Standard Monomial Theory and Applications

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1. Introduction

In the theory of finite dimensional representations of complex reductive algebraic groups, the group $GL_n(\mathbb{C})$ is singled out by the fact that besides the usual language of weight lattices, roots and characters, there exists an additional important combinatorial tool: the Young tableaux. To construct objects like the tableaux in a more general setting, consider the weight lattice X of a complex semisimple Lie algebra (or, more general, symmetrizable Kac-Moody algebra) $\mathfrak{g}$, and denote by Π the set of all piecewise linear paths $\pi : [0,1]_{\mathbb{Q}} \to X_{\mathbb{Q}}$ starting in 0 and ending in an integral weight. We associate to a simple root α operators e_α and f_α on Π, and, using these operators, we construct for a dominant weight λ a set of paths $B(\lambda)$ that can be viewed as a generalization of the Young tableaux: For example, the sum over the endpoints of all paths in $B(\lambda)$ is the character of $V(\lambda)$, and the Littlewood-Richardson rule can be generalized in a straightforward way. Though the theory of the paths is completely independent of the theory of quantum groups, they can be viewed as a geometric realization of the theory of crystals of representations.

The next step is then to associate a basis of the representation to the paths. The starting point for the theory was a series of articles in which Lakshmibai, Musili

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and Seshadri initiated a program to construct a basis for the space $H^0(G/B, \mathcal{L}_\lambda)$ with some particularly nice geometric properties. Here we suppose that G is a reductive algebraic group defined over an algebraically closed field k, B is a fixed Borel subgroup, and $\mathcal{L}_\lambda$ is the line bundle on the flag variety G/B associated to a dominant weight. The purpose of the program is to extend the Hodge-Young standard monomial theory for the group $GL(n)$ to the case of any semisimple linear algebraic group and, more generally, to Kac-Moody algebras.

Using quantum groups at a root of unity, we define a basis of the representation such that each element of the basis can be viewed in some sense as a ℓ-th root of a product of extremal weight vectors. As applications we get a straightforward construction of Standard Monomial Theory, a representation theoretic proof of the normality of Schubert varieties, a combinatorial proof of the Demazure character formula, the "Good filtration" property for tensor products in positive characteristic [38a], a reduced Groebner basis for the defining ideal of Schubert varieties in terms of generalized Plücker relations, ...

In the first two sections we recall the main facts concerning the path model. In the third and fourth section we give an introduction into the construction of the path vectors, the main tool here is the quantum Frobenius map for quantum groups at roots of unity. For simplicity we restrict ourself to the finite dimensional case, but the proofs hold, with the appropriate adaptions, also for arbitrary symmetrizable Kac-Moody algebras. For details see [39]. In the next three sections we discuss the application to the geometry of Schubert varieties, we generalize the results in [39] to unions of Schubert varieties.

In addition to the geometric consequences for Schubert varieties such as normality, vanishing theorems, ideal theory etc, the Standard Monomial Theory has also led to the determination of the singular loci of Schubert varieties (cf. [22], [23], [29], [31], [34]), and the results are recalled in §9.

As a further application of Standard Monomial Theory, one obtains ([13], [28]) the normality and Cohen-Macaulayness for two classes of affine varieties - certain ladder determinantal varieties (cf. §10.15) and certain quiver varieties (cf. §10.19). These results are proved by identifying them with the "opposite cells" in suitable Schubert varieties $X_Q(w)$ in suitable $SL(n)/Q$.

Standard Monomial Theory grew out of, and applies to, the algebraic geometry of flag varieties and their subvarieties. We sketch an extension of the theory to a larger class of spaces, the Bott-Samelson varieties and configuration varieties. These varieties (like Schubert varieties) have a natural B-action, and the spaces of global sections of ample line bundles provide a class of B-representations whose description is the main goal of the theory. We have a generalized Bruhat order and a set of L-S paths fitting into a path model, which provide an indexing system for bases. Moreover, although we do not yet have a direct generalization of the basis $\{p_\pi\}$ corresponding to L-S paths, we do describe an analog of the "standard tableau" bases of Section 8.

2. An indexing system for a basis: The L-S paths

For a complex semisimple Lie algebra $\mathfrak{g}$ fix a Cartan subalgebra $\mathfrak{h}$, a Borel subalgebra $\mathfrak{b}$, and denote by X the weight lattice of $\mathfrak{g}$. Corresponding to the choice of $\mathfrak{b}$ let X^+ be the set of dominant weights. On $X_\mathbb{R} := X \otimes_\mathbb{Z} \mathbb{R}$ denote by $(\cdot,\cdot)$ the Killing form, and for a root β let $\beta^\vee = 2\beta/(\beta,\beta)$ be the co-root.

Let $V(\lambda)$ be the simple $\mathfrak{g}$-module of highest weight λ. The aim of this section is to describe an indexing system for a basis of $V(\lambda)$ of $\mathfrak{h}$-eigenvectors. Denote by $\pi_\lambda : t \to t\lambda$ the path that connects the origin with λ by a straight line.

