Standard Monomial Theory for Bott-Samelson Varieties *

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Bott-Samelson varieties are an important tool in geometric representation theory [1], [3], [25], [10]. They were originally defined as desingularizations of Schubert varieties and share many of the properties of Schubert varieties. They have an action of a Borel subgroup, and the projective coordinate ring of a Bott-Samelson variety splits into certain generalized Demazure modules (which also appear in other contexts [22], [23]).

Standard Monomial Theory, developed by Seshadri and the first author [15], [16], and recently completed by the second author [20], gives explicit bases for the Demazure modules associated to Schubert varieties. In this paper, we extend the techniques of [20] to give explicit bases for the generalized Demazure modules associated to Bott-Samelson varieties, thus proving a strengthened form of the results announced by the first and third authors in [12]. (See also [13].) We also obtain more elementary proofs of the cohomology vanishing theorems of Kumar [10] and Mathieu [25]; of the projective normality of Bott-Samelson varieties; and of the Demazure character formula.

1 Basis Theorem

In this section we state the main results which we prove in the rest of the paper.

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1.1 Demazure modules

Let G be a reductive algebraic group of rank n over an algebraically closed field $\mathbf{k}$, $\mathbf{g}$ its Lie algebra, and $U(\mathbf{g})$ its universal enveloping algebra. (To avoid technicalities, we deal with finite-dimensional $\mathbf{g}$, but our results extend straightforwardly to symmetrizable Kac-Moody algebras.) Choose a Cartan subgroup $H \subset G$, and a Borel subgroup $B \supset H$. For $i = 1, \dots, n$, we then have positive and negative simple root vectors E_i and F_i generating $\mathbf{g}$; the Cartan subalgebra $\mathbf{h}$; the Borel subalgebra $\mathbf{b}$ generated by E_i and $\mathbf{h}$; the simple roots α_i and coroots $\alpha_i^\vee = [E_i, F_i]$; the fundamental weights ϖ_i with $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$ and the weight lattice $X = \bigoplus_{i=1}^n \mathbb{Z}\varpi_i$; and the simple reflections $s_i : X \to X,\ \lambda \mapsto \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$, generating the Weyl group W.

An element $\lambda \in X$ is a one-dimensional representation $\lambda : \mathbf{h} \to \mathbf{k}$, and it extends to a one-dimensional $\mathbf{b}$-module denoted $\mathbf{k}_\lambda$. For a dominant weight $\lambda \in X^+ := \bigoplus_{i=1}^n \mathbb{Z}_+ \varpi_i$, we let V_λ denote the Weyl module of highest weight vector v_λ with $b \cdot v_\lambda = \lambda(b)\, v_\lambda$ for $b \in \mathbf{b}$. (That is: over the rational numbers, $V_{\lambda,\mathbb{Q}}$ is the irreducible $G_\mathbb{Q}$-module; $V_{\lambda,\mathbb{Z}} \subset V_{\lambda,\mathbb{Q}}$ is the smallest $\mathbb{Z}$-submodule containing v_λ and closed under the operations $E_i^l/l!$ and $F_i^l/l!$ for $l \geq 0$, $i \in [1, n]$; and in general $V_\lambda = V_{\lambda,\mathbf{k}} := V_{\lambda,\mathbb{Z}} \otimes_\mathbb{Z} \mathbf{k}$.) We also have the dual module V_λ^*. For $\mathbf{k}$ of characteristic zero, V_λ and V_λ^* are irreducible G-modules.

Given an arbitrary *word*, meaning a sequence $\mathbf{i} = (i_1, \dots, i_r)$ with $i_j \in \{1, 2, \dots n\}$; as well as a *multiplicity list* $\mathbf{m} = (m_1, \dots, m_r)$ with $m_j \in \mathbb{Z}_+$; we let

$$\lambda_1 := m_1 \varpi_{i_1}, \dots, \lambda_r = m_r \varpi_{i_r}.$$

We define the *generalized Demazure module* $V_{\mathbf{i},\mathbf{m}}$ as a certain B-submodule of the tensor product $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}$:

$$V_{\mathbf{i},\mathbf{m}} := \mathbf{u}_{i_1} \cdot (v_{\lambda_1} \otimes \mathbf{u}_{i_2} \cdot (v_{\lambda_2} \otimes \cdots \otimes \mathbf{u}_{i_{r-1}} \cdot (v_{\lambda_{r-1}} \otimes \mathbf{u}_{i_r} \cdot (v_{\lambda_r})) \cdots)),$$

where $\mathbf{u}_i = \bigoplus_{l \geq 0} \mathbf{k} F_i^l / l!$ denotes the hyperalgebra of a single negative root vector. (By convention, if $r = 0$, so that $\mathbf{i}$ is the empty word, we set $V_{\mathbf{i},\mathbf{m}} = \mathbf{k}_0$, the trivial one-dimensional B-module.) The dual B-module $V_{\mathbf{i},\mathbf{m}}^*$ is a quotient of $V_{\lambda_1}^* \otimes \cdots \otimes V_{\lambda_r}^*$. We will explain in §1.4 how these modules arise from Bott-Samelson varieties.

The ordinary Demazure module $V_\lambda(w) \subset V_\lambda$ is essentially a special case. Given $w \in W$, choose the word $\mathbf{i}$ so that $w = s_{i_1} \cdots s_{i_r}$ is a reduced decomposition, and hence

$$V_\lambda(w) := \mathbf{u}_{i_1} \cdots \mathbf{u}_{i_r} \cdot v_\lambda\,.$$

Given a weight $\lambda = l_1 \varpi_1 + \cdots + l_n \varpi_n$, choose multiplicities $(m_1, \dots, m_r)$ as follows. Suppose the rightmost occurrence of $i = 1$ in the word $\mathbf{i}$ is at position k: that is, $i_k = 1, i_j \neq 1$ for $j > k$. Then let $m_k = l_1$. (If $i = 1$ does not occur in $\mathbf{i}$, let $m_1 = 0$.) Next let k' be the rightmost occurrence of $i = 2$ in $\mathbf{i}$, and let $m_{k'} = l_2$. Proceed in this way for each i, then let $m_j = 0$ if it has not already been defined. Finally, let $\lambda' = \sum_{i \notin \mathbf{i}} l_i \varpi_i$, where the sum runs over those i which do not occur anywhere in $\mathbf{i}$. Then $V_\lambda(w) \cong \mathbf{k}_{\lambda'} \otimes V_{\mathbf{i},\mathbf{m}}$, with $w(\lambda') = \lambda'$.

This paper revolves around the following

Problem: Find explicit bases for the generalized Demazure module $V_{\mathbf{i},\mathbf{m}}$ and its dual $V^*_{\mathbf{i},\mathbf{m}}$.

From now on, we will assume $\mathbf{i}$ and $\mathbf{m}$ are fixed, so that we can refer to i_i, or r, or $\lambda_1, \ldots, \lambda_r$ without ambiguity.

1.2 Standard tableaux

We recall the machinery of Lakshmibai-Seshadri paths ([17], [18]) needed to index our bases of $V_{\mathbf{i},\mathbf{m}}$.

Let $X_{\mathbb{R}} := X \otimes_{\mathbb{Z}} \mathbb{R}$ be the real form of the weight lattice. A *path* is a piecewise-linear map $\pi : [0,1] \to X_{\mathbb{R}}$ (up to reparametrization) with $\pi(0) = 0$. For a weight $\lambda \in X$, we let π^λ denote the straight-line path: $\pi^\lambda(t) := t\lambda$; and $\pi_1 * \pi_2$ denotes the concatenation of two paths. The *weight* of a path is its endpoint, $\mathrm{wt}(\pi) := \pi(1)$.

Let $W_\lambda \subset W$ be the stabilizer of a weight $\lambda \in X$, and use $>$ to denote the Chevalley-Bruhat order on W and on the coset space W/W_λ. Let $\lambda \in X^+$ be a dominant weight. An *LS-chain* of shape λ is a pair of lists

$$(\tau_1 > \cdots > \tau_q;\ 0 = a_0 < a_1 < \cdots < a_q = 1),$$

where $\tau_j \in W/W_\lambda$ and $a_j \in \mathbb{Q}$, such that for each j there exists a chain in the Bruhat order $\tau_j = \sigma_0 > \sigma_1 > \cdots > \sigma_p = \tau_{j+1}$ with $\ell(\sigma_{k+1}) = \ell(\sigma_k) + 1$ and $a_j(\sigma_{k+1}\lambda - \sigma_k\lambda) \in \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$ for each k. An LS-chain corresponds to a path $\pi : [0,1] \to X_{\mathbb{R}}$, whose linear pieces are defined by:

$$\pi(t) = \sum_{j=1}^{k-1}(a_j - a_{j-1})\tau_j\lambda + (t - a_{k-1})\tau_k\lambda \quad \text{for} \quad a_{k-1} \le t \le a_k.$$

We call a path an *LS-path* if it can be so constructed from a (necessarily unique) LS-chain. We will frequently refer to LS-paths by their defining LS-chains, and abuse notation by writing: $\pi = (\tau_1 > \cdots > \tau_q;\ a_0 < \cdots < a_q)$.

The *lowering root operators* f_i act on a path π in the usual way [18]. Since we will only consider f_i acting on an LS-path or a concatenation of LS-paths, we may equivalently define these operators as in [17]. That is, let Q be the minimum value of the function $t \mapsto \langle \pi(t), \alpha_i^\vee \rangle$ for $t \in [0,1]$. Let t_1 be the largest t for which this minimum Q is attained, and let $t_2 \in [0,1]$ be the smallest $t > t_1$ where the function attains $Q+1$ (if there exists any such t). Now split our path π into three segments $\pi = \pi_1 * \pi_2 * \pi_3$, corresponding to $t \in [0, t_1],\ t \in [t_1, t_2]$, and $t \in [t_2, 1]$. Define the root operator

$$f_i(\pi) := \pi_1 * s_i\pi_2 * \pi_3,$$

where $s_i\pi_2$ is the path $t \mapsto s_i(\pi_2(t))$. If there exists no t_2 as above, then $f_i(\pi)$ is undefined.

We may also define the *raising root operators* e_i analogously, so that e_i reflects the portion of π between t_1' and t_2', where t_2' is the *smallest* t for which $\langle \pi(t), \alpha_i^\vee \rangle$ attains its minimum Q, and t_1' is the *largest* $t < t_2'$ where the function

attains $Q+1$ (if there exists such t). We thus have $e_i(f_i(\pi)) = \pi$ whenever $f_i(\pi)$ is defined, and $\mathrm{wt}(f_i\pi) = \mathrm{wt}(\pi) - \alpha_i,\ \mathrm{wt}(e_i\pi) = \mathrm{wt}(\pi) + \alpha_i$.

Recall our notation $\lambda_1 = m_1\omega_{i_1}, \dots, \lambda_r = m_r\omega_{i_r}$ for our fixed data $\mathbf{i}, \mathbf{m}$. A *tableau* or *LS-monomial* of shape $(\lambda_1, \dots, \lambda_r)$ is a concatenation $\Pi = \pi_1 * \cdots * \pi_r$, where π_j is an LS-path of shape λ_j. A given tableau can be divided in only one way into such a concatenation of LS-paths of the proper shapes. Indeed, all LS-paths of shape λ_j have length $|\lambda_j|$ (in the W-invariant metric); so if we divide the path Π into pieces of length $|\lambda_1|, \dots, |\lambda_r|$, and successively translate the division points to the origin, we obtain the unique LS-path factors $\pi_1, \dots, \pi_r$.

We will usually refer to the tableau Π by the corresponding r-tuple of LS-chains, and write: $\Pi = (\pi_1, \dots, \pi_r)$. If we list all the Weyl group cosets in all the chains $\pi_1, \cdots, \pi_r$, we obtain a long list which we denote

$$\overline{\Pi} = (\tau_{11}, \tau_{12}, \dots, \tau_{1p_1}, \tau_{21}, \dots, \tau_{rp_r}),$$

where τ_{jp} is a coset modulo W_{λ_j} in the LS-chain π_j. When convenient we will reindex this long list as $\overline{\Pi} = (\tau_1, \tau_2, \dots, \tau_N)$.

Denote by $[1, r]$ the set of integers $\{1, 2, \dots, r\}$. For any subset of indices $J = \{a < b < \cdots\} \subset [1, r]$, we have a *subword* $\mathbf{i}(J) = (i_a, i_b, \dots)$ of our fixed word $\mathbf{i} = (i_1, \dots, i_r)$. We also define $w(\mathbf{i}) = s_{i_1} s_{i_2} \cdots s_{i_r} \in W$, so that $w(\mathbf{i}(J)) = s_{i_a} s_{i_b} \cdots$, the Weyl group element corresponding to the subword $\mathbf{i}(J)$. We say that $\mathbf{i}$ is *reduced* if $r = \ell(w(\mathbf{i}))$, the Bruhat length; and similarly for the subword $\mathbf{i}(J)$. Further, we write $J^{(j)} := J \cap [1, j]$, so that $w(\mathbf{i}(J^{(j)}))$ is an initial subword of $w(\mathbf{i}(J))$. If $\mathbf{i}(J)$ is reduced, then so is $\mathbf{i}(J^{(j)})$.

