathbf{m} = (m_1, \dots, m_l)$ with $m_j \in \mathbb{Z}^+$, we define a *tableau of shape* $(\mathbf{i}, \mathbf{m})$ to be a concatenation of m_1 columns of size i_1, m_2 columns of size i_2, etc:

$$\tau = \kappa_{11} \star \kappa_{12} \star \cdots \star \kappa_{1m_1} \star \kappa_{21} \star \cdots \star \kappa_{2m_2} \star \cdots \star \kappa_{l1} \star \cdots \star \kappa_{lm_l},$$

where κ_{km} is a column of size i_k for each k, m. (If $m_k = 0$, there are *no* columns in the corresponding position of τ.)

Remarks. (a) This terminology is suggested by the classical notion of a column-strict Young tableau with m_1 columns of size i_1 followed by m_2 columns of size i_2, etc., transcribed in terms of its column reading word. For example, take $\mathbf{i} = (3, 2, 3)$, $\mathbf{m} = (0, 2, 1)$, which corresponds to the Young diagram at left below. Note that $m_1 = 0$ means there are zero columns of size $i_1 = 3$ on the left end of our diagram.

$$\lambda = \begin{matrix} \times & \times & \times \\ \times & \times & \times \\ & & \times \end{matrix} \qquad \tau = \begin{matrix} 3 & 1 & 1 \\ 4 & 3 & 2 \\ & & 3 \end{matrix}$$

The filling at right is transcribed in our notation as $\tau = 34 \star 13 \star 123$. One can define a generalized Young diagram corresponding to any reduced $(\mathbf{i}, \mathbf{m})$. (See [Mg2,Mg3,RS2].)

(b) In the model of Littelmann, bases are parametrized by piecewise-linear paths in the weight lattice $\mathbb{Z}^n \mod \mathbb{Z}(1, \dots, 1)$ of G. Our tableaux encode such paths if we consider a column $(r_1, \dots, r_i)$ as denoting a weight $\pi = e_{r_1} + \cdots + e_{r_i}$ of the ith fundamental representation of G, so that a tableau is a sequence of weights $\pi_1 \star \pi_2 \star \cdots$. The associated path goes in linear steps from from 0 to π_1 to $\pi_1 + \pi_2$ etc.

1.2 Liftable-standard tableaux. Let us once and for all **arbitrarily fix a word**

$$\mathbf{i} = (i_1, \dots, i_l),$$

reduced or non-reduced. From now on we will assume the presence of this chosen ambient word. For $k \in \mathbb{Z}^+$, we will frequently use the notation

$$[k] = \{1, 2, \dots, k\},$$

as well as $[k, l] = \{k, k+1, \dots, l\}$.

A *subword* of $\mathbf{i}$ is a subsequence $\mathbf{i}' = (i_{j_1}, i_{j_2}, \dots, i_{j_r})$ for some indices $1 \leq j_1 < \cdots < j_r \leq l$. We say the set $J = \{j_1, \dots, j_r\} \subseteq [l]$ is the *subword index* of $\mathbf{i}'$, and we write $\mathbf{i}' = \mathbf{i}(J)$. Note that we consider subwords $\mathbf{i}(J_1)$, $\mathbf{i}(J_2)$ to be different whenever $J_1 \neq J_2$, so that there is a total of 2^l distinct subwords. Abusing notation, we will frequently identify an indexing set $J \subseteq [l]$ with the corresponding subword $\mathbf{i}(J)$ of our fixed ambient word $\mathbf{i}$, and we will call J itself a subword. The intersection, union, and complement of two subwords are defined in the obvious way in terms of their indexing sets. For $k \leq l$, the interval $[k] \subseteq [l]$ indexes an initial subword of $\mathbf{i}$.

Given any subword $J \subseteq [l]$, define $w(J)$, the *permutation generated by* J, as the partial product of $s_{i_1} s_{i_2} \cdots s_{i_l}$ containing only those factors which appear in $J = \{j_1 < \cdots < j_r\}$:

$$w(J) \;=\; \prod_{j \in J} s_{i_j} \;=\; s_{i_{j_1}} \cdots s_{i_{j_r}}\,.$$

Again, the subword J is *reduced* if the above is a minimal-length decomposition of $w(J)$ into s_i's. Also define the *column generated by J up to position k* to be

$$w(J \cap [k]) \cdot \varpi_{i_k}.$$

Now, consider a decreasing nest of subwords of $\mathbf{i}$,

$$[l] \supseteq J_{11} \supseteq \cdots \supseteq J_{1m_1} \supseteq J_{21} \supseteq \cdots \supseteq J_{lm_l},$$

We say it is a *reduced nest* if

$$J_{km} \cap [k] \quad \text{is a reduced word for all } k, m.$$

We say that a tableau τ of shape $(\mathbf{i}, \mathbf{m})$ is *generated* by the reduced nest of subwords (or that the reduced nest is a *lifting* of the tableau) if each column κ_{km} of $\tau = \kappa_{11} \star \cdots \star \kappa_{lm_l}$ is generated by the subword J_{km} up to the position k:

$$\kappa_{km} = w(J_{km} \cap [k]) \cdot \varpi_{i_k}.$$

Definition. A tableau τ of shape $(\mathbf{i}, \mathbf{m})$ is *liftable-standard* (or just *standard*) if there exists a reduced nest of subwords of $\mathbf{i}$ which generates τ. The set of all standard tableaux of shape $(\mathbf{i}, \mathbf{m})$ is denoted $\mathcal{T}(\mathbf{i}, \mathbf{m})$.

A tableau τ is called *standard with respect to a subword* $J \subseteq [l]$ if all of the subwords J_{km} in the lifting are subwords of J: $J \supseteq J_{11} \supseteq \cdots \supseteq J_{lm_l}$. The set of such tableaux is denoted $\mathcal{T}(J, \mathbf{m})$.

While useful to deduce general properties of standard tableaux, this definition is quite difficult to work with in specific cases. We will give a description of $\mathcal{T}(\mathbf{i}, \mathbf{m})$ allowing efficient computations in Section 1.4.

Examples. (a) Let $\mathbf{i} = (1, 2, 1) = 121$, $\mathbf{m} = (1, 1, 1)$ as in the introduction. A typical subword index is $J = \{1, 3\}$, associated to the subword $\mathbf{i}(J) = (i_1, i_3) = (1, 1)$. In order to emphasize that the position of the letters is essential to distinguish subwords, we will write $\circ$ in place of a letter of $\mathbf{i}$ which is missing in $\mathbf{i}(J)$. That is,

$\mathbf{i}(J) = i_1 \circ i_3 = 1 \circ 1$. For $J_1 = \{1\}$, $J_2 = \{3\}$, we have $\mathbf{i}(J_1) = 1 \circ \circ \neq \mathbf{i}(J_2) = \circ \circ 1$, and for the empty word we have $\mathbf{i}(\emptyset) = \circ \circ \circ$.

The nest of sets $J_{11} = \{1,2\} \supseteq J_{21} = \{1,2\} \supseteq J_{31} = \{2\}$ indexes the nest of subwords $12\circ \supseteq 12\circ \supseteq \circ 2\circ$, which generates the standard tableau $\tau = s_1\varpi_1 \star s_1 s_2 \varpi_2 \star s_2 \varpi_1 = 2 \star 23 \star 1$. Another lifting for the same tableau is $121 \supseteq 121 \supseteq \circ\circ\circ$.

(b) Consider a $GL(n)$ Demazure module $V_w(\lambda)$ for a permutation $w \in W$ and a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$. (The character of $V_w(\lambda)$ is called a *key polynomial*.) Then $H(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{m}})$ is isomorphic to the dual module $V_w^*(\lambda)$ if $(\mathbf{i}, \mathbf{m})$ are taken as follows.