We are going to describe a set of paths obtained by bending π_λ: the Lakshmibai-Seshadri paths. The definition given here is a "translation" of the definition in [33] into the language of paths. Let W be the Weyl group of $\mathfrak{g}$, and for a dominant weight λ denote by W_λ the stabilizer of λ in W. Let "$\leq$" be the Bruhat order on W/W_λ. We identify a pair $\pi = (\underline{\tau}, \underline{a})$ of sequences:

- $\underline{\tau} : \tau_1 > \tau_2 > \ldots > \tau_r$ is a sequence of linearly ordered cosets in W/W_λ and
- $\underline{a} : a_0 := 0 < a_1 < \ldots < a_r := 1$ is a sequence of rational numbers.

with the path $\pi : [0,1] \to X_\mathbb{R}$ defined by:

$$\pi(t) := \sum_{i=1}^{j-1} (a_i - a_{i-1})\tau_i(\lambda) + (t - a_{j-1})\tau_j(\lambda) \quad \text{for} \quad a_{j-1} \leq t \leq a_j.$$

Note that $\lambda - \pi(1) = (\lambda - \tau_r(\lambda)) + \sum_{i=1}^{r-1} a_i\big(\tau_{i+1}(\lambda) - \tau_i(\lambda)\big)$, so if the a_i are chosen such that the $a_i(\tau_{i+1}(\lambda) - \tau_i(\lambda))$ are still in the root lattice, then $\pi(1) \in X$. To ensure this, we introduce now the notion of an a-chain. Let $l(\cdot)$ be the length function on W/W_λ and denote by $\beta^\vee$ the coroot of a positive real root β.

Let $\tau > \sigma$ be two elements of W/W_λ and let $0 < a < 1$ be a rational number. By an *a-chain* for the pair (τ, σ) we mean a sequence of cosets in W/W_λ:

$$\kappa_0 := \tau > \kappa_1 := s_{\beta_1}\tau > \kappa_2 := s_{\beta_2}s_{\beta_1}\tau > \ldots > \kappa_s := s_{\beta_s} \cdot \ldots \cdot s_{\beta_1}\tau = \sigma,$$

where $\beta_1, \ldots, \beta_s$ are positive real roots and $l(\kappa_i) = l(\kappa_{i-1}) - 1$, $a(\kappa_i(\lambda), \beta_i^\vee) \in \mathbb{Z}$ for all $i = 1, \ldots, s$.

Definition 2.1. A pair $(\underline{\tau}, \underline{a})$ is called a *Lakshmibai-Seshadri* path of shape λ if for all $i = 1, \ldots, r-1$ there exists an a_i-chain for the pair (τ_i, τ_{i+1}).

Example 2.2. For $\sigma \in W/W_\lambda$ let $\pi_{\sigma(\lambda)}$ be the path $t \mapsto t\sigma(\lambda)$ that connects 0 with $\sigma(\lambda)$ by a straight line. Then $\pi_{\sigma(\lambda)}$ is the Lakshmibai-Seshadri path $(\sigma; 0, 1)$.

Example 2.3. Let α be a simple root and suppose $\sigma \in W/W_\lambda$ is such that $n = (\sigma(\lambda), \alpha^\vee) > 0$. Then $(s_\alpha\sigma, \sigma; 0, i/n, 1)$ is an L-S path for $1 \leq i < n$.

Example 2.4. Suppose $\mathfrak{g} = \mathfrak{sl}_2$, $\mathfrak{h} = \{\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} | a \in \mathbb{C}\}$, and $\lambda = n\epsilon$ (where ϵ denotes the projection of a diagonal matrix onto its first entry). Then the set $B(\lambda)$ of L-S paths of shape $\lambda = n\epsilon$ is equal to:

$$B(n\epsilon) = \big\{(s_\alpha; 0, 1),\ (s_\alpha, id; 0, \frac{1}{n}, 1),\ \ldots,\ (s_\alpha, id; 0, \frac{n-1}{n}, 1),\ (id; 0, 1)\big\}.$$

Note that $\sum_{\pi \in B(n\epsilon)} e^{\pi(1)}$ is the character of the irreducible representation $V(n\epsilon)$.

Example 2.5. Let $\mathfrak{g}$ be a semisimple Lie algebra and suppose that ω is a minuscule fundamental weight (i.e., $(\omega, \beta^\vee) = 0$ or 1 for a positive root β). Then

$$B(\omega) = \big\{(\sigma; 0, 1) \mid \sigma \in W/W_\omega\big\}.$$

Recall that the weight spaces in $V(\omega)_\mu$ are at most one-dimensional, and $V(\omega)_\mu \neq 0$ if and only if $\mu = \sigma(\omega)$ for some $\sigma \in W/W_\omega$. Since $\pi(1) = \sigma(\omega)$ for $\pi = (\sigma; 0, 1)$, we get hence $\mathrm{Char}\, V(\omega) = \sum_{\pi \in B(\omega)} e^{\pi(1)}$.

Example 2.6. Suppose $\mathfrak{g}$ is a simple Lie algebra of simply laced type and let $V(\beta) = \mathfrak{g}$ be the adjoint representation, where β is the highest root. The set of L-S paths consists then of two types: There are the ones of the form $(\tau; 0, 1)$, $\tau \in W/W_\beta$, which correspond to the straight line that connects the origin by a straight line with the root $\tau(\beta)$. The others are of the form $(s_\alpha\tau, \tau; 0, \frac{1}{2}, 1)$, where τ is such that $\tau(\beta) = \alpha$ is a simple root. Note again that the L-S paths provide a way to calculate the character of the representation: For every root we have exactly one path ending in the root, and we have as many paths ending in the origin as we have simple roots, which is the same as the dimension of $\mathfrak{h}$.

The fact that the L-S paths provide a tool to calculate characters holds in general. This was conjectured (and proved in many special cases) by V. Lakshmibai, and first proved in the general case in [35]. It turns out that this character formula for L-S paths is a special case of a much more general formula which will be explained in the next section.