Let Π be a tableau of shape $(\lambda_1, \dots, \lambda_r)$, considered as a sequence $(\pi_1, \dots, \pi_r)$ of LS-chains π_j, producing the long list of cosets $\overline{\Pi} = (\tau_{11}, \dots, \tau_{rp_r})$.

Definition. We say Π is a *liftable-standard tableau* (or just *liftable*) if there exists a long chain of position-sets $(J_{11} \supset \cdots \supset J_{rp_r})$ such that for all j, p, the subword $\mathbf{i}(J_{jp}^{(j)})$ is reduced and

$$w(\mathbf{i}(J_{jp}^{(j)})) \equiv \tau_{jp} \mod W_{\lambda_j}.$$

Now consider the tableau Π as a concatenation of LS-paths $\pi_1 * \cdots * \pi_r$.

Definition. We say a tableau Π is a *constructable-standard* tableau (or is *constructable*) if it can be written as

$$\Pi = f_{i_1}^{l_1}(\pi^{\lambda_1} * f_{i_2}^{l_2}(\pi^{\lambda_2} * \cdots f_{i_r}^{l_r}(\pi^{\lambda_r}) \cdots))$$

for some $l_1, \dots, l_r \in \mathbb{Z}_+$.

Note that for *any* $l_1, \dots, l_r \in \mathbb{Z}_+$, the path Π defined by previous formula is a concatenation of LS-paths of the correct shapes, and is thus a tableau, which is constructable-standard by the definition. (See [18, §§2.6, 4.2].)

Given an arbitrary path Π of shape $(\lambda_1, \dots, \lambda_r)$, we may test whether it is a standard tableau as follows. Define the highest raising $e^{\mathrm{top}}(\Pi) := e^l(\Pi)$, where

l is maximal such that $e^l(\Pi)$ is defined. Let $e_{i_1}^{\rm top}(\Pi) = \pi'_1 * \cdots * \pi'_r$ (division into pieces of length $|\lambda_1|, \ldots, |\lambda_r|$). If $\pi'_1 \neq \pi^{\lambda_1}$, then Π is not standard. Otherwise, let $\Pi' = \pi'_2 * \cdots * \pi'_r$ and let $e_{i_2}^{\rm top}(\Pi') = \pi''_2 * \cdots * \pi''_r$. If $\pi''_2 \neq \pi^{\lambda_2}$, then Π is not standard. If we can continue in this way, raising by e_{i_j} and removing intial factors π^{λ_j}, until we obtain the one-point path (of length 0), then Π is standard; otherwise it is not.

Example. Let $G = GL_3$, $\mathbf{i} = (1,2,1,2)$ a non-reduced word, $\mathbf{m} = (2,1,0,1)$, $X = \mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2$, $\varpi_1 = \epsilon_1$, $\varpi_2 = \epsilon_1 + \epsilon_2$, $\lambda_1 = 2\epsilon_1$, $\lambda_2 = \epsilon_1 + \epsilon_2$, $\lambda_3 = 0$, $\lambda_4 = \epsilon_1 + \epsilon_2$.

A typical LS-path of shape λ is: $\pi = (s_1 > e; 0 < \frac{1}{2} < 1) = \pi^{\epsilon(2)} * \pi^{\epsilon(1)}$, a concatenation of two length-one segments. (For legibility we write $\epsilon(1), \epsilon(2)$ instead of ϵ_1, ϵ_2; and π^μ denotes a straight-line path.)

Now consider the tableau $\Pi = \pi^{\epsilon(2)} * \pi^{\epsilon(1)} * \pi^{\epsilon(1)+\epsilon(3)} * \pi^{\epsilon(2)+\epsilon(3)}$. Dividing Π into segments of lengths $2,1,0,1$, we get the LS-path factors of shapes $\lambda_1, \ldots, \lambda_4$: $\Pi = \pi_1 * \pi_2 * \pi_3 * \pi_4$, where:

$$\begin{aligned}
\pi_1 &= \pi^{\epsilon(2)} * \pi^{\epsilon(1)} = (s_1 > e; 0 < \tfrac{1}{2} < 1), \\
\pi_2 &= \pi^{\epsilon(1)+\epsilon(3)} = (s_2; 0 < 1), \\
\pi_3 &= \pi^0 = (e; 0 < 1), \text{ the one-point path}, \\
\pi_4 &= \pi^{\epsilon(2)+\epsilon(3)} = (s_2 s_1; 0 < 1).
\end{aligned}$$

This path has the lifting: $J_{11} = \{1,2,3,4\}$, $J_{12} = \{2,3,4\}$, $J_{21} = \{2,3,4\}$, $J_{31} = \{3,4\}$, $J_{41} = \{3,4\}$; since:

$$\begin{aligned}
w(\mathbf{i}(J_{11}^{(1)})) &= w(i_1) = s_1 &&\equiv \tau_{11} \mod W_{\lambda_1} \\
w(\mathbf{i}(J_{12}^{(1)})) &= w(\emptyset) = e &&\equiv \tau_{12} \mod W_{\lambda_1} \\
w(\mathbf{i}(J_{21}^{(2)})) &= w(i_2) = s_2 &&\equiv \tau_{21} \mod W_{\lambda_2} \\
w(\mathbf{i}(J_{31}^{(3)})) &= w(i_3) = s_1 \equiv e &&\equiv \tau_{31} \mod W_{\lambda_3} \\
w(\mathbf{i}(J_{41}^{(4)})) &= w(i_3 i_4) = s_1 s_2 &&\equiv \tau_{41} \mod W_{\lambda_4}
\end{aligned}$$

Furthermore, Π is constructable, since:

$$\begin{aligned}
\Pi &= f_1(\, \pi^{2\epsilon(1)} * f_2(\, \pi^{\epsilon(1)+\epsilon(2)} * f_1 f_2(\pi^{\epsilon(1)+\epsilon(2)})\,)\,) \\
&= f_1(\, \pi^{\lambda_1} * f_2(\, \pi^{\lambda_2} * f_1(\, \pi^{\lambda_3} * f_2(\pi^{\lambda_4})\,)\,)\,)
\end{aligned}$$

Theorem 1 *A tableau* Π *is liftable if and only if it is constructable. We call such* Π *a* standard tableau *of shape* $(\lambda_1, \ldots, \lambda_r)$.

Any set $\mathcal{T}$ of paths possesses a natural structure of *crystal graph*: namely, the graph with vertex set $\mathcal{T}$, and with i-colored edges $\{\Pi, f_i\Pi\}$ (whenever both Π and $f_i\Pi$ lie in $\mathcal{T}$). For example, the crystal graph of an ordinary Demazure module $V_\lambda(w)$ is associated to the set of tableaux $\{f_{i_1}^{l_1} \cdots f_{i_r}^{l_r} \pi^\lambda \mid l_1, \ldots, l_r \geq 0\}$, where $w = s_{i_1} \cdots s_{i_r}$ (reduced). Many important crystal graphs reduce to this basic case.

Theorem 2 *The crystal graph on the set* $\{\Pi\}$ *of standard tableaux of shape* $(\lambda_1, \ldots, \lambda_r)$ *is isomorphic (as an edge-colored graph) to a disjoint union of crystal graphs of ordinary Demazure modules. For* $\mu \in X^+$ *a dominant weight, the same is true of the set of paths* $\{\pi^\mu * \Pi\}$*, where* Π *runs over the standard tableaux of shape* $(\lambda_1, \ldots, \lambda_r)$*.*

This is a combinatorial version of excellent filtration for the B-modules $V^*_{\mathbf{i},\mathbf{m}}$ and $\mathbf{k}^*_\mu \otimes V^*_{\mathbf{i},\mathbf{m}}$. (See §2.4.)

1.3 Standard monomial bases

We use tableaux to index bases of B-modules, starting with the Weyl modules V_λ, then proceeding to $V_{\mathbf{i},\mathbf{m}}$.

For any Weyl module V_λ, the second author has constructed [20] a basis $\{v_\pi\}$ indexed by LS-chains π of shape λ, in which v_π is a weight vector with weight $\mathrm{wt}(\pi)$. (See §3.1 below for details.) The basis $\{v_\pi\}$, inspired by the work of Raghavan and Sankaran [26], is highly non-canonical, depending on several arbitrary choices. However $\{v_\pi\}$ is related to most "reasonable" bases of V_λ by a triangular matrix. Actually, we shall find it more convenient to pair $\{v_\lambda\}$ with bases of the dual module V^*_λ.

To be more specific, define the following *lexicographic partial order* on LS-chains. Given $\pi = (\tau_1 > \cdots ; 0 < a_1 < \cdots)$ and $\theta = (\sigma_1 > \cdots ; 0 < b_1 < \cdots)$, we say $\pi < \theta$ if: $\tau_1 < \sigma_1$ (in Bruhat order); or $\tau_1 = \sigma_1$, $a_1 < b_1$; or $\tau_1 = \sigma_1$, $a_1 = b_1$, $\tau_2 < \sigma_2$; etc. Note that the highest weight path is minimal in this order, and a path is large in this order if it is far from the highest weight path. We can extend this to tableaux by defining $(\pi_1, \ldots, \pi_r) < (\theta_1, \ldots, \theta_r)$ to mean: $\pi_1 < \theta_1$ (in the above order); or $\pi_1 = \theta_1$ and $\pi_2 < \theta_2$; etc.

Given a basis $\{p_\pi\}$ of V^*_λ indexed by LS-chains of shape λ, we say $\{p_\pi\}$ is *triangular* to $\{v_\pi\}$ if

$$p_\pi = v^*_\pi + \sum_{\theta > \pi} \propto v^*_\theta$$

where $\propto$ indicates an appropriate scalar coefficient (possibly different in each term). That is, $\langle p_\pi, v_\pi \rangle = 1$, and $\langle p_\theta, v_\pi \rangle = 0$ for all $\theta \not\geq \pi$. A certain basis $\{p_\pi\}$ defined in [20] obeys this triangular property for all groups G, as do most of the other known bases of Weyl modules, at least for classical groups G (types A_n, B_n, C_n, D_n).

Theorem 3 *The following bases of* V^*_λ *are triangular to* $\{v_\pi\}$*, for the specified classes of reductive groups:*
(a) Littelmann's canonically defined LS-path basis [20], for all G;
(b) Lusztig's dual canonical basis [21] (= Kashiwara's upper crystal basis [8], [6]), for classical G;
(c) Lakshmibai's standard PBW basis [11], for classical G.
(d) Lakshmibai-Seshadri's standard monomial basis [15], [16], for classical G.

Proof. For (a), see [20]. It is established in [26], [24] that, for the fundamental representations $V^*_{\varpi_i}$, the bases (a), (b), and (d) coincide. But this implies the

triangularity for an arbitrary V_λ^*. The triangularity between (b) and (c) follows from [11]. •

We expect that bases (b)–(d) possess the triangularity property for all G (including the Kac-Moody case).

Any such system of bases $\{p_\pi\}$ for each V_λ^* gives a basis of $V_{\lambda_1}^* \otimes \cdots \otimes V_{\lambda_r}^*$ whose indexing set consists of *all* tableaux of shape $(\lambda_1, \ldots, \lambda_r)$. The *standard* tableaux pick out a subset of this basis which restricts to a basis of the quotient $V_{\mathbf{i},\mathbf{m}}^*$.

Theorem 4 *For every $\lambda \in X^+$, let $\{p_\pi\}$ be a basis of V_λ^* which is triangular to $\{v_\pi\}$. For $\Pi = (\pi_1, \ldots, \pi_r)$ a tableau of shape $(\lambda_1, \ldots, \lambda_r)$, define*

$$p_\Pi := p_{\pi_1} \otimes \cdots \otimes p_{\pi_r} \in V_{\lambda_1}^* \otimes \cdots \otimes V_{\lambda_r}^*.$$

Then $\{p_\Pi\}$, where Π runs over the standard *tableau of shape $(\lambda_1, \ldots, \lambda_r)$, restricts to a basis of $V_{\mathbf{i},\mathbf{m}}^*$. We call this a* standard monomial basis *of $V_{\mathbf{i},\mathbf{m}}^*$.*

Since we may assume p_Π to have weight $\mathrm{wt}(\Pi)$, we can use the combinatorics of tableaux to compute the character of $V_{\mathbf{i},\mathbf{m}}$. Let $R = \mathbb{Z}[X] = \bigoplus_{\lambda \in X} \mathbb{Z} e^\lambda$ be the group ring of the weight lattice X. The Weyl group acts $\mathbb{Z}$-linearly on characters by $w(e^\lambda) = e^{w(\lambda)}$. We may define the *Demazure operator* $\Lambda_i : R \to R$ by

$$\Lambda_i(f) := \frac{f - e^{-\alpha_i} s_i(f)}{1 - e^{-\alpha_i}},$$

which can be interpreted uniquely as an element of R. We may also characterize Λ_i as the unique linear operator with $\Lambda_i^2 = \Lambda_i$ and

$$\Lambda_i(e^\lambda) = e^\lambda + e^{\lambda - \alpha_i} + e^{\lambda - 2\alpha_i} + \cdots + e^{s_i \lambda}.$$

for any $\lambda \in X$ with $\langle \lambda, \alpha_i^\vee \rangle \geq 0$.