Let $s_{i_1} \ldots s_{i_l} = w$ be a reduced decomposition. Further suppose that if the last occurence of each letter $k = 1, \ldots, n-1$ in $\mathbf{i}$ is at position j_k, so that $i_{j_k} = k$, then $j_1 < j_2 < \ldots < j_{n-1}$. Now let $\lambda' = (\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n)$ be the conjugate partition, and take $\mathbf{m} = (m_1, \ldots, m_l)$ with $m_{j_k} = \lambda'_k - \lambda'_{k+1}$ and $m_j = 0$ otherwise. That is, m_{j_k} is the number of columns of size k in the Young diagram of λ.

It is easily seen that a classical Young tableau is semi-standard exactly if its column reading word is liftable-standard with respect to the above $(\mathbf{i}, \mathbf{m})$ for $V_{w_0}(\lambda)$, where w_0 is the longest permutation. The liftable-standard tableaux for the $(\mathbf{i}, \mathbf{m})$ corresponding to a general $V_w(\lambda)$ are exactly the standard tableaux on the Schubert variety X_w in classical Standard Monomial Theory [LkSd1,LkSd2].

For example, the pair $\mathbf{i} = (3,2,3)$, $\mathbf{m} = (0,2,1)$ of the previous section give the Demazure module $V_{s_3 s_2 s_3}(3,3,1)$. The filling pictured is a semi-standard Young tableau in the classical sense and is standard on $X_{s_3 s_2 s_3}$. Its column word $\tau = 34 \star 13 \star 123$ has several liftings, such as $32\circ \supseteq \circ 2\circ \supseteq \circ\circ\circ$ and $323 \supseteq \circ 23 \supseteq \circ\circ 3$.

(c) For any permutation $w \in W$ we have a kind of generalized Young diagram called a Rothe diagram. In [Mg3] we explain how to relate this to a pair $(\mathbf{i}, \mathbf{m})$ so that $H^0(\mathbf{i}, \mathbf{m})$ is the dual Schubert module of Kraskiewicz and Pragacz [KP], whose character is a Schubert polynomial. In this case our standard tableaux are essentially identical to the non-commutative Schubert polynomials of Lascoux and Schutzenberger [LcSb2].

1.3. Standard basis.

In the Introduction we defined the Bott-Samelson variety $Z_{\mathbf{i}}$, the embedding $\mu : Z_{\mathbf{i}} \to \mathrm{Gr}(\mathbf{i})$, and the line bundle $\mathcal{L}_{\mathbf{m}} = \mu^* \mathcal{O}(\mathbf{m})$.

Let

$$x = \begin{pmatrix} x_{11} & \cdots & x_{1i} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{ni} \end{pmatrix} \in \mathrm{Gr}(i)$$

be the homogeneous coordinates on the Grassmannian, so that x represents the subspace spanned by the column vectors of the matrix. Then any column $\kappa = (r_1 < \cdots < r_i)$ is associated to a Plucker coordinate, the minor on rows $r_1, \ldots, r_n$ of x:

$$\Delta_\kappa(x) \;=\; \det_{i \times i} \begin{pmatrix} x_{r_1 1} & \cdots & x_{r_1 i} \\ \vdots & \ddots & \vdots \\ x_{r_i 1} & \cdots & x_{r_i i} \end{pmatrix} \;\in\; H^0(\mathrm{Gr}(i), \mathcal{O}(1)).$$

Furthermore, the set of all tableaux of shape $(\mathbf{i}, \mathbf{m})$ parametrize a spanning set of $H^0(\mathrm{Gr}(\mathbf{i}), \mathcal{O}(\mathbf{m}))$ consisting of monomials in the Plucker coordinates. That is, for $\tau = \kappa_{11} \star \cdots \star \kappa_{lm_l}$, let

$$\Delta_\tau \;=\; \prod_{j=1}^{l} \prod_{m=1}^{m_j} \Delta_{\kappa_{jm}}(x^{(j)}) \;\in\; H^0(\mathrm{Gr}(\mathbf{i}), \mathcal{O}(\mathbf{m})),$$

where $x^{(j)}$ denotes the homogeneous coordinates on the jth factor of $\mathrm{Gr}(\mathbf{i})$. We let $\Delta_\emptyset = 1$. Denote the restriction of the section Δ_τ to $Z_\mathbf{i} \subseteq \mathrm{Gr}(\mathbf{i})$ by the same symbol Δ_τ. Under this restriction the Plucker monomials still form a spanning set of $H^0(Z_\mathbf{i}, \mathcal{L}_\mathbf{m})$ by the following "Borel-Weil-Bott" theorem:

Proposition (Mathieu [Mt1,Mt2]**, Kumar** [Ku]**).**
(i) The map

$$\mu^* : H^0(\mathrm{Gr}(\mathbf{i}), \mathcal{O}(\mathbf{m})) \to H^0(Z_\mathbf{i}, \mathcal{L}_\mathbf{m})$$

is a surjective homomorphism of B-modules.
(ii) $H^i(Z_\mathbf{i}, \mathcal{L}_\mathbf{m}) = 0$ for all $i > 0$.

If a tableau τ is standard, we call Δ_τ a *standard monomial*.

Theorem 1. *The standard monomials of shape $(\mathbf{i}, \mathbf{m})$ form a basis of the space of sections of $\mathcal{L}_\mathbf{m}$ over $Z_\mathbf{i}$:*

$$H^0(Z_\mathbf{i}, \mathcal{L}_\mathbf{m}) = \bigoplus_{\tau \in \mathcal{T}(\mathbf{i},\mathbf{m})} \mathbb{F}\,\Delta_\tau.$$

The proof will be given in Section 3.

Writing a diagonal matrix as $\mathrm{diag}(x_1, \ldots, x_n) \in T$, we obtain the coordinate ring $\mathbb{F}[T] = \mathbb{F}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ (modulo the relation $x_1 \cdots x_n = 1$ in case $G = SL(n)$). By the (dual) character of a B-module M, we mean

$$\mathrm{char}^* M = \mathrm{tr}(\mathrm{diag}(x_1, \ldots, x_n) | M^*) \;\in\; \mathbb{F}[T].$$

(We take duals in order to get polynomials as characters.) Now, given any tableau $\tau = (r_1, \ldots, r_N)$, we define its *weight monomial*

$$x^\tau = x_{r_1} \cdots x_{r_N} \in \mathbb{F}[T].$$

Then $\mathrm{char}^* \mathbb{F}\Delta_\tau = x^\tau$, and we obtain:

Corollary.

$$\mathrm{char}^* H^0(Z_\mathbf{i}, \mathcal{L}_\mathbf{m}) = \sum_{\tau \in \mathcal{T}(\mathbf{i},\mathbf{m})} x^\tau.$$

1.4. Demazure operations on tableaux.
Define Demazure's isobaric divided difference operator

$$\Lambda_i : \mathbb{F}[T] \to \mathbb{F}[T],$$

$$\Lambda_i f = \frac{x_i f - x_{i+1} s_i f}{x_i - x_{i+1}}.$$

Example. Let $f(x_1, x_2, x_3) = x_1^2 x_2^2 x_3$, so that

$$\begin{aligned} \Lambda_2 f(x_1, x_2, x_3) &= \tfrac{x_2(x_1^2 x_2^2 x_3) - x_3(x_1^2 x_3^2 x_2)}{x_2 - x_3} \\ &= x_1^2 x_2 x_3 (x_2 + x_3). \end{aligned}$$

Let $\varpi_i = x_1 x_2 \cdots x_i \in \mathbb{F}[T]$, the ith fundamental weight of $GL(n)$.