Theorem 2.7. *The character $Char\, B(\lambda)$ of the set of L-S paths of shape λ is equal to the character of of the irreducible representation $V(\lambda)$ of highest weight λ.*

The character formula above can be refined in the following way: For a Lakshmibai-Seshadri path $\pi = (\tau_1, \dots, \tau_r; 0, a_1, \dots, 1)$ denote by $i(\pi) := \tau_1$ the "first direction" of the path. For $\tau \in W/W_\lambda$, let $B(\lambda)_\tau$ be the subset of all L-S paths of shape λ such that $i(\pi) \leq \tau$ in the Bruhat ordering. Denote by Λ_α the Demazure operator on $\mathbb{Z}[X]$:

$$\Lambda_\alpha(e^\mu) := \frac{e^{\mu+\rho} - e^{s_\alpha(\mu+\rho)}}{1 - e^{-\alpha}} e^{-\rho}$$

For a proof of the following formula see [35]:

Demazure Type Character Formula . *For any reduced decomposition $\tau = s_{\alpha_1} \dots s_{\alpha_r}$ one has $\Lambda_{\alpha_1} \circ \cdots \circ \Lambda_{\alpha_r}(e^\lambda) = \sum_{\eta \in B(\lambda)_\tau} e^{\eta(1)}$.*

3. Path models of a representation

The L-S paths can be thought of as an example of a much more general theory, the theory of path models. Though not everything is needed in the following, we present a short survey of the main results concerning this combinatorial tool.

Definition 3.1. A rational piecewise linear path in $X_\mathbb{R}$ is a piecewise linear, continuous map $\pi : [0, 1] \to X_\mathbb{R}$ such that all turning points are rational. We consider two paths π, η as identical if there exists a piecewise linear, nondecreasing, continuous, surjective map $\phi : [0, 1] \to [0, 1]$ such that $\pi = \eta \circ \phi$. Denote by Π the set of all rational piecewise linear paths such that $\pi(0) = 0$ and $\pi(1) \in X$.

Example 3.2. *i)* For $\lambda \in X$ set $\pi_\lambda(t) := t\lambda$, then $\pi_\lambda \in \Pi \Leftrightarrow \lambda \in X$.

ii) Let π_1, π_2 be two rational piecewise linear paths starting in 0. By $\pi := \pi_1 * \pi_2$ we mean the path defined by

$$\pi(t) := \begin{cases} \pi_1(2t), & \text{if } 0 \leq t \leq 1/2; \\ \pi_1(1) + \pi_2(2t - 1), & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

iii) The set $B(\lambda)$ of L-S paths of shape λ is a subset of Π.

For a finite set of paths $B \subset \Pi$ denote by $\operatorname{Char} B$ the character of B, i.e., the formal sum: $\operatorname{Char} B := \sum_{\pi \in B} e^{\pi(1)}$.

Example 3.3. For $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and $\lambda = n\epsilon$ we get:

$$\operatorname{Char} B(n\epsilon) = \sum_{i=0}^{n} e^{(is_\alpha(\epsilon)+(n-i)\epsilon)} = \sum_{i=0}^{n} e^{(n-2i)\epsilon} = \operatorname{Char} V(n\epsilon).$$

To obtain combinatorial character formulas and multiplicity formulas as in the example above, we define "lowering" and "raising" operators f_α, e_α for any simple root. The definition of the operators is elementary, it is a cutting and glueing procedure. Fix $\pi \in \Pi$, and denote by h_α the function:

$$h_\alpha : [0,1] \to \mathbb{R}, \quad t \mapsto (\pi(t), \alpha^\vee).$$

Let m_α be the minimal value attained by this function. We define non-decreasing functions $l, r : [0,1] \to [0,1]$:

$$l(t) := \min\{1, h_\alpha(s) - m_\alpha \mid t \leq s \leq 1\}, \; r(t) := \max\{0, 1 + m_\alpha - h_\alpha(s) \mid 0 \leq s \leq t\}$$

Note that $l(t) = 0$ for $0 \leq t \leq s$, where s is maximal such that $h(s) = m_\alpha$, and $r(t) = 1$ for $s' \leq t \leq 1$, where s' is minimal such that $h(s') = m_\alpha$.

In the following we consider the set $\Pi \cup \{0\}$, where we define the value of the operators on 0 to be $e_\alpha(0) = f_\alpha(0) := 0$.

Definition 3.4. If $r(0) = 0$, then $(e_\alpha\pi)(t) := \pi(t) + r(t)\alpha$, otherwise we define $e_\alpha(\pi) := 0$. If $l(1) = 1$, then let $f_\alpha\pi$ be the path defined by $(f_\alpha\pi)(t) := \pi(t) - l(t)\alpha$, and if $l(1) < 1$, then we define $f_\alpha(\pi) := 0$.

If we think of a path as a concatenation of "smaller" paths $\pi = \pi_1 * \ldots * \pi_r$, then we can view e_α and f_α as operators that replace some of the π_j by $s_\alpha(\pi_j)$.

Example 3.5. Suppose $\mathfrak{g}$ is a simple Lie algebra of simply laced type, and let β be the highest root. The paths obtained from $\pi_\beta : t \mapsto t\beta$ by applying the operators f_α, e_α are exactly the L-S paths of shape λ.