Theorem 5 *The character of the B-module $V_{\mathbf{i},\mathbf{m}}$ is:*

$$\Lambda_{i_1}(e^{\lambda_1} \Lambda_{i_2}(e^{\lambda_2} \cdots \Lambda_{i_r}(e^{\lambda_r}) \cdots)).$$

Our strategy of proof for the above theorems is as follows. Theorem 1 is an elementary combinatorial fact, proved in §2.2. Also in §2.3 we prove that the Demazure formula computes the formal character of the set of standard tableaux, and we prove Theorem 2 in §2.4, all by the combinatorics of paths. Next we prove Theorem 4 in two steps. First, in §3 we show that the set $\{p_\Pi\}$ is linearly independent in $V_{\mathbf{i},\mathbf{m}}^*$, so that

$$\dim V_{\mathbf{i},\mathbf{m}}^* \geq \dim \mathrm{Span}\langle p_\Pi \rangle = \#\text{ standard tableaux}.$$

Then in §4 we compare the combinatorial Demazure formula with the geometric version to conclude

$$\#\text{ standard tableaux} \geq \dim V_{\mathbf{i},\mathbf{m}}^*,$$

which proves the Theorem. Also, Theorem 5 follows in the course of this proof, as does Theorem 6 below.

1.4 Bott-Samelson varieties

We now give a Borel-Weil-type result for producing our Demazure modules inside the projective coordinate rings of certain varieties. We will prove this, along with the corresponding analog of Bott's vanishing theorem. (These theorems were originally proved in our case in [25] and [10].)

Recall our fixed word $\mathbf{i} = (i_1, \ldots, i_r)$ from §1.1, and our Borel subgroup B of G. For each i let $P_i \supset B$ be the minimal parabolic subgroup with Weyl group $\langle s_i \rangle$, and define the *Bott-Samelson variety* as the quotient

$$Z_{\mathbf{i}} := (P_{i_1} \times \cdots \times P_{i_r}) \,/\, B^r,$$

where B^r acts on the right by:

$$(p_1, \ldots, p_r) \cdot (b_1, \ldots, b_r) := (p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_{r-1}^{-1} p_r b_r).$$

This is a smooth algebraic variety of dimension r. (If $r = 0$ and $\mathbf{i}$ is the empty word, we let $Z_{\mathbf{i}}$ be a point.)

For $\lambda \in X$, let e^λ denote the multiplicative character of B associated to λ, and let $\mathbf{k}^*_{(\lambda_1,\ldots,\lambda_r)}$ be the one-dimensional representation of B^r defined by

$$(b_1, \ldots, b_r)^{-1} \cdot k := e^{\lambda_1}(b_1) \cdots e^{\lambda_r}(b_r) k.$$

Define a line bundle on $Z_{\mathbf{i}}$ by

$$\mathcal{L}_{\mathbf{i},\mathbf{m}} := (P_{i_1} \times \cdots \times P_{i_r}) \overset{B^r}{\times} \mathbf{k}^*_{(\lambda_1,\ldots,\lambda_r)}$$

so that we identify

$$(\tilde{p}, k) \sim (\tilde{p} \cdot \tilde{b},\ \tilde{b}^{-1} \cdot k)$$

for $\tilde{p} \in P_{i_1} \times \cdots \times P_{i_r}$, $\tilde{b} \in B^r$, and $k \in \mathbf{k}$. Unraveling the definitions, we may concretely describe the space of regular global sections of this bundle as:

$$H^0(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{i},\mathbf{m}}) \quad = \quad \left\{ f : P_{i_1} \times \cdots \times P_{i_r} \to \mathbf{k} \;\middle|\; \begin{array}{c} \forall\ b_j \in B,\ p_j \in P_{i_j} \\ f((p_1, \ldots, p_r) \cdot (b_1, \ldots, b_r)) = \\ e^{\lambda_1}(b_1) \cdots e^{\lambda_r}(b_r) f(p_1, \ldots, p_r) \end{array} \right\},$$

where f denotes a polynomial function on the linear algebraic group $P_{i_1} \times \cdots \times P_{i_r}$.

The Borel subgroup acts on $Z_{\mathbf{i}}$ and $\mathcal{L}_{\mathbf{i},\mathbf{m}}$ by left multiplication: for $b \in B$,

$$b \cdot (p_1, \ldots, p_r) := (bp_1, p_2, \ldots, p_r) \quad \text{and} \quad b \cdot (p_1, \ldots, p_r, k) := (bp_1, p_2, \ldots, p_r, k)\,.$$

The space $H^0(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{i},\mathbf{m}})$ of regular global sections of $\mathcal{L}_{\mathbf{i},\mathbf{m}}$ over $Z_{\mathbf{i}}$ is naturally a B-module under translation.

Our analysis extends to certain varieties desingularized by Bott-Samelson varieties, called *configuration varieties* in [22], [23]. For a given $\mathbf{m} = (m_1, \ldots, m_r)$, the line bundle $\mathcal{L}_{\mathbf{i},\mathbf{m}}$ is very ample (resp. semi-ample) exactly when all $m_j > 0$ (resp. $m_j \geq 0$). In the latter case, define $Z_{\mathbf{i},\mathbf{m}}$ as the image of the natural map $Z_{\mathbf{i}} \to \mathbb{P}^* H^0(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{i},\mathbf{m}})$. This variety is singular in general, and can be of smaller dimension than $Z_{\mathbf{i}}$. If we take $\mathbf{i}$, $\mathbf{m}$ so that $\mathbf{k}_{\lambda'} \otimes V_{\mathbf{i},\mathbf{m}} \cong V_\lambda(w)$ is a Demazure module (see §1.1), then $Z_{\mathbf{i},\mathbf{m}} \cong \overline{B \cdot wB} \subset G/B$, a Schubert variety.

Theorem 6 *The B-module of regular global sections is isomorphic to the dual of a generalized Demazure module:*

$$H^0(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{i},\mathbf{m}}) \cong V^*_{\mathbf{i},\mathbf{m}}.$$

Also, $Z_{\mathbf{i}}$ is projectively normal with respect to the bundle $\mathcal{L}_{\mathbf{i},\mathbf{m}}$, and $H^i(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{i},\mathbf{m}}) = 0$ for $i \geq 1$. Furthermore, all the above statements hold for $Z_{\mathbf{i},\mathbf{m}}$ in place of $Z_{\mathbf{i}}$.

We give the proof in §4.2.

1.5 The Symplectic group

In [13], we work out the above constructions at length in the case of the general linear group $G = GL_{n+1}$. (Our treatment there is more elementary, avoiding the technicalities of the basis $\{v_\Pi\}$.) In this section, we give the example of the symplectic group $G = Sp_{2n}$, in the spirit of De Concini [2]. (The orthogonal case is similar, but slightly more complicated.)

In general, the main obscurity in the above constructions is that most bases $\{p_\pi\}$ of V^*_λ are difficult to write explicitly, so that writing the corresponding basis $\{p_\Pi\}$ of $V^*_{\mathbf{i},\mathbf{m}}$ is equally difficult. However for G a classical group, it is easier to construct bases of the fundamental representations $V^*_{\varpi_i}$, and to obtain from these a standard monomial basis $\{p_\pi\}$ of the quotient $(V^*_{\varpi_i})^{\otimes m} \to V^*_{m\varpi_i} = V^*_{\lambda_i}$. (See [15], [16].) Thus we will obtain bases of $V^*_{\mathbf{i},\mathbf{m}}$ via the composite restriction map:

$$\rho : (V^*_{\varpi_{i_1}})^{\otimes m_1} \otimes \cdots \otimes (V^*_{\varpi_{i_r}})^{\otimes m_r} \to V^*_{\lambda_1} \otimes \cdots \otimes V^*_{\lambda_r} \to V^*_{\mathbf{i},\mathbf{m}}$$

This is the formulation announced in [12].

Now, for $i \in [1, 2n]$, let us denote $\bar{i} := 2n+1-i$ and $|i| := \min(i, \bar{i})$. The standard basis of $\mathbf{k}^{2n}$ is $\{e_1, \ldots, e_{2n}\} = \{e_1, \ldots, e_n, e_{\bar{n}}, \ldots, e_{\bar{1}}\}$. Let $G = Sp_{2n}(\mathbf{k})$ be the linear isometries of the symplectic form $\langle e_i, e_{\bar{j}} \rangle = -\langle e_{\bar{j}}, e_i \rangle = \delta_{ij}$, $\langle e_i, e_j \rangle = \langle e_{\bar{i}}, e_{\bar{j}} \rangle = 0$ for $i, j \in [1, n]$. That is, $G = \{A \in GL_{2n} \mid AEA^t = E\}$, where E is the matrix with ij-coordinate $\langle e_i, e_j \rangle$ for $i, j \in [1, 2n]$.

We may write the weight lattice as $X = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$; with simple roots $\alpha_1 = \epsilon_1 - \epsilon_2$, $\ldots$, $\alpha_{n-1} = \epsilon_{n-1} - \epsilon_n$, $\alpha_n = 2\epsilon_n$; fundamental weights $\varpi_i = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_i$; and simple coroots $\alpha_1^\vee = \epsilon_1^* - \epsilon_2^*$, $\ldots$, $\alpha_{n-1}^\vee = \epsilon_{n-1}^* - \epsilon_n^*$, $\alpha_n^\vee = \epsilon_n^*$. An element of the Weyl group W may be indexed by a *signed permutation*: a map $w : [1, n] \to [1, 2n] = \{1, \cdots, n, \bar{n}, \cdots, \bar{1}\}$ such that $|w| : i \mapsto |w(i)|$ is a permutation of $[1, n]$. Such a w acts on X by $w(\epsilon_i) := \epsilon_{w(i)}$, where we write $\epsilon_{\bar{i}} := -\epsilon_i$ for $i \in [1, n]$.

We can realize the fundamental representations $V^*_{\varpi_i}$ inside the coordinate ring $\mathbf{k}[G]$ of polynomial functions on the affine variety G. (The group acts on functions via left translation: $(g \cdot f)(A) := f(g^{-1}A)$ for $f \in \mathbf{k}[G]$, $g, A \in G$.) That is, we have

$$V^*_{\varpi_i} \cong \mathrm{Span}\langle\, p_\tau \mid \tau \subset [1, 2n],\ \#\tau = i \,\rangle,$$

where $p_\tau = p_\tau(A)$ is the minor of the matrix A on the first i columns and on the rows τ.

To use the above model, it is most convenient to index the basis of the fundamental-weight modules V_{ϖ_i} not by LS-paths, but by certain *lattice paths*, concatenations of coordinate steps

$$\pi^i := \pi^{\epsilon(i)}, \qquad \pi^{\bar{i}} := \pi^{-\epsilon(i)}$$

for $i \in [1,n]$. A subset $\tau = \{\tau(1) < \cdots < \tau(i)\} \subset [1,2n]$ corresponds to the lattice path $\pi(\tau) := \pi^{\tau(1)} * \cdots * \pi^{\tau(r)}$. We write:

$$\pi[1,i] := \pi([1,i]) = \pi^1 * \pi^2 * \cdots * \pi^i.$$

Any path obtained from $\pi[1,i]$ by repeated application of the lowering operators $f_1, \ldots, f_n$ is of the form $\pi(\tau)$ for some τ, and we say such paths $\pi(\tau)$ (or subsets τ) are *lattice-standard* of shape ϖ_i. A basis for $V^*_{\varpi_i}$ is given by $\{p_\tau \mid \tau \text{ lattice-standard}\}$. This basis has the triangularity property of Theorem 3. For more details, see [2].

Similarly, a path Π is lattice-standard for $V^*_{\mathbf{i},\mathbf{m}}$ if it can be constructed by the usual formula, with the π^{λ_i} replaced by dominant lattice-paths:

$$\Pi = f_{i_1}^{l_1}(\,\pi[1,i_1]^{*m_1} * f_{i_2}^{l_2}(\,\pi[1,i_2]^{*m_2} * \cdots f_{i_r}^{l_r}(\pi[1,i_r]^{*m_r})\cdots)),$$

where $\pi[1,i]^{*m} := \pi[1,i] * \cdots * \pi[1,i]$ (m factors). Every such path is a concatenation of lattice-standard tableaux for fundamental weights:

$$\Pi = \pi(\tau^{(11)}) * \pi(\tau^{(12)}) * \cdots * \pi(\tau^{(1m_1)}) * \pi(\tau^{(21)}) * \cdots * \pi(\tau^{(rm_r)}),$$

where $\tau^{(jm)} \subset [1,2n],\ \#\tau^{(jm)} = i_j$ for $1 \le j \le r,\ 1 \le m \le m_j$. A basis of $V^*_{\mathbf{i},\mathbf{m}}$ is given by:

$$\{\rho(\,p_{\tau^{(11)}} \otimes \cdots \otimes p_{\tau^{(rm_r)}}\,) \ \mid\ (\tau^{(11)}, \cdots, \tau^{(rm_r)}) \text{ is lattice-standard}\},$$

the restriction of standard tensor-monomials via the map ρ. Alternatively, we can realize $V^*_{\mathbf{i},\mathbf{m}}$ inside $\mathbf{k}[G]$, with a basis of monomials in the minors p_τ:

$$V^*_{\mathbf{i},\mathbf{m}} \cong \mathrm{Span}_{\mathbf{k}}\{p_{\tau^{(11)}} \cdots p_{\tau^{(rm_r)}} \ \mid\ (\tau^{(11)}, \cdots, \tau^{(rm_r)}) \text{ is lattice-standard}\}.$$

Example. Let $G = Sp_4$, $\mathbf{i} = (2,1,2)$, $\mathbf{m} = (1,1,1)$. For conciseness, we denote the lattice-path $\pi^a * \pi^b * \pi^c * \cdots$ by the list $abc\cdots$. The lattice-standard tableaux for $V^*_{\varpi_1}$ are $\{1, 2, \bar{2}, \bar{1}\}$, meaning $\{\pi^1, \pi^2, \pi^{\bar{2}}, \pi^{\bar{1}}\}$; with crystal graph: $1 \xrightarrow{f_1} 2 \xrightarrow{f_2} \bar{2} \xrightarrow{f_1} \bar{1}$. For $V^*_{\varpi_2}$, the lattice-standard tableaux are: $\{12, 1\bar{2}, 2\bar{2}, 2\bar{1}, \bar{2}\bar{1}\}$, meaning $\{\pi^1 * \pi^2,\ \pi^1 * \pi^{\bar{2}}, \cdots\}$; with crystal graph: $12 \xrightarrow{f_2} 1\bar{2} \xrightarrow{f_1} 2\bar{2} \xrightarrow{f_1} 2\bar{1} \xrightarrow{f_2} \bar{2}\bar{1}$.