Proposition (Demazure's formula [D2,Mt1,Ku]**).**

$$\operatorname{char}^* H^0(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{m}}) = \Lambda_{i_1}(\varpi_{i_1}^{m_1} \Lambda_{i_2}(\varpi_{i_2}^{m_2} \cdots \Lambda_{i_l}(\varpi_{i_l}^{m_l}) \cdots)).$$

Now we define analogs of the Demazure operations acting on tableaux instead of on characters. This will allow us to "lift" the Demazure formula from characters to tableaux, thus reconciling the two character formulas above. It also gives an efficient algorithm for generating the standard monomial basis.

We will need the *root operators* on tableaux first defined by Lascoux and Schutzenberger [LcSb1], and later generalized by Littelmann [Lt1, Lt2, Lt3]. For $i \in \{1, \ldots, n-1\}$, the *lowering operator* f_i takes a tableau $\tau = (r_1, r_2, \ldots)$ either to a formal null symbol $\mathbf{O}$, or to a new tableau $\tau' = (r'_1, r'_2, \ldots)$ by changing a single entry $r_j = i$ to $r'_j = i+1$ and leaving the other entries alone ($r'_j = r_j$), according to the following rule.

First, we ignore all the entries of τ except those equal to i or $i+1$; if an i is followed by an $i+1$ (not counting any ignored entries in between), then henceforth we ignore that pair of entries; we look again for an i followed (up to ignored entries) by an $i+1$, and henceforth ignore this pair; and iterate until we ignore everything but a subsequence of the form $i{+}1, i{+}1, \ldots, i{+}1, i, i, \ldots, i$. If there are *no* i entries in this subsequence, then $f_i(\tau) = \mathbf{O}$, the null symbol. If there *are* some i entries, then the *leftmost* is changed to $i{+}1$.

This is identical to Littelmann's minimum-point definition [Lt1,Lt2] if we think of tableaux as paths in the weight lattice.

Example. We apply f_2 to the tableau

τ	=	1	2	2	1	3	2	1	4	2	2	3	3
		.	2	2	.	3	2	.	.	2	2	3	3
		.	2	.	.	.	2	.	.	2	.	.	3
		.	**2**	.	.	.	**2**	.	.	.	.	.	.
$f_2(\tau)$	=	1	**3**	2	1	3	**2**	1	4	2	2	3	3
$(f_2)^2(\tau)$	=	1	**3**	2	1	3	**3**	1	4	2	2	3	3
$(f_2)^3(\tau)$	=	**O**											

We also have the *raising operator* defined by $e_i(\tau) = (f_i)^{-1}(\tau)$ if this exists, $e_i(\tau) = \mathbf{O}$ otherwise. One can describe the action of e_i identically to that of f_i except that e_i changes the rightmost non-ignored $i{+}1$ into i.

Now define the plactic Demazure operator Λ_i taking a tableau τ to a set of tableaux:

$$\Lambda_i(\tau) = \{\tau, f_i(\tau), f_i^2(\tau), \dots\} - \{\mathbf{O}\}.$$

To apply Λ_i to a set of tableaux $\mathcal{T}$, apply it to each element and take the union:

$$\Lambda_i(\mathcal{T}) = \bigcup_{\tau\in\mathcal{T}} \Lambda_i(\tau).$$

We will also need a tableau analog of multiplying by a monomial in the x_i's. For a column κ, define $\kappa^{\star m} = \kappa \star \cdots \star \kappa$ (m factors). Then the multiplication by the monomial ϖ_i^m in the character formula will correspond to concatenating with the tableau $\varpi_i^{\star m} = (1, 2, \dots, i, \dots, 1, 2, \dots, i)$.

Now we can build up the set of standard tableaux using the above operations.

Theorem 2. *The set of liftable-standard tableaux is generated by the Demazure and concatenation operations:*

$$\mathcal{T}(\mathbf{i},\mathbf{m}) = \Lambda_{i_1}\left(\varpi_{i_1}^{\star m_1} \star \Lambda_{i_2}\left(\varpi_{i_2}^{\star m_2} \star \cdots \Lambda_{i_l}\left(\varpi_{i_l}^{\star m_l}\right)\cdots\right)\right).$$

The proof is given in Section 2.

Example. Let $\mathbf{i} = 121$, $\mathbf{m} = (1,1,1)$, so that $\mathcal{T}(\mathbf{i},\mathbf{m}) = \Lambda_1(1\star\Lambda_2(12\star\Lambda_1(1)))$. To *generate* the standard tableaux, we start with the empty tableau $\emptyset$, and proceed from the right end of the above Demazure formula:

$$\{\emptyset\}\xrightarrow{1\star}\{1\}\xrightarrow{\Lambda_1}\{1,2\}\xrightarrow{12\star}\{121,122\}\xrightarrow{\Lambda_2}$$

$$\{121,131,122,132,133\}\xrightarrow{1\star}\{1121,1131,1122,1132,1133\}\xrightarrow{\Lambda_1}$$

$$\{1121,2121,2122,1131,2131,2231,2232,1122,1132,2132,1133,2133,2233\}$$

The last set is $\mathcal{T}(\mathbf{i},\mathbf{m})$.

To *test* whether a given tableau is standard, say $\tau = 2123$, we invert the above operations: at the kth step we raise the tableau as far as possible using $f_{i_k}^{-1}$, then strip off the initial word ϖ_{i_k}, then go on to the next step. That is,

$$\tau = 2123\xrightarrow{f_1^{-1}}1123\xrightarrow{(1\star)^{-1}}123\xrightarrow{(12\star)^{-1}}3$$

This algorithm will terminate in the empty tableau $\emptyset$ exactly if the the original τ is standard. But in this case we end with a tableau $\tau' = 3$ for which we can invert neither $f_{i_3} = f_1$ nor $(\varpi_{i_3}\star) = (1\star)$, so the original τ is *not* standard. This process is closely related to the keys of Lascoux and Schutzenberger [LcSb1,LcSb2].

The root operators f_i and e_i also define a crystal graph structure on $\mathcal{T}(\mathbf{i},\mathbf{m})$, which suggests that our standard basis will deform to a crystal basis inside the quantum function ring of B.

2. Proof of Theorem 2

For a subword $J \subseteq [l]$, the set of *constructible tableaux* is

$$\begin{aligned}\mathcal{C}(J,\mathbf{m}) &= \Lambda_{i_1}^{\delta_1}\left(\varpi_{i_1}^{\star m_1} \star \Lambda_{i_2}^{\delta_2}\left(\varpi_{i_2}^{\star m_2} \star \cdots \Lambda_{i_l}^{\delta_l}\left(\varpi_{i_l}^{\star m_l}\right)\cdots\right)\right) \\ &= \left\{ f_{i_1}^{a_1}(\varpi_{i_1}^{\star m_1} \star f_{i_2}^{a_2}(\varpi_{i_2}^{m_2} \star \cdots f_{i_l}^{a_l}(\varpi_{i_l}^{m_l})\cdots)) \;\middle|\; \begin{array}{c} a_1,\dots,a_l \geq 0, \\ a_j = 0 \text{ for } j \notin J \end{array} \right\}\end{aligned}$$

where $\delta_k = 1$ if $k \in J$, $\delta_k = 0$ if $k \notin J$. We defined $\mathcal{T}(J,\mathbf{m})$ in §1.2. (By convention, for $\mathbf{m} = (0,\dots,0)$ we set $\mathcal{C}(J,\mathbf{m}) = \mathcal{T}(J,\mathbf{m}) = \{\emptyset\}$, containing only the empty tableau.)

We will show that

$$\mathcal{T}(J,\mathbf{m}) = \mathcal{C}(J,\mathbf{m})$$

for all subwords $J \subseteq [l]$. We proceed by a series of elementary lemmas establishing identical recursions for the two sides.