The following properties of the operators are easy to prove [36], [38]:

Lemma 3.6. *Let α be a simple root and suppose $\pi \in \Pi$.*

i) *If $e_\alpha\pi \neq 0$, then $e_\alpha(\pi)(1) = \pi(1) + \alpha$, and if $f_\alpha(\pi) \neq 0$, then $f_\alpha(\pi)(1) = \pi(1) - \alpha$.*
ii) *If $e_\alpha(\pi) \neq 0$, then $f_\alpha e_\alpha(\pi) = \pi$, and if $f_\alpha(\pi) \neq 0$, then $e_\alpha f_\alpha(\pi) = \pi$.*
iii) *Let π^* be the dual path, i.e. $\pi^*(t) := \pi(1-t) - \pi(1)$. Then $(f_\alpha\pi)^* = e_\alpha(\pi^*)$ and $(e_\alpha\pi)^* = f_\alpha(\pi^*)$.*
iv) *Let n be maximal such that $f_\alpha^n(\pi) \neq 0$, and let m be maximal such that $e_\alpha^m(\pi) \neq 0$. Then $(\pi(1), \alpha^\vee) = n - m$.*
v) *For $k \in \mathbb{N}$ let $k\pi$ be the path obtained by stretching π: $(k\pi)(t) := k\pi(t)$. Then $k(f_\alpha\pi) = f_\alpha^k(k\pi)$ and $k(e_\alpha\pi) = e_\alpha^k(k\pi)$*

Corollary 3.7. *Suppose B is a finite subset of paths such that $B \cup \{0\}$ is stable under the root operators. Denote by B_μ the subset of paths in B ending in μ. Then $|B_\mu| = |B_{w(\mu)}|$ for any $w \in W$.*

Proof of the corollary . It suffices to prove that $|B_\mu| = |B_{s_\alpha(\mu)}|$ for a simple root α. If $(\mu, \alpha^\vee) = 0$, then there is nothing to prove. Suppose $(\mu, \alpha^\vee) = n > 0$. Then *ii*) and *iv*) implies that that the map $\pi \mapsto f_\alpha^n\pi$ induces a bijection $B_\mu \to B_{s_\alpha(\mu)}$. Similarly, if $(\mu, \alpha^\vee) = n < 0$, then $\pi \mapsto e_\alpha^{|n|}\pi$ induces a bijection $B_\mu \to B_{s_\alpha(\mu)}$. $\square$

Denote by $C := \{\nu \in X_{\mathbb{R}} \mid (\nu, \beta^\vee) \geq 0 \textit{ for all positive roots } \beta\}$ the dominant Weyl chamber, and let C^0 be the interior of C. Let $\Pi^+ \subset \Pi$ the set of paths η such that the image $\operatorname{Im}\eta$ is contained in the dominant Weyl chamber C. Denote by $\rho \in X^+$ half the sum of the positive roots. If $B \subset \Pi$ is a finite subset such that $B \cup \{0\}$ is stable under the root operators e_α, f_α, then we have already seen that its character $\operatorname{Char} B := \sum_{\eta \in B} e^{\eta(1)}$ is stable under the action of the Weyl group W.

For $\pi \in \Pi^+$ let $B(\pi) \subset \Pi$ be the subset of paths which can be obtained from π by applying the operators, and let $G(\pi)$ be the colored, directed graph having as vertices the elements of $B(\pi)$, and we put an arrow $\eta \xrightarrow{\alpha} \eta'$ with color a simple root α between $\eta, \eta' \in B_\pi$ if and only if $f_\alpha(\eta) = \eta'$ (or, equivalently, $e_\alpha(\eta') = \eta$).

The structure of the set of paths $B(\pi)$ generated by a path $\pi \in \Pi^+$ is described by the following theorem (see [36] for proofs):

Theorem 3.8. *Suppose $\pi, \pi_1, \pi_2 \in \Pi^+$.*

a) *Integrality: $B(\pi)$ is integral, i.e., the minimum of the function $t \mapsto (\eta(t), \alpha^\vee)$ is an integer for all $\eta \in B(\pi)$ and all simple roots α.*
b) *Highest weight path: π is the only path in $B(\pi)$ such that $e_\alpha \pi = 0$ for all simple roots.*
c) *Isomorphism: $G(\pi_1) \simeq G(\pi_2)$ if and only if $\pi_1(1) = \pi_2(1)$.*
d) *Weyl group: The action of the simple reflections s_α on Π defined by:*

$$s_\alpha(\eta) := \begin{cases} f_\alpha^p(\eta), & \textit{if } p := (\eta(1), \alpha^\vee) \geq 0, \\ e_\alpha^p(\eta), & \textit{if } -p := (\eta(1), \alpha^\vee) < 0, \end{cases}$$

extends to an action of the Weyl group W on Π such that the $w(\eta)(1) = w(\eta(1))$.

The independence of the graph structure of the choice of the starting path has as consequence that the graph is isomorphic to the crystal graph of the representation, see [16] or [18] for details.

Weyl Character Formula . *Let $\rho \in X$ be half the sum of the positive roots, and suppose $\pi \in \Pi^+$. Then*

$$\sum_{\sigma \in W} \operatorname{sgn}(\sigma) e^{\sigma(\rho)} \, \mathit{Char}\, B(\pi) = \sum_{\sigma \in W} \operatorname{sgn}(\sigma) e^{\sigma(\rho + \pi(1))}.$$

In particular, $\mathit{Char}\, B(\pi)$ is equal to the character of the irreducible $\mathfrak{g}$-module $V(\lambda)$ of highest weight $\lambda := \pi(1)$.

The integrality property stated above seems at the first instance to be a "technical" fact without further consequences. But using the lemma above, it follows easily from the definition of the operators that: If π, η are paths having the integrality property, then $e_\alpha(\pi * \eta) = \pi * (e_\alpha \eta)$ if there exists an $n > 0$ such that $e_\alpha^n \eta \neq 0$ but $f_\alpha^n \pi = 0$, and it is equal to $(e_\alpha \pi) * \eta$ otherwise. Similarly, $f_\alpha(\pi * \eta) = (f_\alpha \pi) * \eta$ if there exists an $n > 0$ such that $f_\alpha^n \pi \neq 0$ but $e_\alpha^n \eta = 0$, and it is equal to $\pi * (f_\alpha \eta)$ otherwise.