For $V^*_{\mathbf{i},\mathbf{m}}$, we can construct the 17 lattice-standard tableaux of the form $\Pi = f_2^\bullet(12 * f_1^\bullet(1 * f_2^\bullet 12))$ in steps, starting from the right end of the expression for Π:

$$\{12\} \xrightarrow{f_2^\bullet} \{12, 1\bar{2}\} \xrightarrow{1*} \{121, 11\bar{2}\} \xrightarrow{f_1^\bullet} \{112, 11\bar{2}, 212, 21\bar{2}, 22\bar{2}, 22\bar{1}\}$$

$$\xrightarrow{12*} \{12112, 1211\bar{2}, 12212, 1221\bar{2}, 1222\bar{2}, 1222\bar{1}\}$$

$$\stackrel{f_2^\bullet}{\longrightarrow} \{\, 12112, 1211\bar{2}, 12212, 1221\bar{2}, 1222\bar{2}, 1222\bar{1},$$
$$1\bar{2}112, 1211\bar{2}, 1\bar{2}212, 1\bar{2}\bar{2}12, 1\bar{2}\bar{2}1\bar{2}, 1\bar{2}21\bar{2},$$
$$1\bar{2}222\bar{2}, 1\bar{2}\bar{2}22\bar{2}, 1\bar{2}22\bar{1}, 1\bar{2}\bar{2}2\bar{1}, 1\bar{2}\bar{2}\bar{2}\bar{1}\,\} \ .$$

A list like $1\bar{2}\bar{2}2\bar{1}$ represents the lattice-path $\Pi = \pi^1 * \pi^{\bar{2}} * \pi^{\bar{2}} * \pi^2 * \pi^{\bar{1}}$, which can be divided into lattice-standard paths for fundamental weights as: $\Pi = \pi(1\bar{2}) * \pi(\bar{2}) * \pi(2\bar{1})$. To illustrate the action of the lowering operator f_1, we write under each tableau the value of $\langle \pi^j, \alpha_1^\vee \rangle = -1, 0$, or $+1$ for each coordinate step π^j, and we emphasize the step which is flipped by f_1 (that is, the step where the path rises for the last time from the minimum value of $\langle \pi(t), \alpha_1^\vee \rangle$):

$$\begin{array}{c} 1\,2\,\mathbf{1}\,1\,\bar{2} \\ +-++\,+ \end{array} \stackrel{f_1}{\longrightarrow} \begin{array}{c} 1\,2\,2\,\mathbf{1}\,\bar{2} \\ +--++ \end{array} \stackrel{f_1}{\longrightarrow} \begin{array}{c} 1\,2\,2\,2\,\mathbf{\bar{2}} \\ +---+ \end{array} \stackrel{f_1}{\longrightarrow} \begin{array}{c} 1\,2\,2\,2\,\bar{1} \\ +----\end{array} \stackrel{f_1}{\longrightarrow} \text{undefined}$$

The 17 standard tableaux index a basis for $V^*_{\mathbf{i},\mathbf{m}}$, which we can realize inside $\mathbf{k}[G]$, the coordinate ring of $G \subset GL_4$. That is, $\mathbf{k}[G]$ is the polynomial ring $\mathbf{k}[x_{ij}]_{i,j\in[1,4]}$, where $X = (x_{ij})$ a generic matrix, modulo the ideal generated by the supra-diagonal coordinates of $XEX^t - E$: that is, modulo the polynomials

$$-x_{41}x_{12} - x_{31}x_{22} + x_{21}x_{32} + x_{11}x_{42}, \quad -x_{41}x_{13} - x_{31}x_{23} + x_{21}x_{33} + x_{11}x_{43},$$
$$-x_{41}x_{14} - x_{31}x_{24} + x_{21}x_{34} + x_{11}x_{44} - 1, \quad -x_{42}x_{13} - x_{32}x_{23} + x_{22}x_{33} + x_{12}x_{43} - 1,$$
$$-x_{42}x_{14} - x_{32}x_{24} + x_{22}x_{34} + x_{12}x_{44}, \quad -x_{43}x_{14} - x_{33}x_{24} + x_{23}x_{34} + x_{13}x_{44}.$$

(This ideal has a small square-free Gröbner basis in the degree-lex order, so it is reduced.) For example, the tableau $1\bar{2}\bar{2}2\bar{1}$ corresponds to the following polynomial in $\mathbf{k}[G]$:

$$p_{1\bar{2}\bar{2}2\bar{1}} = p_{1\bar{2}} p_{\bar{2}} p_{2\bar{1}} = (x_{11}x_{32} - x_{31}x_{12}) \cdot x_{31} \cdot (x_{21}x_{42} - x_{41}x_{22}).$$

(Recall $\bar{2} = 3$, $\bar{1} = 4$.)

2 Combinatorics of tableaux

In this section, we regard the word $\mathbf{i} = (i_1, \ldots, i_r)$ as given, and for $J \subset [1, r]$, we abbreviate $w(\mathbf{i}(J))$ as $w(J)$. For example, $sw(J)$ means $s \cdot w(\mathbf{i}(J))$.

2.1 The main lemma

Lemma 7 *Consider a long list of Weyl group cosets*

$$(\tau_{11}, \ldots, \tau_{rp_r}) = (\tau_1, \ldots, \tau_N)$$

possessing a lifting

$$(J_{11} \supset \cdots \supset J_{rp_r}) = (J_1 \supset \cdots \supset J_N),$$

meaning that each subword $\mathbf{i}(J^{(j)}_{jp})$ *is reduced and* $\tau_{jp} \equiv w(J^{(j)}_{jp}) \bmod W_{\lambda_j}$. *Suppose that for some consecutive elements* $\tau_{jp} = \tau_{K-1}$, $\tau_{j'p'} = \tau_K$, $\tau_{j''p''} = \tau_{K+1}$, *and for some simple reflection* s *we have either:*

$$\begin{array}{lllll} (a) & sw(J^{(j)}_{K-1}) < w(J^{(j)}_{K-1}) & \text{and} & s\tau_K \geq \tau_K; & \text{or} \\ (b) & sw(J^{(j'')}_{K+1}) > w(J^{(j'')}_{K+1}) & \text{and} & s\tau_K \leq \tau_K. & \end{array}$$

Then the long list of cosets

$$(\tau_1, \ldots, \tau_{K-1}, s\tau_K, \tau_{K+1}, \ldots, \tau_N)$$

also possesses a lifting $(\tilde{J}_1 \supset \ldots \supset \tilde{J}_N)$, *with (respectively):*

$$\begin{array}{ll} (a') & sw(\tilde{J}^{(j')}_K) < w(\tilde{J}^{(j')}_K); \quad \text{or} \\ (b') & sw(\tilde{J}^{(j')}_K) > w(\tilde{J}^{(j')}_K). \end{array}$$

Proof. Cf. [19, Pf of Thm 10.1]. We will prove the lemma under assumption (a). For (b), replace $<$ by $>$ and $K-1$ by $K+1$. First suppose $sw(J^{(j')}_K) < w(J^{(j')}_K)$. Then we must have $s\tau_K = \tau_K$, and we may take $\tilde{J}_k = J_k$ for all k.

On the other hand, suppose $sw(J^{(j')}_K) > w(J^{(j')}_K)$. We have

$$sw(J^{(j)}_{K-1}) < w(J^{(j)}_{K-1}) \geq w(J^{(j)}_K),$$

and by the Zigzag Lemma [5, Prop 5.9],

$$sw(J^{(j)}_K) \leq w(J^{(j)}_{K-1}).$$

Thus by the subword definition of Bruhat order, there is a reduced $J'_K \subset J^{(j)}_{K-1}$ with

$$w(J'_K) = sw(J^{(j)}_K).$$

Furthermore for all $L > K$ there are reduced subwords $J'_L \subset J^{(j)}_{K-1}$ with

$$w(J'_L) = w(J^{(j)}_L),$$

and we may take the sets J'_L to be decreasing as L increases. Now define $\widetilde{J}_L = J_L$ for $L < K$ and $\widetilde{J}_L = (J_L \setminus [1, j]) \cup J'_L$ for $L \geq K$.

Now

$$w(\widetilde{J}^{(j')}_K) = w(J'_K)w(J_K \cap [j+1, j']) = s \cdot w(J^{(j)}_K) \cdot w(J_K \cap [j+1, j']),$$

and by our supposition the latter product is reduced (length of product = sum of lengths). Hence $\widetilde{J}^{(j')}_K$ is a reduced word, and $w(\widetilde{J}^{(j')}_K) = sw(J^{(j')}_K) \equiv s\tau_K$. Similarly the appropriate initial segment of $\widetilde{J}_L$ for any other L is a reduced lifting of τ_L. Therefore $(\widetilde{J}_1 \supset \cdots \supset \widetilde{J}_N)$ is a lifting of $(\tau_1, \ldots, s\tau_K, \ldots, \tau_N)$ as required. Property (a') follows from the above along with our supposition. •

2.2 Root operators

Lemma 8 *Let $i \in [1,n]$.*
(a) If Π is a liftable tableau, and $e_i(\Pi)$ exists, then $e_i(\Pi)$ is liftable.
(b) If Π is a liftable tableau, and $e_i(\Pi)$, $f_i(\Pi)$ both exist, then $f_i(\Pi)$ is liftable.
(c) If Π is a liftable tableau with respect to $\mathbf{i}$, and $f_{i_1}(\Pi)$ exists, where i_1 is the first *letter of $\mathbf{i}$, then $f_{i_1}(\Pi)$ is liftable.*

Proof. It is clear by [18] that $e_i(\Pi)$, $f_i(\Pi)$ are always tableaux if they are defined, so in each case we need only show liftability.
(a) Suppose

$$\overline{\Pi} = (\tau_{11}, \dots, \tau_{rp_r}) = (\tau_1, \dots, \tau_N)$$

with lifting

$$(J_{11} \supset \cdots \supset J_{rp_r}) = (J_1 \supset \cdots \supset J_N).$$

It is easily seen (cf. [17, Prop 4.2]) that

$$\overline{e_i(\Pi)} = \begin{cases} (\tau_1, \dots, \tau_{K-1}, s\tau_K, \dots, s\tau_L, \tau_{L+1}, \dots \tau_N) & \text{or} \\ (\tau_1, \dots, \tau_{K-1}, \tau_K, s\tau_K, \dots, s\tau_L, \tau_{L+1}, \dots \tau_N) \end{cases}$$

for some indices $1 \le K \le L \le N$ with

$$s\tau_K \le \tau_K, \dots, s\tau_L \le \tau_L \qquad \text{and} \qquad s\tau_{L+1} > \tau_{L+1}.$$

We must show that this list of cosets has a lifting.

First, suppose $L \ne N$, so that $\tau_{L+1} = \tau_{j''p''}$ exists, and let $\tau_L = \tau_{j'p'}$. Then $sw(J_{L+1}^{j''}) > w(J_{L+1}^{j''})$ and $s\tau_L \le \tau_L$, so we can apply Lemma 7(b) to the lifting $(J_1 \supset \cdots)$ at positions L, $L+1$. Using condition (b′) we can repeat this at positions $L-1$, L, and so on leftward, thus producing a lifting of $e_i(\Pi)$.

On the other hand, suppose $L = N$. If $sw(J_L^{(j')}) > w(J_L^{(j')})$, then $\tau_N = s\tau_N$, and we may again repeatedly apply the Lemma starting at positions $N-1$, N. If $sw(J_N^{(j')}) < w(J_N^{(j')})$, we cannot directly apply the Lemma, but instead take $\tilde{J}_N \subset J_N$ so that $w(\tilde{J}_N^{(j')}) = sw(J_N^{(j')})$. Then $(J_1 \supset \cdots \supset J_{N-1} \supset \tilde{J}_N)$ is a lifting of $(\tau_1, \dots, \tau_{N-1}, s\tau_N)$, to which we can apply the Lemma starting at positions $N-1$, N. In each case, we produce a lifting of $e_i(\Pi)$.