2.1 Recursion for $\mathcal{T}(J,\mathbf{m})$.

Definition-Lemma 1. *If $J \subseteq [l]$ is any subword and $J', J'' \subseteq J$ are maximal-length reduced subwords of J, then $w(J') = w(J'')$. We denote $w_{\max}(J) \stackrel{\text{def}}{=} w(J')$ for any maximal reduced $J' \subseteq J$.*

Proof. By the subword definition of Bruhat order, the lemma is equivalent to saying that $S(J) = \{w(J') \mid J' \subseteq J\}$ is an interval in the Bruhat order: $S(J) = [e, w_{\max}]$ for some $w_{\max} \in W$. By induction, we suppose this is true for J and show it holds for $J_0 = \{j_0\} \cup J$, where $j_0 < j$ for all $j \in J$. Let $w_J = w_{\max}(J)$. If $s_{j_0} w_J > w_J$, then $S(J_0) = [e, s_{j_0} w_J]$. If $s_{j_0} w_J < w_J$, then $[e, s_{j_0} w_J] \subseteq [e, w_J]$, and $S(J_0) = S(J)$, by the Zigzag Lemma [Hu §5.9].

Note that if J is reduced then $w_{\max}(J) = w(J)$.

Given $J' \subset J \subseteq [l]$, we say J' is *less than J with respect to column k* if the maximal kth column generated by J' is smaller in Bruhat order than the maximal kth column generated by J:

$$J' \stackrel{k}{<} J \quad\Leftrightarrow\quad \begin{array}{c} J' \subset J \quad \text{and} \\ w_{\max}(J' \cap [k]) \cdot \varpi_{i_k} \;<\; w_{\max}(J \cap [k]) \cdot \varpi_{i_k}. \end{array}$$

Now, let $\epsilon(k) = (0,\dots,1,\dots,0)$, a sequence of length l with a 1 in the kth place. Then for $\mathbf{m} = (0,\dots,0,m_k,\dots,m_l)$, we have $\mathbf{m} - \epsilon(k) = (0,\dots,0,m_k - 1,\dots,m_l)$.

Lemma 2. *For $\mathbf{m} = (0,\dots,0,m_k,\dots,m_l)$ with $m_k > 0$, $J \subseteq [l]$, and $\kappa_{\max} = w_{\max}(J \cap [k]) \cdot \varpi_{i_k}$, we have*

$$\mathcal{T}(J,\mathbf{m}) = \kappa_{\max} \star \mathcal{T}(J, \mathbf{m} - \epsilon(k)) \;\sqcup \bigcup_{J' \stackrel{k}{<} J} \mathcal{T}(J',\mathbf{m}).$$

Proof. (a) First, it is evident that $\mathcal{T}(J',\mathbf{m}) \subseteq \mathcal{T}(J,\mathbf{m})$ for any $J' \subseteq J$.
(b) Also, $\kappa_{\max} \star \mathcal{T}(J, \mathbf{m} - \epsilon(k)) \subseteq \mathcal{T}(J,\mathbf{m})$ as follows. If $\tau' = \kappa_1 \star \kappa_2 \star \cdots \in$

$\mathcal{T}(J, \mathbf{m} - \epsilon(k))$, by definition there exists a lifting $J_1 \supseteq J_2 \supseteq \cdots$ with $J_1 \subseteq J$ and $J_1 \cap [k]$ reduced.

Now let $\widetilde{J} \subseteq [k]$ be a maximal reduced subword of $J \cap [k]$. By the Definition-Lemma,

$$w(\widetilde{J}) \geq w(J_1 \cap [k]) \geq w(J_2 \cap [k]) \geq \cdots,$$

so we may take reduced words $\widetilde{J}_k \subseteq \widetilde{J}$ with $w(\widetilde{J}_j) = w(J_j \cap [k])$ and $\widetilde{J}_1 \supseteq \widetilde{J}_2 \supseteq \cdots$. Set $J'_j = J_j \cap [k+1, l]$. Then

$$\widetilde{J} \cup [k+1, l] \supseteq \widetilde{J}_1 \cup J'_1 \supseteq \widetilde{J}_2 \cup J'_2 \supseteq \cdots$$

is a lifting of $\kappa_{\max} \star \tau' = \kappa_{\max} \star \kappa_1 \star \kappa_2 \cdots$.

(c) Finally, suppose $\tau = \kappa_0 \star \kappa_1 \star \kappa_2 \cdots \in \mathcal{T}(J, \mathbf{m})$, with lifting $J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots$. Then we must have exactly one of the following. Either $\kappa_0 = \kappa_{\max}$ and $J_1 \supseteq J_2 \supseteq \cdots$ is a lifting of $\kappa_1 \star \kappa_2 \star \cdots$, so that $\tau \in \kappa \star \mathcal{T}(J, \mathbf{m} - \epsilon(k))$. Or $\kappa_0 \neq \kappa_{\max}$, meaning

$$w_{\max}(J_0 \cap [k]) \cdot \varpi_{i_k} \neq w_{\max}(J \cap [k]) \cdot \varpi_{i_k},$$

and hence

$$w_{\max}(J_0 \cap [k]) \cdot \varpi_{i_k} < w_{\max}(J \cap [k]) \cdot \varpi_{i_k}.$$

Therefore $J_0 \overset{k}{<} J$ and $\tau \in \mathcal{T}(J_0, \mathbf{m})$.

Lemma 3. *Let $\mathbf{m} = (0, \ldots, 0, m_k, \ldots, m_l)$. If $J, J' \subseteq [l]$ are subwords with $J \cap [k+1, l] = J' \cap [k+1, l]$ and $w_{\max}(J \cap [k]) = w_{\max}(J' \cap [k])$, then $\mathcal{T}(J, \mathbf{m}) = \mathcal{T}(J', \mathbf{m})$.*

Proof. If $\kappa_1 \star \kappa_2 \cdots \in \mathcal{T}(J, \mathbf{m})$ has a lifting $J_1 \supseteq J_2 \cdots$ with $J_1 \subseteq J$, then

$$w_{\max}(J \cap [k]) \geq w(J_1 \cap [k]) \geq w(J_2 \cap [k]) \geq \cdots,$$

so by the subword definition of Bruhat order we can find reduced words with

$$J' \cap [k] \supseteq \widetilde{J}_1 \supseteq \widetilde{J}_2 \supseteq \cdots$$

with $w(\widetilde{J}_j) = w(J_j \cap [k])$. Setting $J'_j = \widetilde{J}_j \cup (J_j \cap [k+1, l])$ for all j gives a lifting $J'_1 \supseteq J'_2 \supseteq \cdots$ for $\kappa_1 \star \kappa_2 \star \cdots$ which shows it to lie in $\mathcal{T}(J', \mathbf{m})$.

Reversing the roles of J and J' we obtain the reverse inclusion, which completes the proof.

2.2. Head-string property.

Definition. For $i \in \{1, \ldots, n-1\}$, the *i-string through τ* is defined as

$$S_i(\tau) \overset{\text{def}}{=} \{\cdots, e_i^2\tau, e_i\tau, \tau, f_i\tau, f_i^2\tau, \cdots\} - \{\mathbf{O}\}.$$

We say a set of tableaux $\mathcal{T}$ has the *head-string property* if for any i and any $\tau \in \mathcal{T}$, we have either :

(i) $S_i(\tau) \subseteq \mathcal{T}$ (the entire i-string of τ lies in $\mathcal{T}$); or

(ii) $S_i(\tau) \cap \mathcal{T} = \{\tau\}$ and $e_i\tau = \mathbf{O}$ (only the head of the string lies in $\mathcal{T}$).

A key step in our proof of Theorems 1 and 2 will be to show that the set of standard tableaux has this property.