So if $\pi_1, \pi_2 \in \Pi^+$, then let $B(\pi_1) * B(\pi_2)$ be the set of all concatenations $\eta_1 * \eta_2$, where $\eta_1 \in B(\pi_1)$ and $\eta_2 \in B(\pi_2)$. The rules above show that $B(\pi_1) * B(\pi_2)$ is stable under the root operators. It follows by the theorem above:

Concatenation . *Suppose $\pi_1, \pi_2 \in \Pi^+$. Then*

$$B(\pi_1) * B(\pi_2) = \bigcup B(\pi_1 * \eta),$$

where the union runs over all paths $\eta \in B(\pi_2)$ such that $\pi_1 * \eta \in \Pi^+$.

Since $\mathrm{Char}\, B(\pi_1) * B(\pi_2) = \mathrm{Char}\, B(\pi_1)\, \mathrm{Char}\, B(\pi_2) = \mathrm{Char}\, V(\pi_1(1)) \otimes V(\pi_2(1))$, we get as an immediate consequence of the theorem above and the character formula:

Generalized Littlewood-Richardson rule . *For dominant weights λ, μ, let π_1, π_2 in Π^+ be such that $\pi_1(1) = \lambda$ and $\pi_2(1) = \mu$. Then the tensor product of the irreducible representations $V(\lambda)$, $V(\mu)$ of $\mathfrak{g}$ of highest weight λ, μ is isomorphic to the direct sum*

$$V(\lambda) \otimes V(\mu) \simeq \bigoplus V(\lambda + \eta(1)),$$

where the sum runs over all paths $\eta \in B(\pi_2)$ such that $\pi_1 * \eta \in \Pi^+$.

The L-S paths discussed in the previous section are an example of such a set of paths, stable under the root operators. The character formula stated in the preceding section is an immediate consequence of the proposition below and the character formula above. For a proof of the following proposition see [35].

Proposition 3.9. *Let $\pi_\lambda : t \mapsto t\lambda$ be the path that joins the origin with the dominant weight λ by a straight line. Then the set of paths $B(\pi_\lambda)$, obtained from π_λ by applying all possible combinations of the root operators, is equal to $B(\lambda)$, the set of L-S paths of shape λ.*

4. A basis associated to the L-S paths

The character formula shows that we can use the set of L-S paths as an indexing system of a basis of $\mathfrak{h}$-eigenvectors of $V(\lambda)$. The next aim is to attach in a canonical way such a basis to $B(\lambda)$.

The idea of the construction is the following: Suppose for simplicity that $\mathfrak{g}$ is of simply laced type. For a dominant weight λ, let $V(\lambda)$ be the irreducible module of $\mathfrak{g}$ of highest weight λ. Denote by $U_{\mathbf{v}}(\mathfrak{g})$ the quantum group at an ℓ-th root of unity $\mathbf{v}$, let $N(\lambda)$ be the Weyl module and by $L(\lambda)$ the simple module for $U_{\mathbf{v}}(\mathfrak{g})$ of highest weight λ.

Lusztig [40] has constructed a Frobenius map $Fr : U_{\mathbf{v}}(\mathfrak{g}) \to U(\mathfrak{g})$ between the quantum group and the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Further, he has shown that if we consider $V(\lambda)$ via the Frobenius Fr as an $U_{\mathbf{v}}(\mathfrak{g})$-module, then this is the simple module $L(\ell\lambda)$. This identification provides a $U_{\mathbf{v}}(\mathfrak{g})$-equivariant map $p : N(\ell\lambda) \to V(\lambda)$, the quotient of $N(\ell\lambda)$ by its maximal proper $U_{\mathbf{v}}(\mathfrak{g})$-submodule.

We are going to define a subspace $N(\ell\lambda)^\ell$ which is naturally equipped with a $U(\mathfrak{g})$-action, and we will use this to define a section $s : V(\lambda) \to N(\ell\lambda)^\ell \subset N(\ell\lambda)$ of the projection defined above. The dual map $s^* : N(\ell\lambda)^{\ell,*} \to V(\lambda)^*$ induces a map:

$$(N(\lambda)^*)^{\otimes \ell} \to N(\ell\lambda)^* \to N(\ell\lambda)^{\ell,*} \to V(\lambda)^*$$

Now once a highest weight vector $m_\lambda \in N(\lambda)$ is fixed, there is a canonical choice of extremal weight vectors $m_\tau \in N(\lambda)$ of weight $\tau(\lambda)$, $\tau \in W/W_\lambda$, and corresponding dual vectors $b_\tau \in N(\lambda)^*$ of weight $-\tau(\lambda)$.

Let now $\pi = (\tau_1, \dots, \tau_r; 0, a_1, a_2, \dots, 1)$ is an L-S path of shape λ, and suppose ℓ is such that $\ell a_i \in \mathbb{Z}$ for all $i = 1, \dots, r$. Then the vector b_π is well defined:

$$b_\pi := \underbrace{b_{\tau_1} \otimes \ldots \otimes b_{\tau_1}}_{\ell a_1} \otimes \underbrace{b_{\tau_2} \otimes \ldots \otimes b_{\tau_2}}_{\ell(a_2 - a_1)} \otimes \ldots \otimes \underbrace{b_{\tau_r} \otimes \ldots \otimes b_{\tau_r}}_{\ell(1-a_{r-1})} \in (N(\lambda)^*)^{\otimes \ell}$$

Denote by $p_\pi \in V(\lambda)^*$ its image, the *path vector* associated to π. To make the construction canonical we assume that the ℓ above is minimal with the property that $\ell a_i \in \mathbb{Z}$ for all i and for all π of shape λ.