(b) We have

$$\overline{f_i(\Pi)} = \begin{cases} (\tau_1, \dots, \tau_{K-1}, s\tau_K, \dots, s\tau_L, \tau_{L+1}, \dots \tau_N) & \text{or} \\ (\tau_1, \dots, \tau_{K-1}, s\tau_K, \dots, s\tau_L, \tau_L, \tau_{L+1}, \dots \tau_N) \end{cases}$$

for $1 \le K \le L \le N$ with

$$s\tau_{K-1} < \tau_{K-1} \qquad \text{and} \qquad s\tau_K \ge \tau_K, \dots, s\tau_L \ge \tau_L.$$

Since $e_i(\Pi)$ also exists, we must have $K > 1$, so that we may repeatedly apply Lemma 7(a) analogously to the previous argument, starting at positions $K-1$,

K and proceeding rightward to produce a lifting of $f_i(\Pi)$.

(c) Let $J_K = J_{j'p'}$. As before, if $K > 1$ or $sw(J_1^{(j')}) < w(J_1^{(j')})$, we use Lemma 7(a) immediately. Otherwise if $K = 1$ and $sw(J_1^{(j')}) > w(J_1^{(j')})$, take $\tilde{J}_1 = J_1 \cup \{1\}$ (reduced): this gives a lifting $(\tilde{J}_1 \supset J_2 \supset \cdots \supset J_N)$ of $(s\tau_1, \tau_2, \ldots, \tau_N)$, to which we apply Lemma 7(a). •

Proof of Theorem 1. We use induction on r, the number of letters of the word $\mathbf{i} = (i_1, \ldots, i_r)$. For $r = 0$ and $\mathbf{i}$ the empty word, the only constructable or liftable tableau is the trivial path π^0. Now assume the Theorem for the word $(i_2, \ldots, i_r)$.

Constructable $\Rightarrow$ liftable. Suppose $\Pi = f_{i_1}^{l_1}(\pi^{\lambda_1} * f_{i_2}^{l_2}(\pi^{\lambda_2} \cdots))$ is constructable. By induction $\Pi' = f_{i_2}^{l_2}(\pi^{\lambda_2} \cdots)$ is liftable. Then $\pi^{\lambda_1} * \Pi'$ is clearly also liftable, and so is $\Pi = f_{i_1}^{l_1}(\pi^{\lambda_1} * \Pi')$ by Lemma 8(c).

Liftable $\Rightarrow$ constructable. Suppose $\Pi = (\pi_1, \ldots, \pi_r)$ has lifting $(J_{11} \supset \cdots)$. Since $w(J_{1p}^{(1)}) = s_{i_1}$ or id, the cosets $\bmod W_{\lambda_1}$ in the LS-chain π_1 must be $\overline{\pi_1} = (s_{i_1}, \ldots, s_{i_1}, \mathrm{id}, \ldots, \mathrm{id})$. Thus we may write $e_{i_1}^{\mathrm{top}}(\Pi) = e_{i_1}^{l_1}(\Pi) = (\pi^{\lambda_1}, \pi_2', \ldots, \pi_r')$ for some l_1, since if the initial segment were not π^{λ_1}, we could apply e_{i_1} once more (§1.2). But $\Pi' = (\pi_2', \ldots, \pi_r')$ is liftable by Lemma 8(a), and is therefore constructable by induction. Hence $\Pi = f_{i_1}^{l_1}(\pi^{\lambda_1} * \Pi')$ is also constructable. •

2.3 Demazure operators

We show that the number of liftable tableaux is given by the Demazure character formula. This is the combinatorial version of Theorem 5. For a set $\mathcal{T}$ of tableaux, define the formal character (or multi-variate generating function)

$$\mathrm{char}(\mathcal{T}) := \sum_{\Pi \in \mathcal{T}} e^{\mathrm{wt}(\Pi)}.$$

For $i \in [1, n]$, a tableau Π_0 with $e_i(\Pi_0)$ undefined is called an *i-head*. An *i-string* is the set S of all tableaux generated under f_i by some i-head Π_0: $S = \{\Pi_0, f_i(\Pi_0), f_i^2(\Pi_0), \ldots, f_i^l(\Pi_0)\}$, where l is maximal with $f_i^l(\Pi_0)$ defined. In fact $l = \langle \mathrm{wt}(\Pi_0), \alpha_i^\vee \rangle \geq 0$. (See [18].) Thus any i-string S with head Π_0 has character

$$\mathrm{char}(S) = e^{\mathrm{wt}(\Pi_0)} + e^{\mathrm{wt}(\Pi_0) - \alpha_i} + \ldots + e^{s_i\,\mathrm{wt}(\Pi_0)}.$$

Suppose a set of tableau $\mathcal{T}$ is a disjoint union of i-heads and i-strings. Let

$$f_i^\bullet \mathcal{T} := \{f_i^l(\Pi) \mid \Pi \in \mathcal{T},\ l \geq 0\}.$$

Then it is clear that

$$\mathrm{char}(f_i^\bullet \mathcal{T}) = \Lambda_i(\mathrm{char}\, \mathcal{T}).$$

Lemma 9 *Let $\mathcal{T}$ be the set of standard tableau of shape $(\lambda_2, \ldots, \lambda_r)$. Then the set of concatenations $\pi^{\lambda_1} * \mathcal{T}$ is a disjoint union of i_1-heads and i_1-strings.*

Proof. Let $\pi_1 = \pi^{\lambda_1}$, $e = e_{i_1}$, $f = f_{i_1}$. First we show: if $\pi_1 * \Pi \in \pi * \mathcal{T}$ with $e(\pi_1 * \Pi)$ defined, then $e(\pi_1 * \Pi)$, $f(\pi_1 * \Pi) \in \pi_1 * \mathcal{T}$. By [18, §2.6] we have $e(\pi_1 * \Pi) = \pi_1 * (e\Pi)$ or $(e\pi_1) * \Pi$; but $e\pi_1$ is undefined, so the first alternative must hold. Also $e\Pi \in \mathcal{T}$ by Lemma 8(a), so $e(\pi_1 * \Pi) \in \pi_1 * \mathcal{T}$.

Next we show: if $\pi_1 * \Pi \in \pi_1 * \mathcal{T}$ with $e(\pi_1 * \Pi)$ and $f(\pi_1 * \Pi)$ are both defined, then $f(\pi_1 * \Pi) \in \pi_1 * \mathcal{T}$. First, recall that the operator e reflects a part of $\pi_1 * \Pi$ before the first minimum point of the function $t \mapsto \langle(\pi_1 * \Pi)(t), \alpha^\vee\rangle$, and the operator f reflects a part of $\pi_1 * \Pi$ after the last minimum point of the function. Thus f acts on the path at a *later* point than e acts.

Now consider $f(\pi_1 * \Pi) = (f\pi_1) * \Pi$ or $\pi_1 * (f\Pi)$. We know that $e(\pi_1 * \Pi) = \pi_1 * (e\Pi)$, so that e acts after the first segment of $\pi_1 * \Pi$; and f acts later than e. Thus we must have the second alternative: $f(\pi_1 * \Pi) = \pi_1 * (f\Pi)$. But $f\Pi \in \mathcal{T}$ by Lemma 8(b), so $f(\pi_1 * \Pi) \in \pi_1 * \mathcal{T}$. •

Corollary 10 *The formal character of the set of standard tableaux of shape* $(\lambda_1, \dots, \lambda_r)$ *is*

$$\Lambda_{i_1}(e^{\lambda_1}\Lambda_{i_2}(e^{\lambda_2}\cdots\Lambda_{i_r}(e^{\lambda_r})\cdots)).$$

Proof. Let $\mathcal{T}$ be the set of standard tableaux for the $(i_2, \dots, i_r)$. The set of standard tableaux for $(i_1, \dots, i_r)$ is $\mathcal{T}' = f_{i_1}^\bullet(\pi^{\lambda_1} * \mathcal{T})$, so that $\operatorname{char}\mathcal{T}' = \Lambda_{i_1}(e^{\lambda_1}\operatorname{char}\mathcal{T})$. The result now follows by induction. •

2.4 Combinatorial excellent filtration

The ordinary Demazure modules $V_\lambda(w)$ and their duals play a central role in the theory of B-modules. For example, consider the twisted dual Demazure module $\mathbf{k}_\mu^* \otimes V_\nu^*(y)$ for dominant weights $\mu, \nu \in X^+$. It was conjectured by A. Joseph and proved by O. Mathieu [25] (in the general case, and by P. Polo in some special cases) that this module has an *excellent filtration*, namely a filtration by B-modules whose quotients are isomorphic to $V_\lambda^*(w)$ for various $\lambda \in X^+$, $w \in W$. This implies that $\mathbf{k}_\mu^* \otimes V_\nu^*(y)$ inherits the favorable homological properties of the $V_\lambda^*(w)$. More generally, Mathieu proved that twists of our generalized Demazure modules $\mathbf{k}_\mu^* \otimes V_{\mathbf{i},\mathbf{m}}^*$ have excellent filtration. For a survey, see [27].

In this section, we prove Theorem 2, which is a combinatorial analogue of Mathieu's result. We first formulate and prove a more precise result for the special case of a twisted ordinary Demazure module $\mathbf{k}_\mu^* \otimes V_\nu^*(y)$. (Note that we do not distinguish between tableaux for this module and for its dual.)

Let $C(\lambda, w)$ denote the crystal graph on the set $\{f_{i_1}^\bullet \cdots f_{i_r}^\bullet \pi^\lambda\}$ of standard tableaux for $V_\lambda(w)$, where $w = s_{i_1} \cdots s_{i_r}$ is any reduced decomposition. This set of tableaux is known to be independent of the choice of reduced decomposition. In fact, it is precisely the set of LS-paths $\pi = (\tau_1 > \cdots ; 0 < a_1 \cdots)$ such that $w \geq \tau_1 \mod W_\lambda$. See [7], [17].

Further, let $C(\mu, \nu, y)$ be the crystal graph on the set $\{\pi^\mu * \pi\}$, where π runs over standard tableaux for $V_\nu(y)$. Recall that each edge in a crystal graph is assigned a color $i = 1, \dots, n$. We will show that for dominant μ, ν, the $C(\mu, \nu, y)$ is isomorphic as an edge-colored graph to a disjoint union of various

$C(\lambda, w)$. There is one such component $C(\lambda, w)$ for each path $\pi^\mu * \pi$ which is *dominant*, i.e., which stays completely within the dominant Weyl chamber: that is, $\langle \mu + \pi(t), \alpha_i^\vee \rangle \geq 0$ for all $t \in [0,1]$ and all i. The λ corresponding to a dominant path is its weight: $\lambda = \mathrm{wt}(\pi^\mu * \pi) = \mu + \mathrm{wt}(\pi)$. To compute the corresponding w requires some definitions.

The second part of the following lemma is due to Deodhar [14, Lemma 11.1].

Lemma 11 *(i) For $u, w \in W$, the set $\{u'w \mid u' \leq u\}$ has unique maximal and minimal elements.*
(ii) Let $W' \subset W$ be the parabolic subgroup generated by some subset of the simple roots. For $z \leq y \in W$, the set $\{u'z \mid u' \in W',\ u'z \leq y\}$ has a unique maximal element.

Proof. For a set $A \subset W$, we write $w = \max A$ if $w \in A$ and $w \geq y$ for all $y \in A$ (that is, w is the unique Bruhat-maximal element of A). Similarly for $\min A$. We will repeatedly use the Zigzag Lemma [5, §5.11]:

$$\text{If } x \leq y, \text{ then } sx \leq \max(y, sy) \ \text{ and } \ \min(x, sx) \leq sy,$$

where s is any simple reflection, and similarly for xs and ys.

(i) Induction on $\ell(u)$. The case $\ell(u) = 0$ is trivial. Take $su > u$, and assume $u'w \leq u_1 w$ for all $u' \leq u$. We claim $u''w \leq \max(u_1 w, su_1 w)$ for all $u'' \leq su$, so that $\max(u_1 w, su_1 w)$ is the unique maximum of $\{u''w \mid u'' \leq su\}$.

From the subword definition of Bruhat order, any $u'' \leq su$ has either: $u'' \leq u$, so that $u''w \leq u_1 w$; or $su'' \leq u$, so that $su''w \leq u_1 w$, and $u''w = s(su''w) \leq \max(u_1 w, su_1 w)$ by the Zigzag Lemma. This proves the claim.

The proof that $\{u'w \mid u' \leq u\}$ has a unique minimum is almost the same: Again take $su > u$. If $u_0 w \leq u'w$ for all $u' \leq u$, then $\min(u_0 w, su_0 w) \leq u''w$ for $u'' \leq su$.