We will use the following properties special to groups of type A: for any i and any column κ,

$$e_i\,\kappa = \mathbf{O} \ \text{ or } \ f_i\,\kappa = \mathbf{O} \qquad \text{and} \qquad e_i^2\,\kappa = f_i^2\,\kappa = \mathbf{O}.$$

Lemma 4. *For κ a column, τ' a tableau, and $a > 0$, we have*

$$f_i^a(\kappa \star \tau') = \begin{cases} (f_i\kappa) \star (f_i^{a-1}\tau') & \text{if } f_i\kappa \neq \mathbf{O},\ e_i\tau' = \mathbf{O} \\ \kappa \star (f_i^a \tau') & \text{otherwise,} \end{cases}$$

$$e_i^a(\kappa \star \tau') = \begin{cases} \kappa \star (e_i^a\tau') & \text{if } (f_i\kappa = \mathbf{O} \text{ and } e_i\tau' \neq \mathbf{O}) \\ & \text{or } e_i^2\tau' \neq \mathbf{O} \\ (e_i\kappa) \star (e_i^{a-1}\tau') & \text{otherwise.} \end{cases}$$

Here we use the convention that $\tau \star \mathbf{O} = \mathbf{O} \star \tau = \mathbf{O}$.

Proof. This follows from the well-known (and easily checked) formulas [Lt2]

$$f_i(\tau \star \tau') = \begin{cases} (f_i\tau) \star \tau' & \text{if } \exists n > 0,\ f_i^n\tau \neq \mathbf{O},\ e_i^n\tau' = \mathbf{O} \\ \tau \star (f_i\tau') & \text{otherwise,} \end{cases}$$

$$e_i(\tau \star \tau') = \begin{cases} \tau \star (e_i\tau') & \text{if } \exists n > 0,\ f_i^n\tau = \mathbf{O},\ e_i^n\tau' \neq \mathbf{O} \\ (e_i\tau) \star \tau' & \text{otherwise,} \end{cases}$$

together with $f_i^2\kappa = \mathbf{O}$.

2.3. Recursion for $\mathcal{C}(J, \mathbf{m})$.

Theorem 2$^+$.
(i) As in Lemma 2, let $\mathbf{m} = (0, \ldots, 0, m_k, \ldots, m_l)$ with $m_k > 0$, $J \subseteq [l]$, and $\kappa_{\max} = w_{\max}(J \cap [k]) \cdot \varpi_{i_k}$. Then we have

$$\mathcal{C}(J, \mathbf{m}) = \kappa_{\max} \star \mathcal{C}(J, \mathbf{m} - \epsilon(k)) \ \sqcup \bigcup_{J' \overset{k}{<} J} \mathcal{C}(J', \mathbf{m}).$$

(ii) $\mathcal{C}(J, \mathbf{m}) = \mathcal{T}(J, \mathbf{m})$.

(iii) $\mathcal{C}(J, \mathbf{m})$ has the head-string property.

Proof. By induction on $|J|$ (the order of J) and $|\mathbf{m}| = m_1 + \cdots + m_l$ (the number of columns in a tableau). The initial cases $J = \emptyset$ or $\mathbf{m} = (0, \ldots, 0)$ are trivial. Now assume (i)-(iii) for all $\mathcal{C}(J', \mathbf{m}')$ with $|J'| < |J|$ or $|\mathbf{m}'| < |\mathbf{m}|$. We will prove (i)-(iii) for $\mathcal{C}(J, \mathbf{m})$.

(i) If $J \cap [k] = \emptyset$, the righthand side of the equation (i) reduces to $\varpi_{i_k} \star \mathcal{C}(J, \mathbf{m} - \epsilon(k))$, and the claim is clear.

Otherwise, let j_1 be the smallest element of $J \cap [k]$, and write

$$\widetilde{J} = J - \{j_1\}, \qquad i = i_{j_1}, \qquad \widetilde{\kappa}_{\max} = w_{\max}(\widetilde{J} \cap [k]) \cdot \varpi_{i_k} \ \leq \ \kappa_{\max}.$$

Then $\mathcal{C}(J, \mathbf{m}) = \Lambda_i \mathcal{C}(\widetilde{J}, \mathbf{m})$, so that every tableau in $\mathcal{C}(J, \mathbf{m})$ may be written as $\kappa \star \tau = f_i^a(\widetilde{\kappa} \star \tilde{\tau})$ for $\widetilde{\kappa} \star \tilde{\tau} \in \mathcal{C}(\widetilde{J}, \mathbf{m})$. In fact, we have $\tau \in \mathcal{C}(J, \mathbf{m} - \epsilon(k))$, which follows easily by induction and Lemma 4.

By induction, we have

$$\mathcal{C}(\widetilde{J},\mathbf{m}) = \widetilde{\kappa}_{\max} \star \mathcal{C}(\widetilde{J},\mathbf{m}-\epsilon(k)) \;\sqcup \bigcup_{\widetilde{J}' \overset{k}{<} \widetilde{J}} \mathcal{C}(\widetilde{J}',\mathbf{m}).$$

(a) First, it is evident that $\mathcal{C}(J,\mathbf{m}) \supseteq \mathcal{C}(J',\mathbf{m})$ whenever $J \supseteq J'$.
(b) We show $\mathcal{C}(J,\mathbf{m}) \supseteq \kappa_{\max} \star \mathcal{C}(J,\mathbf{m}-\epsilon(k))$ as follows.
For $\tau \in \mathcal{C}(J,\mathbf{m}-\epsilon(k))$, we can write $\tau = f_i^a\tilde{\tau}$ for $\tilde{\tau} \in \mathcal{C}(\widetilde{J},\mathbf{m}-\epsilon(k))$. In fact, by raising $\tilde{\tau}$ with e_i and increasing a, we may assume that $e_i\tilde{\tau} = \mathbf{O}$. (We know $e_i^a\tilde{\tau} \in \mathcal{C}(\widetilde{J},\mathbf{m}-\epsilon(k)) \cup \{\mathbf{O}\}$ by the head-string property (iii).)

In case $\kappa_{\max} = \widetilde{\kappa}_{\max}$, we have $f_i\widetilde{\kappa}_{\max} = \mathbf{O}$, and

$$\kappa_{\max}\star\tau \;=\; \kappa_{\max}\star(f_i^a\tilde{\tau}) \;=\; f_i^a(\widetilde{\kappa}_{\max}\star\tilde{\tau}) \;\in\; \Lambda_i\mathcal{C}(\widetilde{J},\mathbf{m}) \;=\; \mathcal{C}(J,\mathbf{m}).$$

In case $\kappa_{\max} > \widetilde{\kappa}_{\max}$, we have $\kappa_{\max} = f_i\widetilde{\kappa}_{\max}$, and

$$\kappa_{\max}\star\tau = \kappa_{\max}\star(f_i^a\tilde{\tau}) = f_i^{a+1}(\widetilde{\kappa}_{\max}\star\tilde{\tau}) \;\in\; \Lambda_i\mathcal{C}(\widetilde{J},\mathbf{m}) \;=\; \mathcal{C}(J,\mathbf{m}).$$

This completes the $\supseteq$ direction of formula (i).
(c) Now we show the $\subseteq$ direction of formula (i). We suppose $\kappa \star \tau \in \mathcal{C}(J,\mathbf{m})$ with $\kappa < \kappa_{\max}$, and proceed to show $\kappa \star \tau \in \mathcal{C}(J',\mathbf{m})$ as in the Theorem. Let us write $\kappa \star \tau = f_i^a(\widetilde{\kappa} \star \tilde{\tau})$ with $\tilde{\tau} \in \mathcal{C}(\widetilde{J},\mathbf{m}-\epsilon(k))$.