The construction presented here is actually characteristic free and works over the ring $\tilde{R}$ obtained from $\mathbb{Z}$ by adjoining all roots of unity.

To make the construction more precise, we need first to fix some notation: Let $A = (a_{i,j})$ be the Cartan matrix of $\mathfrak{g}$, and let $\mathfrak{g}^t$ be the semisimple Lie algebra associated to the transposed matrix A^t. We fix $\underline{d} = (d_1, \dots, d_n)$ minimal such that $(d_i a_{i,j})$ is a symmetric matrix, and let d be the smallest common multiple of the d_j.

In the following we will often attach a $(\ )^t$ to some object associated to $\mathfrak{g}^t$ to distinguish it from the corresponding $\mathfrak{g}$ object. Let $\alpha_1, \dots, \alpha_n$ be the simple roots of $\mathfrak{g}$, and for $\mathfrak{g}^t$ let the corresponding roots be $\gamma_1 = \alpha_1/d_1, \dots, \gamma_n = \alpha_n/d_n$.

Let $U_q(\mathfrak{g}^t)$ be the quantum group associated to $\mathfrak{g}^t$ over the field $\mathbb{Q}(q)$, with generators $E_{\gamma_i}, F_{\gamma_i}, K_{\gamma_i}$ and $K_{\gamma_i}^{-1}$. We use the usual abbreviations ($\overline{d}_i := d/d_i$)

$$[n]_i := \frac{q^{\overline{d}_i n} - q^{-\overline{d}_i n}}{q^{\overline{d}_i} - q^{-\overline{d}_i}},\ [n]_i! := [1]_i \cdots [n]_i,\ \begin{bmatrix} n \\ m \end{bmatrix}_i := \frac{[n]_i!}{[m]_i![n-m]_i!},$$

where we define the latter to be zero for $n < m$. We will sometimes just write $E_i, K_i, \dots$ for $E_{\gamma_i}, K_{\gamma_i}, \dots$. In addition, we use the following abbreviations:

$$q_i := q^{\overline{d}_i} = q^{\frac{(\gamma_i,\gamma_i)^t}{2}},\ \begin{bmatrix} K_i; c \\ p \end{bmatrix} := \prod_{s=1}^{p} \frac{K_i q^{\overline{d}_i(c-s+1)} - K_i^{-1} q^{\overline{d}_i(-c+s-1)}}{q^{\overline{d}_i s} - q^{-\overline{d}_i s}}.$$

Let $U_{q,A}$ be the Lusztig-form of U_q defined over the ring of Laurent polynomials $A := \mathbb{Z}[q, q^{-1}]$ and generated by the divided powers $E_i^{(n)} := \frac{E_i^n}{[n]_i!}$ and $F_i^{(n)} := \frac{F_i^n}{[n]_i!}$. Let U_q^+ (respectively U_q^-) be the subalgebra generated by the E_i (respectively F_i), and denote by $U_{q,A}^+$ (respectively $U_{q,A}^-$) the corresponding A-form generated by the divided powers.

For an A-algebra R, let $U_{q,R}^+$ be the algebra $U_{q,A}^+ \otimes_A R$ and denote by $U_{q,R}^-$ the algebra $U_{q,A}^- \otimes_A R$.

We use a similar notation for the enveloping algebra $U(\mathfrak{g})$. To distinguish better between the elements of $U(\mathfrak{g})$ and $U_q(\mathfrak{g}^t)$, we denote the generators of $U(\mathfrak{g})$ by $X_\alpha, H_\alpha, Y_\alpha$ or X_i, H_i, Y_i. Let $U = U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$ defined over $\mathbb{Q}$, let $U_\mathbb{Z}$ be the Kostant-$\mathbb{Z}$-form of U, set $U_R := U_\mathbb{Z} \otimes_\mathbb{Z} R$, etc.

We suppose in the following always that ℓ is divisible by $2d$ and set $\overline{\ell} := \ell/d$. Denote by R the ring A/I, where I is the ideal generated by the 2ℓ-th cyclotomic polynomial, let $\mathbf{v}$ be the image of q in R, and set $U_\mathbf{v} := U_{q,A} \otimes_A R$, $U_\mathbf{v}^+ := U_{q,A}^+ \otimes_A R$ and $U_\mathbf{v}^- := U_{q,A}^- \otimes_A R$. Let $\ell_i := \frac{\ell d_i}{d}$, then, by the definition of d, ℓ_i is minimal such that

$$\ell_i \frac{(\gamma_i, \gamma_i)^t}{2} = \ell_i \overline{d}_i = \ell_i \frac{d}{d_i} \in \ell\mathbb{Z}.$$

For a dominant weight $\lambda \in X^t$ let $N(\lambda)$ be the simple $U_q(\mathfrak{g}^t)$-module of highest weight λ, fix an A-lattice $N_A(\lambda) := U_{q,A}m_\lambda$ in $N(\lambda)$ by choosing a highest weight vector $m_\lambda \in N(\lambda)$. Set $N_R(\lambda) := N_A(\lambda) \otimes_A R$, then $N_R(\lambda)$ is an $U_{\mathbf{v}}$-module such that its character is given by the Weyl character formula. Consider the weight space decomposition:

$$N_R(\lambda) = \bigoplus_{\mu \in X^t} N_R(\lambda)_\mu \quad \text{and set} \quad N_R(\lambda)^{\overline{\ell}} := \bigoplus_{\mu \in \overline{\ell} X} N_R(\lambda)_\mu.$$

The subspace $N_R(\lambda)^{\overline{\ell}}$ is obviously stable under the subalgebra of $U_{\mathbf{v}}$ generated by the $E_i^{(n\ell_i)}$ and $F_i^{(n\ell_i)}$: If $\mu \in \overline{\ell}X$, then so is $\mu \pm n\ell_i\gamma_i = \mu \pm \frac{nd_i\ell}{d}\gamma_i = \mu \pm n\overline{\ell}\alpha_i$.