(ii) We follow Deodhar [14, Lemma 11.1], correcting several misprints. We denote $(W'z)_{\leq y} := \{u'z \mid u' \in W',\ u'z \leq y\}$. Also let $W^{\min} \subset W$ be the set of minimal coset representatives of $W'\backslash W$, and write $y = wy_0$ with $w \in W'$ and $y_0 \in W^{\min}$. Suppose, without loss of generality, that $z = z_0 \in W^{\min}$ (since $(W'uz)_{\leq y} = (W'z)_{\leq y}$ for any $u \in W'$). Now we proceed by induction on $\ell(y_0)$. For $\ell(y_0) = 0$, we have $y_0 = z_0 = e$, and $\max(W'z)_{\leq y_0 w} = w$. Next suppose $\ell(y_0) > 0$, and choose a simple root s with $y_0 < y_0 s$. Note that $y_0 s \in W^{\min}$, since otherwise there is a simple root $s' \in W'$ with $\ell(s' y_0 s) = \ell(y_0) - 2 = \ell(s' y_0) - 3$. Note also that $wy_0 s < wy_0$, since: $\ell(wy_0 s) = \ell(w) + \ell(y_0 s) = \ell(w) + \ell(y_0) - 1 = \ell(wy_0) - 1$.

Case (a): $z_0 s < z_0$. The facts noted above for y_0 also hold for z_0. We have $z_0 s = \min(z_0 s, z_0) \leq y_0 s$, and by induction we may let $w' z_0 s = \max(W' z_0 s)_{\leq w y_0 s}$. Then we claim $w' z_0 = \max(W' z_0)_{\leq w y_0}$. First, $w' z_0 s \leq w y_0 s$, so $w' z_0 \leq \max(w y_0 s, w y_0) = w y_0$. Now suppose $u z_0 \leq w y_0$ for $u \in W'$. Then $u z_0 s = \min(u z_0 s, u z_0) \leq w y_0 s$ and by definition of w', we have $u z_0 s \leq w' z_0 s$. Thus $u \leq w'$ and $u z_0 \leq w' z_0$.

Case (b): $z_0 s > z_0$ and $z_0 s \in W^{\min}$. We have $z_0 = \min(z_0, z_0 s) \leq y_0 s$, so by induction we may let $w' z_0 = \max(W' z_0)_{\leq w y_0 s}$. We claim $w' z_0 =$

$\max(W'z_0)_{\leq wy_0}$. First, $w'z_0 \leq wy_0$ just as before. Also note that, as before, for any $u \in W'$ we have $uz_0 s > uz_0$. Now suppose $uz_0 \leq wy_0$ for $u \in W'$. Then $uz_0 = \min(uz_0 s, uz_0) \leq wy_0 s$ and by definition of w', we have $uz_0 \leq w'z_0$.

Case (c): $z_0 s > z_0$ and $z_0 s \notin W^{\min}$. First note: there exists a simple root $s' \in W'$ with $s'z_0 s < z_0 s$, so $\ell(s'z_0 s) = \ell(z_0)$; but $sz_0 \geq z_0$, so $sz_0 s' \geq \min(z_0, z_0 s') = z_0$. Hence $sz_0 s' = z_0$. Now, as in case (b), we may let $w' = \max(W'z_0)_{\leq wy_0 s}$. Define $w'' := \max(w', w's')$. Then we claim $w'' = \max(W'z_0)_{\leq wy_0}$. First, $w''z_0 = \max(w'z_0, w'z_0 s) \leq \max(wy_0, wy_0 s) = wy_0$. Now suppose $uz_0 \leq wy_0$ for $u \in W'$. If $us' < u$, then $uz_0 > us'z_0 = uz_0 s$ and $us'z_0 = \min(uz_0, uz_0 s) \leq wy_0 s$, so by the definition of w', we have $us'z_0 \leq w'z_0$. Hence $u \leq w''$, and $uz_0 \leq w''z_0$. On the other hand, if $us' > u$, then $uz_0 = \min(uz_0, uz_0 s) \leq wy_0 s$, so by the definition of w', we have again $uz_0 \leq w'z_0 \leq w''z_0$. •

Now, given μ, ν, y as above, and a path $\pi : [0,1] \to X_{\mathbb{R}}$, let $W(t) := \mathrm{Stab}_W(\mu + \pi(t))$ denote the stabilizer of the point $\mu + \pi(t) \in X_{\mathbb{R}}$. This is the parabolic subgoup of W generated by the simple reflections s_i such that $\mu + \pi(t)$ lies on the corresponding wall of the dominant Weyl chamber. Let $[0,1] = I_1 \sqcup \cdots \sqcup I_q$ be the decomposition of $[0,1]$ into the minimal number of disjoint intervals such that $W(t)$ is constant for all t in each interval I_j. We enumerate the intervals so that for $j < j'$, the t in I_j are smaller than those in $I_{j'}$.

For an LS-path $\pi = (\tau_1 > \cdots ; 0 < a_1 < \cdots)$ with $\pi^\mu * \pi$ dominant, we let $\lambda = \mu + \mathrm{wt}(\pi)$, and we define $w(\pi)$ (modulo W_λ) inductively as follows. Intuitively, we start with $w = \mathrm{id}$; then we travel along the path $\pi^\mu * \pi$ from its endpoint λ to μ, and every time we hit a wall, we multiply w by the corresponding reflection if this makes w longer. However, at the end of our trip, if μ itself is on a wall, we only multiply by the corresponding reflection s if s times the initial direction of π is smaller than y. Formally, we define:

$$w_q := \max(W(I_q)),\ w_{q-1} := \max(W(I_{q-1}) \cdot w_q),\ \ldots,\ w_2 := \max(W(I_2) \cdot w_3),$$

$$w_1 := \max\{uw_2 \mid u \leq u_1\};\quad w(\pi) := w_1\ .$$

Here $u_1 := \max\{u \in W(I_1) \mid u\tau_1 \leq y \mod W_\nu\}$. We explain why u_1 is well-defined. The path $\pi^\mu * \pi$ has a segment in direction $\tau_1(\nu)$ from μ into the fundamental chamber. Thus if $s_i \in W(I_1) = W_\mu$, we have $\langle \tau(\nu), \alpha_i^\vee \rangle \geq 0$, and $s_i \tau_1 \geq \tau_1 \mod W_\nu$. That is, we may take τ_1 to be minimal in the coset $W(I_1)\tau_1$, so that we may use the Lemma to define $u_1\tau_1 = \max\{u'\tau_1 \mid u' \in W(I_1),\ u'\tau_1 \leq y\}$. (Note: In the Kac-Moody setting, in which W may be infinite, we can show that $W(I_j)$ is always finite, and the above definition of $w(\pi)$ is still valid.)

The following result is a refinement of the Littlewood-Richardson rule of [17].

Proposition 12 *The crystal graph of $\mathbf{k}_\mu \otimes V_\nu(y)$ has as its connected components the crystal graphs of ordinary Demazure modules $V_\lambda(w)$. Specifically, it is the disjoint union:*

$$C(\mu, \nu, y) \ = \ \coprod_\pi C(\mu + \mathrm{wt}(\pi), w(\pi)),$$

running over all standard tableaux π for $V_\nu(y)$ such that $\pi^\mu * \pi$ is dominant.

Proof. (a) First the containment $\supset$. Fix a path $\pi = (\tau_1 > \cdots > \tau_q; 0 < a_1 < \cdots < a_q = 1)$ with with $y \geq \tau_1 \mod W_\lambda$, and $\pi^\mu * \pi$ dominant. We must show that for some (and hence for any) reduced word $w(\pi) = s_{i_1} \cdots s_{i_r}$, the path $f_{i_1}^{l_1} \cdots f_{i_r}^{l_r}(\pi^\mu * \pi)$, if defined, is of the form $\pi^\mu * \theta$ with $\theta = (\sigma_1 > \cdots ; 0 < b_1 < \cdots)$ and $y \geq \sigma_1 \mod W_\lambda$. We may choose our reduced word compatible with the partial products w_k in the definition of $w(\pi)$. That is, for any p, we have $w_k \geq s_{i_p} \cdots s_{i_r} \geq w_{k+1}$ for some k.

Claim: $f_{i_p}^{l_p} \cdots f_{i_r}^{l_r}(\pi^\mu * \pi) = \pi^\mu * \theta_p$ for some θ_p with $\theta_p(t) = \pi(t)$ for $t \in I_1 \sqcup \cdots \sqcup I_k$. This follows by descending induction on p. Indeed, assume the claim for a given p (with $p > 1$), and suppose $w_k > s_{i_p} \cdots s_{i_r} \geq w_{k+1}$. Then on $t \in I_1 \sqcup \cdots \sqcup I_k$, the function $t \mapsto \langle \mu + \theta_p(t), \alpha_{i_{p-1}}^\vee \rangle = \langle \mu + \pi(t), \alpha_{i_{p-1}}^\vee \rangle$ is non-negative, and it attains its minimum value 0 at the right endpoint of I_k. Hence by definition, the operator $f_{i_{p-1}}^{l_{p-1}}$ does not change the path $\pi^\mu * \theta_p$ within the interval $t \in I_1 \sqcup \cdots \sqcup I_k$, and the claim holds for $p - 1$.

Finally, $y \geq \sigma$ follows immediately from the claim and the definition of u_1.

(b) Now the opposite containment $\subset$. Consider any path $\pi^\mu * \theta$ (not necessarily dominant), and consider the unique dominant path $\pi^\mu * \pi$ such that $f_{i_1}^{l_1} \cdots f_{i_r}^{l_r}(\pi^\mu * \pi) = \pi^\mu * \theta$ for some reduced word $s_{i_1} \cdots s_{i_r} =: w$ and $l_j > 0$. We must show that $w(\pi) \geq w$.

If $\pi^\mu * \theta$ is itself dominant, there is nothing to prove. Otherwise, let I_{k+1}, $k \geq 1$, be the interval where $\mu + \theta(t)$ first exits the fundamental chamber. We will use decreasing induction on k to show the stronger statement $w_k \geq s_{i_1} \cdots s_{i_r}$.

By the definition of the lowering operators, we have $\mu + \theta(t) = \mu + \pi(t)$ for $t \leq$ the first point of exit of $\pi^\mu * \theta$ from the fundamental chamber. Thus, for $t \in I_k$, $\mu + \theta(t)$ lies on all the walls which are crossed in interval I_{k+1}. Thus there exists a product $e_{j_a}^{m_a} \cdots e_{j_1}^{m_1}$ of e_j with $s_j \in W(I_k)$, such that $\pi^\mu * \theta' := e_{j_a}^{m_a} \cdots e_{j_1}^{m_1}(\pi^\mu * \theta)$ lies inside the dominant chamber for $t \in I_1 \sqcup \cdots \sqcup I_{k+1}$. By induction, $\pi^\mu * \theta' = f_{i_p}^{l_p} \cdots f_{i_r}^{l_r}(\pi^\mu * \pi)$, where $w_{k+1} \geq s_{i_p} \cdots s_{i_r}$. We thus have $\pi^\mu * \theta = f_{j_1}^{m_1} \cdots f_{j_a}^{m_a} f_{i_p}^{l_p} \cdots f_{i_r}^{l_r}(\pi^\mu * \pi)$, where $w_k \geq s_{j_1} \cdots s_{j_a} s_{i_p} \cdots s_{i_r}$, provided only that $k > 1$. If $k = 1$, we may assume that $u_1 \geq s_{j_1} \cdots s_{j_a}$, and the conclusion again follows. •

Examples. We write $(\tau_1 a_1 \tau_2 a_2 \cdots)$ for an LS-chain $(\tau_1 > \cdots ; 0 < a_1 < \cdots)$ of V_ν, so a chain $(\tau_1; 0 < 1)$ with extremal weight $\tau_1(\nu)$ is written simply (τ_1).

(i) $G = SL_3$, $\mu = 2\varpi_2$, $\nu = \varpi_1 + \varpi_2$, $y = s_1 s_2$. There are four components of $C(\mu, \nu, y)$: $\pi = (s_1 s_2)$, $C(\varpi_2, \mathrm{id})$; $\pi = (s_1 s_2 \, \frac{1}{2} \, s_2)$, $C(2\varpi_1, \mathrm{id})$; $\pi = (s_1)$, $C(\varpi_1 + 2\varpi_2, \mathrm{id})$; $\pi = (\mathrm{id})$, $C(\varpi_1 + 3\varpi_2, s_2)$. Note that for $\pi = (s_1)$ we have $W(I_1) = \langle s_2 \rangle$, but $s_2 \tau_1 \not\leq y$, so $w(\pi) = \mathrm{id}$.

(ii) $G = SL_3$, $\mu = 2\varpi_1$, $\nu = 2\varpi_1 + 3\varpi_2$, $y = s_1 s_2$. The path $\pi = (s_1 s_2 \, \frac{2}{5} \, s_1 \, \frac{2}{3} \, e)$ corresponds to $C(2\varpi_1 + \varpi_2, s_1 s_2)$. Note that $\theta = f_2 f_1 f_2(\pi^\mu * \pi) \in C(\mu, \nu, y)$, but this is no contradiction, since $\theta = f_1 f_2^2(\pi^\mu * \pi)$.

Proof of Theorem 2. This follows immediately by the definition of constructable-

standard tableaux and repeated application of the previous Proposition.

2.5 Raghavan-Sankaran operators

We define certain raising operators on liftable tableaux, different from the root operators above, which we will need in our proof of Theorem 4. (Cf. [26], [20, §4].)