In case $\kappa_{\max} = \widetilde{\kappa}_{\max}$ we have $\widetilde{\kappa} < \widetilde{\kappa}_{\max}$, so by (i) applied to $\mathcal{C}(\widetilde{J},\mathbf{m})$, $\widetilde{\kappa} \star \tilde{\tau} \in \mathcal{C}(\widetilde{J}',\mathbf{m})$ for some $\widetilde{J}' \overset{k}{<} \widetilde{J}$. We may assume $\widetilde{\kappa} = \widetilde{\kappa}'_{\max} \overset{\text{def}}{=} w_{\max}(\widetilde{J}' \cap [k]) \cdot \varpi_{i_k}$, since otherwise we would have $\widetilde{\kappa} \star \tilde{\tau}$ in some smaller $\mathcal{C}(\widetilde{J}'',\mathbf{m})$ by induction. If $f_i\widetilde{\kappa}'_{\max} < \kappa_{\max}$, then by definition $\{j_1\} \cup \widetilde{J}' \overset{k}{<} J$, so $\kappa \star \tau \in \mathcal{C}(\{j_1\} \cup \widetilde{J}',\mathbf{m})$ gives the desired result. If $f_i\widetilde{\kappa}'_{\max} = \kappa_{\max}$, then (since $\kappa < \kappa_{\max}$) we must have $f_i^a(\widetilde{\kappa}'_{\max}\star\tilde{\tau}) = \widetilde{\kappa}'_{\max}\star(f_i^a\tilde{\tau})$, and therefore $e_i\tilde{\tau} \neq \mathbf{O}$ by Lemma 4. This means $f_i^a\tilde{\tau} \in \mathcal{C}(\widetilde{J}',\mathbf{m}-\epsilon(k))$ by head-string, and so

$$\kappa \star \tau = \widetilde{\kappa}'_{\max} \star (f_i^a\tilde{\tau}) \in \mathcal{C}(\widetilde{J}',\mathbf{m})$$

by (i) applied to $\mathcal{C}(\widetilde{J}',\mathbf{m})$. Since $\widetilde{J}' \overset{k}{<} J$, we have the desired result.

In case $\kappa_{\max} > \widetilde{\kappa}_{\max}$, we have $\widetilde{J} \overset{k}{<} J$. If $\widetilde{\kappa}_{\max} > \widetilde{\kappa}$, then $\widetilde{\kappa}_{\max} \star \tilde{\tau} \in \mathcal{C}(\widetilde{J}',\mathbf{m})$ for $\widetilde{J}' \overset{k}{<} \widetilde{J}$, so that $\{j_1\} \cup \widetilde{J}' \overset{k}{<} J$, and clearly $\kappa \star \tau \in \mathcal{C}(\{j_1\} \cup \widetilde{J}',\mathbf{m})$, as desired. If $\widetilde{\kappa}_{\max} = \widetilde{\kappa}$, we must have (since $\kappa < \kappa_{\max}$ and $f_i\widetilde{\kappa}_{\max} = \kappa_{\max}$) that $f_i^a(\widetilde{\kappa}_{\max} \star \tilde{\tau}) = \widetilde{\kappa}_{\max} \star (f_i^a\tilde{\tau})$, which means $e_i\tilde{\tau} \neq \mathbf{O}$ by Lemma 4. Thus, by head-string, we have $f_i^a\tilde{\tau} \in \mathcal{C}(\widetilde{J},\mathbf{m}-\epsilon(k))$, so that, as desired,

$$\kappa \star \tau \;=\; \widetilde{\kappa}_{\max} \star (f_i^a\tilde{\tau}) \;\in\; \widetilde{\kappa}_{\max} \star \mathcal{C}(\widetilde{J},\mathbf{m}-\epsilon(k)) \;\subseteq\; \mathcal{C}(\widetilde{J},\mathbf{m}).$$

This completes the proof of (i).

(ii) Follows immediately from the identical recursions for $\mathcal{T}(J,\mathbf{m})$ (Lemma 2) and for $\mathcal{C}(J,\mathbf{m})$ (part (i)), by induction on $|J|$ and $|\mathbf{m}|$. The intitial cases $J=\emptyset$ and $\mathbf{m}=(0,\ldots,0)$ are trivial.

(iii) To show the head-string property for $\mathcal{C}(J,\mathbf{m})$, we need to prove: for all i_0,

$$\tau' \in \mathcal{C}(J,\mathbf{m}) \ \text{ and } \ e_{i_0}\tau' \neq \mathbf{O} \quad \Rightarrow \quad e_{i_0}\tau' \in \mathcal{C}(J,\mathbf{m}) \ \text{ and } \ f_i\tau' \in \mathcal{C}(J,\mathbf{m}) \cup \{\mathbf{O}\}.$$

Take $\tau' = \kappa \star \tau$. If $\kappa < \kappa_{\max}$, then by (i) we have $\tau' \in \mathcal{C}(J',\mathbf{m})$ with $J' \overset{k}{<} J$, and the head-string property follows by induction.

Thus we may assume $\tau' = \kappa_{\max} \star \tau$ with $\tau \in \mathcal{C}(J, \mathbf{m}-\epsilon(k))$.

In case $e_{i_0}\kappa_{\max} = \mathbf{O}$, we have by hypothesis $e_{i_0}(\kappa_{\max} \star \tau) \neq \mathbf{O}$, so we must have $e_{i_0}(\kappa_{\max} \star \tau) = \kappa_{\max} \star (e_{i_0}\tau)$. Hence $e_{i_0}\tau \neq \mathbf{O}$, and by head-string $e_{i_0}\tau \in \mathcal{C}(J,\mathbf{m}-\epsilon(k))$, and

$$\kappa_{\max} \star e_{i_0}\tau \ \in \ \kappa_{\max} \star \mathcal{C}(J,\mathbf{m}-\epsilon(k)) \ \subseteq \ \mathcal{C}(J,\mathbf{m})$$

by (i), as desired. Also by head-string $f_{i_0}\tau \in \mathcal{C}(J,\mathbf{m}-\epsilon(k)) \cup \{\mathbf{O}\}$, and

$$\kappa_{\max} \star f_{i_0}\tau \ \in \ \kappa_{\max} \star \mathcal{C}(J,\mathbf{m}-\epsilon(k)) \cup \{\mathbf{O}\} \ \subseteq \ \mathcal{C}(J,\mathbf{m}) \cup \{\mathbf{O}\}$$

as desired.

In case $f_{i_0}\kappa_{\max} = \mathbf{O}$, if $e_{i_0}(\kappa_{\max} \star \tau) = \kappa_{\max} \star (e_{i_0}\tau) \neq \mathbf{O}$, we may argue as in the previous case. Thus suppose $e_{i_0}(\kappa_{\max} \star \tau) = (e_{i_0}\kappa_{\max}) \star \tau \neq \mathbf{O}$, so that $e_{i_0}\kappa_{\max} \neq \mathbf{O}$. Now let $w_{\max} = w_{\max}(J \cap [k])$, so that $\kappa_{\max} = w_{\max} \cdot \varpi_{i_k}$. Since $f_{i_0}\kappa_{\max} = \mathbf{O}$, $e_{i_0}\kappa_{\max} \neq \mathbf{O}$, we have $s_{i_0}\kappa_{\max} = e_{i_0}\kappa_{\max} < \kappa_{\max}$, and so $s_{i_0}w_{\max} < w_{\max}$. Therefore we may find a reduced word for $w_{\max}$ with first letter equal to i_0:

$$\widetilde{\mathbf{i}} = (i'_1, \ldots, i'_t) \quad \text{with} \quad w(\widetilde{\mathbf{i}}) = w_{\max} \ \text{ and } \ i'_1 = i_0.$$

Let

$$\mathbf{i}' = (i'_1, \ldots, i'_t, i'_{t+1}, i'_{t+2}, \ldots) \quad \text{with} \quad (i'_{t+1}, i'_{t+2}, \ldots) = \mathbf{i}(J \cap [k{+}1, l]).$$

Using (ii) and Lemma 3, we have

$$\mathcal{C}(J,\mathbf{m}) = \mathcal{T}(J,\mathbf{m}) = \mathcal{T}(\mathbf{i}',\mathbf{m}) = \mathcal{C}(\mathbf{i}',\mathbf{m})$$

(Note that $|\,\mathbf{i}'| \leq |J|$, so (ii) holds for $\mathcal{C}(\mathbf{i}',\mathbf{m})$.) But $\mathcal{C}(\mathbf{i}',\mathbf{m}) = \Lambda_{i_0}(\cdots)$, so for any $\tau' \in \mathcal{C}(\mathbf{i}',\mathbf{m})$, we have $e_{i_0}\tau', f_{i_0}\tau' \in \mathcal{C}(\mathbf{i}',\mathbf{m}) \cup \{\mathbf{O}\} = \mathcal{C}(J,\mathbf{m}) \cup \{\mathbf{O}\}$ as desired.