Theorem 4.1. *The map*

$$X_i^{(n)} \mapsto E_i^{(n\ell_i)}|_{N_R(\lambda)^{\overline{\ell}}}, \quad Y_i^{(n)} \mapsto F_i^{(n\ell_i)}|_{N_R(\lambda)^{\overline{\ell}}}, \quad \binom{H_i + m}{n} \mapsto \begin{bmatrix} K_i; m\ell_i \\ n\ell_i \end{bmatrix}|_{N_R(\lambda)^{\overline{\ell}}},$$

extends to a representation map $U_R(\mathfrak{g}) \to \mathrm{End}_R\, N_R(\lambda)^{\overline{\ell}}$.

Some remarks on the proof . One has to prove that the map is compatible with the Serre relations. For U_R^+ and U_R^-, this is a direct consequence of the higher order quantum Serre relations ([40], Chapter 7). For a detailed proof see [40], section 35.2.3. For the proof that also the remaining Serre relations hold see [39].

Let $N = \bigoplus_{\mu \in X^t} N_\mu$ be a finite dimensional $U_q(\mathfrak{g}^t)$-module with a weight space decomposition. If N admits a $U_{q,A}(\mathfrak{g}^t)$-stable A-lattice $N_A = \bigoplus_{\mu \in X^t} N_{A,\mu}$ (where $N_{A,\mu} := N_A \cap N_\mu$), then we denote for any A-algebra R by N_R the $U_{q,R}(\mathfrak{g}^t)$-module $N_A \otimes_A R$. We have a corresponding weight space decomposition $N_R = \bigoplus_{\mu \in X^t} N_{R,\mu}$.

The same arguments as above show that we can make $N_R^{\overline{\ell}} := \bigoplus_{\mu \in \overline{\ell}X} N_{R,\mu}$ into an $U_R(\mathfrak{g})$-module by the same construction. Let S be the antipode, the action of $U_{q,R}(\mathfrak{g}^t)$ on the dual module $N_R^* := \mathrm{Hom}_R(N_R, R)$ is given by: $(uf)(m) := f(S(u)(m))$ for $u \in U_{q,R}(\mathfrak{g}^t)$ and $f \in N_R^*$. It is easy to check that the map $U_R \to \mathrm{End}_R \left(N_R^{\overline{\ell}}\right)^*$ defined by

$$X_i^{(n)} f(m) := f(S(E_i^{(n\ell_i)})m), \quad Y_i^{(n)} f(m) := f(S(F_i^{(n\ell_i)})m),$$

and $\binom{H_i+k}{n} f(m) := f(S(\left[\begin{smallmatrix} K_i; k\ell_i \\ n\ell_i \end{smallmatrix}\right])m)$, is the representation map corresponding to the dual representation of the representation of $U_R(\mathfrak{g})$ on $N_R^{\overline{\ell}}$.

We proceed now as indicated in the introduction of this section. For $\tau \in W/W_\lambda$ fix a reduced decomposition $\tau = s_{i_1} \cdots s_{i_r}$. We associate to τ the vector

$$m_\tau = F_{i_1}^{(n_1)} F_{i_2}^{(n_2)} \ldots F_{i_r}^{(n_r)} m_\lambda$$

where $n_r := (\lambda, \alpha_{i_r}^\vee)$, $\ldots$, $n_1 = (s_{i_2} \cdots s_{i_r}(\lambda), \alpha_{i_1}^\vee)$. It follows from the quantum Verma relations that m_τ is independent of the choice of the reduced decomposition. Denote by $b_\tau \in N_R^*$ the unique eigenvector of weight $-\tau(\lambda)$ such that $b_\tau(m_\tau) = 1$.

For an L-S path $\pi = (\tau_1, \ldots, \tau_r; 0, a_1, a_2, \ldots, 1)$ of shape λ, fix ℓ minimal such that $2d$ divides ℓ and $\overline{\ell}a_i \in \mathbb{Z}$ for all $i = 1, \ldots, r$. (The restriction d divides ℓ is obviously necessary in the construction above, the condition $2d$ divides ℓ is necessary because there are restrictions concerning the existence of the Frobenius map. In certain cases this restriction is not necessary, but to avoid lengthy case by case considerations we prefer to impose this condition because it is sufficient for the existence of the Frobenius map in all cases). Then the vector b_π is well defined:

$$b_\pi := \underbrace{b_{\tau_1} \otimes \ldots \otimes b_{\tau_1}}_{\overline{\ell}a_1} \otimes \underbrace{b_{\tau_2} \otimes \ldots \otimes b_{\tau_2}}_{\overline{\ell}(a_2-a_1)} \otimes \ldots \otimes \underbrace{b_{\tau_r} \otimes \ldots \otimes b_{\tau_r}}_{\overline{\ell}(1-a_{r-1})} \in (N_R(\lambda)^*)^{\otimes \overline{\ell}}.$$

Denote by $p_\pi \in V_R(\lambda)^*$ its image, the *path vector* associated to π.