Given a tableau Π and a simple reflection s, define $\widehat{s}(\Pi)$ as follows. Let $\overline{\Pi} = (\tau_1, \ldots, \tau_N)$. If for some $1 \le K \le N$ we have

$$\tau_1 \ge s\tau_1, \ldots, \tau_K \ge s\tau_K,\ \tau_{K+1} < s\tau_{K+1},$$

take $\widehat{s}(\Pi)$ to have the same rational numbers in its LS-chains as Π has, but change the cosets to

$$\overline{\widehat{s}(\Pi)} = (s\tau_1, \ldots, s\tau_K, \tau_{K+1}, \ldots, \tau_N).$$

(In case $s\tau_K = \tau_{K+1}$ and these cosets form part of the same LS-chain, for consistency of notation we must combine these two into a single segment: that is, omit τ_{K+1} and its corresponding rational number.)

Proposition 13 *If Π is a liftable tableau, then $\widehat{s}(\Pi)$ is also a liftable tableau.*

Proof. First we show that $\widehat{s}(\Pi)$ is a tableau. Suppose the coset τ_K in the definition of $\widehat{s}(\Pi)$ occurs in the jth LS-chain π_j of Π: that is, $\tau_K = \tau_{jp}$. Then $\widehat{s}(\Pi) = (\widehat{s}(\pi_1), \ldots, \widehat{s}(\pi_j), \pi_{j+1}, \ldots, \pi_r)$. We thus need to show that if $\pi = (\tau_1 > \cdots > \tau_q;\ 0 < a_1 < \cdots)$ is an LS-chain, then $\widehat{s}(\pi) = (s\tau_1 > \cdots > s\tau_L > \tau_{L+1} > \cdots > \tau_q;\ 0 < a_1 \cdots)$ is also an LS-chain.

If $L = q$, then $\widehat{s}(\pi) = e^{\mathrm{top}}(\pi)$, where e is the raising root operator corresponding to s. If $L < q$, let $\pi' = (\tau_1 > \cdots > \tau_{L+1}; 0 < a_1 < \cdots < a_L < 1)$, which is an LS-chain by [17, Lemma 3.1]. Then once again $\widehat{s}(\pi') = e^{\mathrm{top}}(\pi')$, so that $\widehat{s}(\pi')$ is an LS-chain, and this easily implies that $\widehat{s}(\pi)$ is also an LS-chain. Therefore $\widehat{s}(\Pi)$ is a tableau.

Now to see that $\widehat{s}(\Pi)$ is liftable, we use Lemma 7(b) repeatedly, starting with the positions L, $L+1$ (or $N-1$, N if $L = N$) and proceeding leftward. (Cf. the proof of Lemma 8(a).) •

3 Linear independence

3.1 LS-path basis

In order to show the independence of the set $\{p_\Pi\}$ of standard monomials in $V^*_{\mathbf{i},\mathbf{m}}$, we first establish independence for a set $\{v_\Pi\}$ in $V_{\mathbf{i},\mathbf{m}}$ which we call the LS-path basis.

First we recall the analogous basis $\{v_\pi\}$ of V_λ referred to in Theorem 3. Let $\widehat{s}_i$ be the operators of §2.5. Note that $\mathrm{wt}\,\widehat{s}_i(\pi) = \mathrm{wt}\,\pi - l\alpha_i$ for $l \in \mathbb{Z}_+$. For an

LS-chain π of shape λ, define integers $l_1(\pi)$, $l_2(\pi)$, ... by

$$l_1(\pi) = \frac{\operatorname{wt}\pi - \operatorname{wt}\widehat{s}_{i_1}(\pi)}{\alpha_{i_1}}, \qquad l_2(\pi) = \frac{\operatorname{wt}\widehat{s}_{i_1}(\pi) - \operatorname{wt}\widehat{s}_{i_2}\widehat{s}_{i_1}(\pi)}{\alpha_{i_2}}, \qquad \cdots .$$

That is, $\operatorname{wt}\pi - \operatorname{wt}\widehat{s}_{i_1}(\pi) = l_1(\pi)\alpha_{i_1}$, etc. Note that this depends on our fixed word $\mathbf{i} = (i_1, \ldots, i_r)$. For $(l_1, \ldots, l_r) = (l_1(\pi), \ldots, l_r(\pi))$, define

$$v_\pi^{\mathbf{i}} = F_{i_1}^{l_1} \cdots F_{i_r}^{l_r} \cdot v_\lambda \in V_\lambda,$$

where $F_i \in \mathbf{g}$ are negative root vectors. Now, let $w_{\max}(\mathbf{i})$ be the unique Bruhat-maximal element in the set $\{w(J) \mid J \subset [1,r]\}$; that is, $w_{\max}(\mathbf{i})$ is the Weyl group element generated by a longest reduced subword of $\mathbf{i}$. (See [13, Lemma 1].) Our definition of $\{v_\pi\}$ is a slight generalization of [20, Definition 3], since $\mathbf{i}$ need not be reduced. Recall [20, Theorem 2]:

Proposition 14 *Let $w = w_{\max}(\mathbf{i})$. The set $\{v_\pi^{\mathbf{i}}\}$, where π runs over all LS-chains $\pi = (\tau_1 > \tau_2 > \cdots ; 0 < a_1 < \cdots)$ with $w > \tau_1$, forms a basis of $V_\lambda(w)$.*

We shall need one technical property of the v_π.

Lemma 15 *Let $\pi' = \widehat{s}_i(\pi)$, $l = (\operatorname{wt}\pi - \operatorname{wt}\pi')/\alpha_i$, and $\theta' < \pi'$ in the lexicographic order on LS-chains (§1.3). Suppose either of the following holds:*
(i) $k \le l$; or (ii) the Raghavan-Sankaran operator coincides with ordinary reflection: $\pi' = \widehat{s}_i(\pi) = s_i(\pi)$. Then

$$F_i^k(v_{\theta'}) = \begin{cases} v_\pi, & \text{if } \theta' = \pi' \text{ and } k = l, \\ \displaystyle\sum_{\theta < \pi} \propto v_\theta, & \text{otherwise,} \end{cases}$$

where θ runs over LS-chains less than π.

Proof. The first case of the conclusion, $F_i^k(v_{\theta'}) = v_\pi$, follows directly from the definition of v_π. The second case of the conclusion follows easily from either hypothesis together with [20, Lemma 3(ii)]. •

Now we extend the above construction to generalized Demazure modules. For a tableau Π of shape $(\lambda_1, \ldots, \lambda_r)$, we have $\widehat{s}_{i_1}(\Pi) = \pi^{\lambda_1} * \Pi'$ for some tableau Π' of shape $(\lambda_2, \ldots, \lambda_r)$; and $\widehat{s}_{i_2}(\Pi') = \pi^{\lambda_2} * \Pi''$, etc. Then define integers

$$L_1(\Pi) = \frac{\operatorname{wt}\Pi - \operatorname{wt}\widehat{s}_{i_1}(\Pi)}{\alpha_{i_1}}, \qquad L_2(\pi) = \frac{\operatorname{wt}\Pi' - \operatorname{wt}\widehat{s}_{i_2}(\Pi')}{\alpha_{i_2}}, \qquad \text{etc.}$$

(Recall that $\operatorname{wt}\Pi$ denotes the endpoint of Π considered as a path, i.e., the sum of the weights of the LS-chains in Π.)

Now for $(L_1, \ldots, L_r) = (L_1(\Pi), \ldots, L_r(\Pi))$, let

$$v_\Pi = F_{i_1}^{L_1}(v_{\lambda_1} \otimes F_{i_2}^{L_2}(v_{\lambda_2} \otimes \cdots F_{i_r}^{l_r}(v_{\lambda_r}) \cdots)).$$

The $\{v_\Pi\}$ coincide with the $\{v_\pi\}$ in the case where $V_{\mathbf{i},\mathbf{m}}$ is an ordinary Demazure module $V_\lambda(w)$.

3.2 Independence of $\{p_\pi\}$

Let $w_j = w_{\max}(i_j, \dots, i_r)$, the longest Weyl group element which can be generated from a tail subword of $\mathbf{i}$. Then clearly $V_{\mathbf{i},\mathbf{m}} \subset V_{\lambda_1}(w_1) \otimes \cdots \otimes V_{\lambda_r}(w_r)$. We will write $v_\pi^{(j)} = v_\pi^{(i_j,\dots,i_r)}$. By Proposition 14, the vectors

$$\{v_{\pi_1}^{(1)} \otimes \cdots \otimes v_{\pi_r}^{(r)}\},$$

where each $\pi_j = (\tau_1 > \cdots)$ varies over all LS-chains with $w_j > \tau_1$, form a basis of $V_{\lambda_1}(w_1) \otimes \cdots \otimes V_{\lambda_r}(w_r)$.

Proposition 16 *For a standard tableau $\Pi = (\pi_1, \dots, \pi_r)$, let us write $v_\Pi \in V_{\mathbf{i},\mathbf{m}}$ in terms of the above basis of $V_{\lambda_1}(w_1) \otimes \cdots \otimes V_{\lambda_r}(w_r)$. Then we have the triangular relation:*

$$v_\Pi = v_{\pi_1}^{(1)} \otimes \cdots \otimes v_{\pi_r}^{(r)} + \sum_{\Theta < \Pi} \propto v_{\theta_1}^{(1)} \otimes \cdots \otimes v_{\theta_r}^{(r)},$$

where $\Theta = (\theta_1, \dots, \theta_r)$ runs over all tableaux less than Π in lexicographic order.

Proof. (Cf. [20, Proof of Lemma 3].) We use induction on r, the number of letters in $\mathbf{i}$. For $r = 1$, there is nothing to prove. Now suppose $r \geq 2$, and we know the Proposition for the word $(i_2, \dots, i_r)$. Let $s = s_{i_1}$ and take $\Pi' = \widehat{s}(\Pi)$, where

$$\Pi = (\pi_1, \dots, \pi_r), \qquad \Pi' = (\widehat{s}(\pi_1), \dots, \widehat{s}(\pi_j), \pi_{j+1}, \dots, \pi_r),$$

and

$$\pi^{\lambda_1} = \widehat{s}(\pi_1) = s(\pi_1),\ \widehat{s}(\pi_2) = s(\pi_2), \dots, \widehat{s}(\pi_{j-1}) = s(\pi_{j-1}).$$

By induction, we may assume

$$v_{\Pi'} = v_{\lambda_1} \otimes v_{\hat{s}(\pi_2)} \otimes \cdots \otimes v_{\hat{s}(\pi_k)} \otimes \cdots \otimes v_{\pi_r} + \sum_{\Theta < \Pi'} \propto v_{\lambda_1} \otimes v_{\theta_2} \otimes \cdots \otimes v_{\theta_r},$$

where $\Theta = (\pi^{\lambda_1}, \theta_2, \dots, \theta_r)$. Then

$$v_\Pi = F_{i_1}^{l_1}(v_{\Pi'}),$$

where $l_1 = h_1 + \cdots + h_k$ for

$$h_1 = \frac{\operatorname{wt} \pi_1 - \operatorname{wt} s(\pi_1)}{\alpha_{i_1}}, \dots, h_{k-1} = \frac{\operatorname{wt} \pi_{k-1} - \operatorname{wt} s(\pi_{k-1})}{\alpha_{i_1}},\ \ h_k = \frac{\operatorname{wt} \pi_k - \operatorname{wt} \widehat{s}(\pi_k)}{\alpha_{i_1}}.$$

The terms of v_Π are obtained by distributing the l_1 operations $F_{i_1}^{l_1}$ arbitrarily among the r factors of each term in $v_{\Pi'}$.

We find the maximal term in v_Π by repeatedly applying Lemma 15. By hypothesis (ii) of the Lemma, the maximal first factor of a term in v_Π is $F_{i_1}^{h_1}(v_{s(\pi_1)}) = v_{\pi_1}$; the maximal second factor is v_{π_2}; and so on through the $(k-1)$th factor. Now, assuming the previous maximal factors have been

achieved, we apply hypothesis (i) of the Lemma to find that the maximal kth factor is $F_{i_1}^{h_k}(v_{\hat{s}(\pi_k)}) = v_{pi_k}$. But then all l_1 operations F_{i_1} have been used, and the subsequent terms are unchanged from $v_{\Pi'}$. •

Recall that the set $\{p_\Pi\}$ inside $V^*_{\mathbf{i},\mathbf{m}}$ consists of monomials in any basis $\{p_\pi\}$ of the V^*_λ which is triangular with respect to $\{v_\pi\}$.

Corollary 17
(i) The set $\{v_\Pi\}$, where Π runs over the standard tableaux of shape $(\lambda_1, \ldots, \lambda_r)$, is linearly independent in $V_{\mathbf{i},\mathbf{m}}$.
(ii) The set $\{p_\Pi\}$, where Π runs over the standard tableaux of shape $(\lambda_1, \ldots, \lambda_r)$, restricts to a linearly independent set in $V^*_{\mathbf{i},\mathbf{m}}$.