The proof of (iii) is finished, the induction proceeds, and the Theorem is proved.

3. Proof of Theorem 1

3.1 Subvarieties.

Given a subword index $J \subseteq [l]$ we may consider $\mathbf{i}(J)$ as a word in its own right, corresponding to a Bott-Samelson variety $Z_{\mathbf{i}(J)}$ which embeds naturally into $Z_{\mathbf{i}}$ via

$$Z_{\mathbf{i}(J)} \cong Z_J \stackrel{\text{def}}{=} Q_1 \times \cdots \times Q_l \ / \ B^l \ \subseteq \ Z_{\mathbf{i}} = P_{i_1} \times \cdots \times P_{i_l} \ / \ B^l,$$

where

$$Q_j = \begin{cases} P_{i_j} & \text{if } j \in J \\ B & \text{if } j \notin J \end{cases}.$$

Let us index Schubert varieties X_κ in a Grassmannian $\mathrm{Gr}(i)$ by columns $\kappa = (r_1, \cdots, r_i)$. That is, let $\mathbb{F}^\kappa$ be the subspace of $\mathbb{F}^n$ spanned by the coordinate vectors e_r for $r \in \kappa$, and define $X_\kappa = \overline{B \cdot \mathbb{F}^\kappa} \subseteq \mathrm{Gr}(i)$. (Under the isomorphism $\mathrm{Gr}(i) \cong G/P$ for a suitable maximal parabolic P, we can write this as $X_\kappa \cong \overline{B \cdot wP} \subseteq G/P$, where $\kappa = w \cdot \varpi_i$.) We have $X_\kappa \subseteq X_{\kappa'}$ if and only if $\kappa \leq \kappa'$ in Bruhat order.

Lemma. *For any $J \subseteq [l]$, the partial multiplication map*

$$\begin{array}{rccc} \mu_k : & Z_J & \to & \mathrm{Gr}(i_k) \\ & (p_1, p_2, \ldots, p_l) & \mapsto & p_1 \cdots p_k \mathbb{F}^{i_k} \end{array}$$

has image equal to the Schubert variety of the column generated by J up to position k:

$$\mathrm{Im}(\mu) = X_\kappa, \qquad \kappa = w_{\max}(J \cap [k]) \cdot \varpi_{i_k}.$$

Proof. This follows from the formula:

$$P_i X_\kappa = \begin{cases} X_{s_i \kappa} & \text{if } s_i\kappa > \kappa \\ X_\kappa & \text{otherwise.} \end{cases}$$

We denote the restriction of the line bundle $\mathcal{L}_{\mathbf{m}}$ from $Z_{\mathbf{i}}$ to Z_J by the same symbol $\mathcal{L}_{\mathbf{m}}$. In order to prove Theorem 1, we will show the more general fact that $\mathcal{T}(J, \mathbf{m})$ indexes a basis of $H^0(Z_J, \mathcal{L}_{\mathbf{m}})$.

3.2 Linear independence.

For any subwords $J_1, J_2, \cdots \subseteq [l]$, we may consider the union of the corresponding Bott-Samelson varieties embedded in $\mathrm{Gr}(\mathbf{i})$:

$$Z_{J_1} \cup Z_{J_2} \cup \cdots \subseteq \mathrm{Gr}(\mathbf{i}).$$

The restriction of $\mathcal{O}(\mathbf{m})$ again defines a line bundle $\mathcal{L}_{\mathbf{m}}$ on the union, and for any tableau τ the Plucker monomial Δ_τ restricts to an element of $H^0(Z_{J_1} \cup Z_{J_2} \cup \cdots, \mathcal{L}_{\mathbf{m}})$.

Definition. A tableau τ of shape $(\mathbf{i}, \mathbf{m})$ is *standard on a union* $Z = Z_{J_1} \cup Z_{J_2} \cup \cdots$ if it is standard on at least one of the components $Z_{J_1}, Z_{J_2}, \ldots$. That is, the set of standard tableaux on Z is

$$\mathcal{T}(Z_{J_1} \cup Z_{J_2} \cup \cdots, \mathbf{m}) \stackrel{\text{def}}{=} \mathcal{T}(J_1, \mathbf{m}) \cup \mathcal{T}(J_2, \mathbf{m}) \cup \cdots .$$

Proposition. *For any subwords $J_1, J_2, \ldots \subseteq [l]$, the standard monomials of shape $(\mathbf{i}, \mathbf{m})$ on the union $Z = Z_{J_1} \cup Z_{J_2} \cup \cdots$ are linearly independent.*

Remark. A statement of this generality holds only for independence: the standard monomials on a union Z do *not* in general span $H^0(Z, \mathcal{L}_{\mathbf{m}})$.

For example, let $\mathbf{i} = (1,2,1)$, $\mathbf{m} = (0,0,1)$, $J_1 = \{1\}$, $J_2 = \{3\}$. Then $\mathcal{T}(J_1, \mathbf{m}) = \mathcal{T}(J_2, \mathbf{m}) = \{1, 2\}$, but $\dim H^0(Z_{J_1} \cup Z_{J_2}, \mathcal{L}_{\mathbf{m}}) = 3$. In fact, in this case the restriction map $H^0(\mathrm{Gr}(\mathbf{i}), \mathcal{O}(\mathbf{m})) \to H^0(Z_{J_1} \cup Z_{J_2}, \mathcal{L}_{\mathbf{m}})$ is not surjective. This is possible because $\mathcal{O}(\mathbf{m})$ is non-ample.

Proof of Proposition. Let $\tau^{(1)}, \ldots, \tau^{(t)}$ be standard tableaux in $\mathcal{T}(Z, \mathbf{m})$. Consider a linear relation among the standard monomials $\Delta_{\tau^{(r)}}$ on the variety Z

$$(*) \qquad a_1 \Delta_{\tau^{(1)}}(z) + \cdots + a_t \Delta_{\tau^{(t)}}(z) = 0 \quad \forall\, z \in Z,$$

where $a_r \in \mathbb{F}$. We will show

$$a_r = 0 \quad \text{for} \quad r = 1, \ldots, t$$

by induction on t (the length of the linear relation) and on $|\mathbf{m}| = m_1 + \cdots + m_l$ (the number of columns in a tableau).