Proof. Part (i) follows immediately from the triangularity of the set $\{v_\Pi\}$ with respect to the basis $\{v^{(1)}_{\pi_1} \otimes \cdots \otimes v^{(r)}_{\pi_r}\}$. Part (ii) follows similarly, using in addition the triangularity of $\{p_\pi\}$ with respect to $\{v_\pi\}$. •

4 Spanning

The independence of the set $\{p_\Pi\}$ in $V^*_{\mathbf{i},\mathbf{m}}$, along with the Demazure formula for standard tableaux (§2.3), gives a lower bound for the dimension of $V^*_{\mathbf{i},\mathbf{m}}$. In this section, we use geometry to find an upper bound for this dimension which coincides with the lower bound, showing that $\{p_\Pi\}$ is a basis.

4.1 The Demazure module as a space of sections

We relate the generalized Demazure module $V_{\mathbf{i},\mathbf{m}}$ and the Bott-Samelson variety $Z_{\mathbf{i}}$ via a succession of three mappings (see [23]). First, let U_i denote the one-dimensional unipotent subgroup of G whose Lie algebra is $\mathbf{k}F_i$. We have an embedding

$$\phi_1 \ : \ U_{i_1} \times \cdots \times U_{i_r} \to (P_{i_1} \times \cdots \times P_{i_r})/B^r = Z_{\mathbf{i}}$$

whose image defines a Zariski-dense open cell in $Z_{\mathbf{i}}$.

Second, let $\widehat{P}_i \supset B$ be the maximal parabolic subgroup whose Weyl group is generated by all the simple reflections except s_i. Let $\mathrm{Gr}(i) = \mathrm{Gr}(i, G) := G/\widehat{P}_i$ be the G-Grassmannian, and define the multiple G-Grassmannian

$$\mathrm{Gr}(\mathbf{i}) := \mathrm{Gr}(i_1) \times \cdots \times \mathrm{Gr}(i_r),$$

on which G acts diagonally (simultaneously on each factor). The Bott-Samelson variety embeds B-equivariantly into this space:

$$\begin{array}{rccc} \phi_2 : & Z_{\mathbf{i}} & \hookrightarrow & \mathrm{Gr}(\mathbf{i}) \\ & (p_1, \ldots, p_r) & \mapsto & (p_1\widehat{P}_{i_1}, p_1p_2\widehat{P}_{i_2}, \ldots, p_1p_2\cdots p_r\widehat{P}_{i_r}). \end{array}$$

(The configuration variety $Z_{\mathbf{i},\mathbf{m}}$ can be realized as the projection of $Z_{\mathbf{i}} \subset \mathrm{Gr}(\mathbf{i})$ to those factors of $\mathrm{Gr}(\mathbf{i})$ for which $m_j > 0$.)

Third, for a weight $\lambda = m\varpi_i$, define a line bundle on $\mathrm{Gr}(i)$ as $\mathcal{L}_\lambda := G \overset{\widehat{P}_i}{\times} \mathbf{k}^*_\lambda$, so that $\mathcal{L}_{\varpi_i}$ is the minimal ample line bundle on $\mathrm{Gr}(i)$. We thus obtain a line bundle $\mathcal{L}_{\mathbf{i},\mathbf{m}} := \mathcal{L}_{\lambda_1} \otimes \cdots \otimes \mathcal{L}_{\lambda_r}$ on $\mathrm{Gr}(\mathbf{i})$, which is very ample (resp. semi-ample) precisely when all $m_j > 0$ (resp. all $m_j \geq 0$). The restriction of $\mathcal{L}_{\mathbf{i},\mathbf{m}}$ to $Z_\mathbf{i} \subset \mathrm{Gr}(\mathbf{i})$ is easily seen to be isomorphic to the line bundle $\mathcal{L}_{\mathbf{i},\mathbf{m}}$ on $Z_\mathbf{i}$ defined in §1.4. Recall that we may identify $H^0(\mathrm{Gr}(i), \mathcal{L}_\lambda) \cong V^*_\lambda$, so that $\mathrm{Gr}(i) \to \mathbb{P}(H^0(\mathrm{Gr}(i), \mathcal{L}_\lambda)^*) \cong \mathbb{P}(V_\lambda)$, $g\widehat{P}_i \mapsto g \cdot v_\lambda$. Thus we have the natural map

$$\begin{array}{rccc} \phi_3 : & \mathrm{Gr}(\mathbf{i}) & \to & \mathbb{P}(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}) \\ & (g_1\widehat{P}_{i_1}, \ldots, g_r\widehat{P}_{i_r}) & \mapsto & g_1 \cdot v_{\lambda_1} \otimes \ldots \otimes g_r \cdot v_{\lambda_r}\,. \end{array}$$

Now, composing $\phi_3 \circ \phi_2$, we have a map

$$Z_\mathbf{i} \to \mathbb{P}(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}),$$

whose image is by definition $Z_{\mathbf{i},\mathbf{m}}$. If we compose all three mappings $\phi_3 \circ \phi_2 \circ \phi_1$, we see that:

$$\begin{array}{rcl} \mathbb{P}(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}) & \supset & \mathrm{Span}_\mathbf{k}\langle Z_{\mathbf{i},\mathbf{m}} \rangle \\ & = & \mathrm{Span}_\mathbf{k}\langle\, U_{i_1}(v_{\lambda_1} \otimes U_{i_2}(v_{\lambda_2} \otimes \cdots U_{i_r} v_{\lambda_r} \cdots))\,\rangle \\ & = & \mathbb{P}(\,\mathbf{u}_{i_1}(v_{\lambda_1} \otimes \mathbf{u}_{i_2}(v_{\lambda_2} \otimes \cdots \mathbf{u}_{i_r} v_{\lambda_r} \cdots))\,) \\ & = & \mathbb{P}(V_{\mathbf{i},\mathbf{m}}) \end{array}$$

That is, we have the map $\phi = \phi_3 \circ \phi_2 : Z_\mathbf{i} \to \mathbb{P}(V_{\mathbf{i},\mathbf{m}})$ whose image spans $\mathbb{P}(V_{\mathbf{i},\mathbf{m}})$. Dually, we have the *injective* linear map ϕ^*, which factors as:

$$V^*_{\mathbf{i},\mathbf{m}} \hookrightarrow H^0(Z_{\mathbf{i},\mathbf{m}}, \mathcal{L}_{\mathbf{i},\mathbf{m}}) \hookrightarrow H^0(Z_\mathbf{i}, \mathcal{L}_{\mathbf{i},\mathbf{m}}).$$

4.2 Geometric Demazure formula

We now use Demazure's character computations with $\mathbb{P}^1$-fibrations [3] to finish our proof of Theorems 4, 5, and 6, on the model of [20, §8].

Our proof proceeds by induction on r, the number of letters in $\mathbf{i}$. For $r = 0$, all statements are trivial. Now let $r \geq 1$, $i = i_1$, and $\mathbf{i}' = (i_2, \ldots, i_r)$, $\mathbf{m}' = (m_2, \ldots, m_r)$.

From the definitions, we easily see that $H^0(Z_\mathbf{i}, \mathcal{L}_{\mathbf{i},\mathbf{m}}) = H^0(P_i/B, \mathcal{E})$, where

$$\mathcal{E} = P_i \overset{B}{\times} (\mathbf{k}^*_{\lambda_1} \otimes H^0(Z_{\mathbf{i}'}, \mathcal{L}_{\mathbf{i}',\mathbf{m}'})) = P_i \overset{B}{\times} (\mathbf{k}^*_{\lambda_1} \otimes V^*_{\mathbf{i}',\mathbf{m}'}),$$

a vector bundle over P_i/B. (The last equality is by induction.) Now restrict the P_i-action on this vector bundle to an action of $G_i \cong SL_2$, the group whose Lie algebra is generated by E_i, F_i. Take $B_i = B \cap G_i$, so that $P_i/B \cong G_i/B_i \cong \mathbb{P}^1$.

The inclusion $v_{\lambda_1} \otimes V_{\mathbf{i}',\mathbf{m}'} \subset V_{\mathbf{i},\mathbf{m}}$ dualizes to a short exact sequence of B_i-modules $0 \to \mathrm{Ker} \to V^*_{\mathbf{i},\mathbf{m}} \to \mathbf{k}^*_{\lambda_1} \otimes V^*_{\mathbf{i}',\mathbf{m}'} \to 0$, which leads to a long exact sequence in cohomology of bundles over $\mathbb{P}^1$:

$$\cdots \to H^i(\mathbb{P}^1, G_i \overset{B_i}{\times} \mathrm{Ker}) \to H^i(\mathbb{P}^1, G_i \overset{B_i}{\times} V_{\mathbf{i},\mathbf{m}}) \to H^i(\mathbb{P}^1, \mathcal{E}) \to \cdots$$

Since $V_{\mathbf{i},\mathbf{m}}$ is a G_i-module, it induces a trivial vector bundle, and $H^1(\mathbb{P}^1, G_i \overset{B_i}{\times} V_{\mathbf{i},\mathbf{m}}) = 0$. Since trivially $H^2(\mathbb{P}^1, G_i \overset{B_i}{\times} \mathrm{Ker}) = 0$, we thus get:

$$H^i(\mathbb{P}^1, \mathcal{E}) = 0 \quad \text{for } i > 0.$$

From the Leray spectral sequence of the fibration $Z_{\mathbf{i}} \to P_i/B$, and induction, it follows that $H^i(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{i},\mathbf{m}}) = 0$ for $i > 0$.

Now, the character ring of G_i is $R_i \cong \mathbf{k}[x]$, a polynomial ring in one variable, with the quotient map $R \to R_i,\ e^\lambda \mapsto x^{\langle \lambda, \alpha_i^\vee \rangle}$. From elementary computations with SL_2-bundles we have Demazure's formula [3] for the G_i-character char_i of the cohomology of $\mathcal{E}$, in terms of the fiber of $\mathcal{E}$ above $eB_i \in G_i/B_i$:

$$\mathrm{char}_i\, H^0(\mathbb{P}^1, \mathcal{E}) - \mathrm{char}_i\, H^1(\mathbb{P}^1, \mathcal{E}) = \Lambda_i\, \mathrm{char}_i(\mathcal{E}|_{eB_i}),$$

where $\Lambda_i : R_i \to R_i$ is the map induced from $\Lambda_i : R \to R$. But the negative H^1 term vanishes, and by specializing the polynomials to $x = 1$ we find

$$\begin{aligned} \dim H^0(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{i},\mathbf{m}}) &= \dim H^0(\mathbb{P}^1, \mathcal{E}) \\ &= \Lambda_i\, \mathrm{char}_i(\mathbf{k}^*_{\lambda_1} \otimes V_{\mathbf{i}',\mathbf{m}'})|_{x=1} \\ &= \Lambda_{i_1}(e^{\lambda_1} \Lambda_{i_2}(e^{\lambda_2} \cdots))|_{e^\lambda = 1}, \end{aligned}$$

where the last equality is by induction. However, we also know:

$$\begin{aligned} \dim H^0(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{i},\mathbf{m}}) &\geq \dim H^0(Z_{\mathbf{i},\mathbf{m}}, \mathcal{L}_{\mathbf{i},\mathbf{m}}) && \text{by §4.2} \\ &\geq \dim V^*_{\mathbf{i},\mathbf{m}} && \text{by §4.2} \\ &\geq \dim \mathrm{Span}\langle p_\Pi \mid \Pi \text{ a standard tableau} \rangle \\ &= \#\{\text{standard tableaux}\} && \text{by Cor 17} \\ &= \Lambda_{i_1}(e^{\lambda_1} \Lambda_{i_2}(e^{\lambda_2} \cdots))|_{e^\lambda = 1} && \text{by Cor 10} \end{aligned}$$

Comparing expressions, we conclude that all the above inequalities are in fact equalities, meaning

$$H^0(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{i},\mathbf{m}}) \cong H^0(Z_{\mathbf{i},\mathbf{m}}, \mathcal{L}_{\mathbf{i},\mathbf{m}}) \cong V^*_{\mathbf{i},\mathbf{m}} = \mathrm{Span}\langle p_\Pi \rangle,$$

and the Demazure character formula holds for all four of these spaces. This implies the projective normality of $Z_{\mathbf{i}}$ and $Z_{\mathbf{i},\mathbf{m}}$ with respect to $\mathcal{L}_{\mathbf{i},\mathbf{m}}$ by [4, Ch II, Ex 5.14].

Finally, the vanishing of the higher cohomology of $\mathcal{L}_{\mathbf{i},\mathbf{m}}$ over $Z_{\mathbf{i},\mathbf{m}}$ follows from a standard argument involving the map $\phi : Z_{\mathbf{i}} \to Z_{\mathbf{i},\mathbf{m}}$ (see, e.g. [20, §8], [22, Prop 28]). Using $H^0(Z_{\mathbf{i},\mathbf{m}}, \mathcal{L}_{\mathbf{i},\mathbf{m}}) = H^0(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{i},\mathbf{m}})$, and the normality of $Z_{\mathbf{i},\mathbf{m}}$, we apply Kempf's Lemma [9] to deduce $H^i(Z_{\mathbf{i},\mathbf{m}}, \mathcal{L}_{\mathbf{i},\mathbf{m}}) = H^i(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{i},\mathbf{m}})$ for all $i > 0$. But we have already shown that the right hand side vanishes.

This completes the proof of Theorems 4, 5, and 6

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