(a) Let us suppose $\mathbf{m} = (0, \ldots, 0, m_k, \ldots, m_l)$ with $m_k > 0$, and write $\tau^{(r)} = \kappa_{k1}^{(r)} \star \cdots \star \kappa_{lm_l}^{(r)}$. Let $I_{k1}^{(r)} \supseteq \cdots \supseteq I_{lm_l}^{(r)}$ be a lifting of $\tau^{(r)}$. By definition, each $I_{k1}^{(r)}$ is contained in one of the subwords $J_1, J_2, \ldots$ defining Z, so the Bott-Samelson variety of the subword $I_{k1}^{(r)}$ is contained in Z:

$$Z_{I_{k1}^{(r)}} \subseteq Z \quad \text{for all } r.$$

(b) Now let κ denote one of the Bruhat-minimal elements among the first columns of $\tau^{(1)}, \ldots, \tau^{(t)}$:

$$\kappa \in \min\{\, \kappa_{k1}^{(1)}, \ldots, \kappa_{k1}^{(t)} \,\}$$

Order the terms of relation $(*)$ so that, for some $1 \leq t_0 \leq t$, we have

$$\kappa = \kappa_{k1}^{(r)} \quad \text{for } r \leq t_0, \qquad\qquad \kappa \not\geq \kappa_{k1}^{(r)} \quad \text{for } r > t_0.$$

(c) We show that $a_r = 0$ for $r \leq t_0$. Let

$$Y = \bigcup_{r \leq t_0} Z_{I_{k1}^{(r)}} \ \subset \ Z.$$

Let us restrict the relation $(*)$ from Z to Y. By the Lemma of §3.1, we have $\mu_k(Y) = X_\kappa \subseteq \mathrm{Gr}(i_k)$. Furthermore, the first factor $\Delta_{\kappa_{11}^{(r)}}(z)$ of $\Delta_{\tau^{(r)}}$ is just the Plucker coordinate of $\kappa_{k1}^{(r)}$ on $\mathrm{Gr}(i_k)$. Since $\kappa \not\geq \kappa_{k1}^{(r)}$ for all $r > t_0$, we have

$$\Delta_{\tau^{(r)}}(y) = 0 \quad \forall\, y \in Y, \qquad \text{for all } r > t_0.$$

so that $(*)$ becomes

$$\Delta_\kappa(y)\ \left(a_1\Delta_{\tilde\tau^{(1)}}(y)+\cdots+a_{t_0}\Delta_{\tilde\tau^{(t_0)}}(y)\right)=0 \qquad \forall y\in Y,$$

where $\tau^{(r)}=\kappa_{k1}^{(r)}\star\tilde\tau^{(r)}$ for some $\tilde\tau^{(r)}\in\mathcal{T}(I_{k1}^{(r)},\mathbf{m}-\epsilon(k))$. However by the same Lemma, Δ_κ is not identically zero on any of the components $Z_{I_{k1}^{(r)}}$ of Y. Hence Δ_κ is not a zero-divisor in the coordinate ring of Y, and we may factor it from the equation to get a linear relation among standard monomials $\tilde\tau^{(r)}$ of shape $(\mathbf{i},\mathbf{m}-\epsilon(k))$ on Y. That is, $\tilde\tau^{(r)}\in\mathcal{T}(Y,\mathbf{m}-\epsilon(k))$, and

$$a_1\Delta_{\tilde\tau^{(1)}}(y)+\cdots+a_{t_0}\Delta_{\tilde\tau^{(t_0)}}(y)=0 \qquad \forall\, y\in Y.$$

By induction on $|\mathbf{m}|$, this relation must be identically zero: $a_r=0$ for $r\le t_0$.
(d) Since $t_0\ge 1$, we have shown that $a_r=0$ for at least a single r. Therefore $(*)$ reduces to a relation with fewer than t terms, which must have $a_r=0$ for all r by induction on t. The proof of the Proposition is finished.

3.3 Dimension counting.

Recall that for $\tau=(r_1,r_2,\dots)$ we define $x^\tau=x_{r_1}x_{r_2}\cdots\in\mathbb{F}[T]$. For any set of tableaux $\mathcal{T}$, let

$$\operatorname{char}^*\mathcal{T}\overset{\text{def}}{=}\sum_{\tau\in\mathcal{T}}x^\tau,$$

Proposition. *For any subword $J\subseteq[l]$ and $\mathbf{m}=(m_1,\dots,m_l)$, $m_j\ge 0$, we have*

$$\operatorname{char}^*\mathcal{T}(J,\mathbf{m})=\Lambda_{i_1}^{\delta_1}(\varpi_{i_1}^{m_1}\Lambda_{i_2}^{\delta_2}(\varpi_{i_2}^{m_2}\cdots\Lambda_{i_l}^{\delta_l}(\varpi_{i_l}^{m_l})\cdots)),$$

where $\delta_j=1$ if $j\in J$ and $\delta_j=0$ if $j\notin J$.

Proof. By Theorem 2^+ of §2.3, we know that

$$\mathcal{T}(J,\mathbf{m})=\Lambda_{i_1}^{\delta_1}(\varpi_{i_1}^{\star m_1}\star\Lambda_{i_2}^{\delta_2}(\varpi_{i_2}^{\star m_2}\star\cdots\Lambda_{i_l}^{\delta_l}(\varpi_{i_l}^{\star m_l})\cdots)),$$

so we need to show that each operation Λ_i and $(\varpi_i\star)$ on tableaux has the corresponding effect on characters.

For $j\in[l]$, let

$$\mathbf{m}\cap[j,l]=(0,\dots,0,m_j,\dots,m_l),\qquad \mathcal{T}_j=\mathcal{T}(J\cap[j,l],\,\mathbf{m}\cap[j,l]).$$

Now,

$$\varpi_{i_k}^{\star m_k}\star\mathcal{T}_{k+1}=\mathcal{T}(J\,\cap\,[k{+}1,l],\mathbf{m}\,\cap\,[k,l]),$$

so by §2.3 this set has the head-string property. That is, we may partition it into i_k-strings

$$\varpi_{i_k}^{\star m_k}\star\mathcal{T}_{k+1}=S^{(1)}\sqcup S^{(2)}\sqcup\cdots$$

so that each $S^{(r)}$ is either a complete i_k-string or only the head of an i_k-string. It is easily verified that

$$\text{char}^* (\Lambda_{i_k} S^{(r)}) = \Lambda_{i_k}(\text{char}^* S^{(r)}),$$

so we have

$$\begin{aligned} \text{char}^* \mathcal{T}_k &= \text{char}^* \Lambda_{i_k}(\varpi_{i_k}^{\star m_k} \star \mathcal{T}_{k+1}) \\ &= \text{char}^* \Lambda_{i_k}(S^{(1)} \sqcup S^{(2)} \sqcup \cdots) \\ &= \Lambda_{i_k} \text{char}^* (S^{(1)} \sqcup S^{(2)} \sqcup \cdots) \\ &= \Lambda_{i_k} \text{char}^* (\varpi_{i_k}^{\star m_k} \star \mathcal{T}_{k+1}) \\ &= \Lambda_{i_k} (\varpi_{i_k}^{m_k} \text{ char}^* \mathcal{T}_{k+1}) \end{aligned}$$

Thus we may build up $\mathcal{T}(\mathbf{i}, \mathbf{m})$ and char* $\mathcal{T}(\mathbf{i}, \mathbf{m})$ in parallel steps, and the Proposition follows.

Proof of Theorem 1. The Demazure character formula of §1.4 applies to the subvarieties Z_J to give

$$\text{char}^* H^0(Z_J, \mathcal{L}_{\mathbf{m}}) = \Lambda_{i_1}^{\delta_1}(\varpi_{i_1}^{m_1} \Lambda_{i_2}^{\delta_2}(\varpi_{i_2}^{m_2} \cdots \Lambda_{i_l}^{\delta_l}(\varpi_{i_l}^{m_l}) \cdots)).$$

Hence by the above Proposition we have (after specializing the characters to $x_1 = \cdots = x_n = 1$):

$$\#\{\Delta_\tau \mid \tau \in \mathcal{T}(J, \mathbf{m})\} = \dim H^0(Z_J, \mathcal{L}_{\mathbf{m}}).$$

But by the Proposition of §3.2, we know that the standard monomials $\{\Delta_\tau \mid \tau \in \mathcal{T}(J, \mathbf{m})\}$ form a linearly independent subset of $H^0(Z_J, \mathcal{L}_{\mathbf{m}})$. Therefore they form a basis.

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