Let $\tilde{R}$ be the ring obtained by adjoining all roots of unity to $\mathbb{Z}$. We fix an embedding $R \hookrightarrow \tilde{R}$. If k is an algebraically closed field and $\operatorname{Char} k = 0$, then we consider k as an $\tilde{R}$-module by the inclusion $\tilde{R} \subset k$. If $\operatorname{Char} k = p > 0$, then we consider k as an $\tilde{R}$-module by extending the canonical map $\mathbb{Z} \to k$ to a map $\tilde{R} \to k$ (where the first map is given by the projection $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ and the inclusion $\mathbb{Z}/p\mathbb{Z} \subset k$). Denote by $V_{\tilde{R}}(\lambda) = V_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} \tilde{R}$ the corresponding Weyl module over the ring $\tilde{R}$, then the collection of vectors

$$\mathbb{B}(\lambda) := \{p_\pi \mid \pi \in B(\lambda)\} \subset V_{\tilde{R}}(\lambda)^*$$

is well defined. By abuse of notation we write also p_π for the image of the vector in $V_k(\lambda)^* := V_{\tilde{R}}(\lambda)^* \otimes_{\tilde{R}} k$ for any algebraically closed field.

Theorem 4.2. *The path vectors $\mathbb{B}(\lambda)$ form a basis for the $U_{\tilde{R}}(\mathfrak{g})$-module $V_{\tilde{R}}(\lambda)^*$.*

Note that $\mathbb{B}(\lambda)$ is a basis for $V_k(\lambda)^*$ for any algebraically closed field k. The proof of the theorem will be given in the next section. The idea is to construct a basis v_π of $V_{\mathbb{Z}}(\lambda)$, indexed by L-S paths, such that $p_\pi(v_\pi) = 1$, and for $\pi \neq \pi'$ we have $p_\pi(v_{\pi'}) = 1$ only if $\pi' \geq \pi$ in some partial order on the set of L-S paths. Note that this implies that $\mathbb{B}(\lambda)$ will be, up to an upper triangular transformation, the dual basis of the basis given by the v_π, in particular, $\mathbb{B}(\lambda)$ is a basis of $V_{\tilde{R}}(\lambda)^*$. The disadvantage of the basis given by the v_π is that it depends heavily on a choice of a reduced decomposition of $i(\pi)$.

5. A basis for $V_{\mathbb{Z}}(\lambda)_\tau$

To construct the basis $D(\lambda)$ of $V_{\mathbb{Z}}(\lambda)$ we have first to introduce a partial order on weight vectors. For extremal weight vectors we write $m_\tau \geq m_\kappa$ if $\tau \geq \kappa$ in the Bruhat order on W/W_λ.

Similarly, we shall write $\pi \geq \eta$ for two L-S paths $\pi = (\tau_1, \ldots, \tau_r; 0, a_1, \ldots, 1)$ and $\eta = (\kappa_1, \ldots, \kappa_s; 0, b_1, \ldots, 1)$ of shape λ if $\tau_1 > \kappa_1$ or $\tau_1 = \kappa_1$ and $a_1 > b_1$, or $\tau_1 = \kappa_1$ and $a_1 = b_1$ and $\tau_2 > \kappa_2, \ldots.$

Recall that $V_{\mathbb{Z}}(\lambda)_\tau$ is the $U_R^+(\mathfrak{g})$ submodule of $V_{\mathbb{Z}}(\lambda)$ generated by m_τ, i.e., $V_{\mathbb{Z}}(\lambda)_\tau = U_R^+(\mathfrak{g})m_\tau$. For an extremal weight vector m_τ and an arbitrary weight vector m we write $m_\tau \geq m$ if $m \in V_{\mathbb{Z}}(\lambda)_\tau$. We denote by "$\geq$" as well the induced lexicographic partial order on tensor products of weight vectors in $N_R(\lambda)^{\otimes \overline{\ell}}$.

We define a weaker order on weight vectors by saying that $m_\mu \succ m_\nu$ for two eigenvectors of weight ν, μ if $\nu \succ \mu$ in the ususal weight ordering (i.e., $\nu - \mu$ is a sum of positive roots). We denote by "$\succ$" as well the induced lexicographic partial order on tensor products.

Suppose $\overline{\ell}a_i \in \mathbb{Z}$ for all i, we denote by m^π the tensor product

$$m^\pi = \underbrace{m_{\tau_1} \otimes \ldots \otimes m_{\tau_1}}_{\overline{\ell}a_1} \otimes \underbrace{m_{\tau_2} \otimes \ldots \otimes m_{\tau_2}}_{\overline{\ell}(a_2-a_1)} \otimes \ldots \otimes \underbrace{m_{\tau_r} \otimes \ldots \otimes m_{\tau_r}}_{\overline{\ell}(1-a_{r-1})} \in (N_R(\lambda)^*)^{\otimes \overline{\ell}}$$

Fix a reduced decomposition $\tau_1 = s_{i_1} \cdots s_{i_t}$. Let $s(\pi) = (n_1, \ldots, n_t)$ be the sequence of integers defined by the following procedure, which has been inspired by the article of K. N. Raghavan and P. Sankaran [48]. Fix j minimal such that $s_{i_1}\tau_j > \tau_j$, and set $j = r+1$ if $s_{i_1}\tau_j \leq \tau_j$ for all j. It is easy to see that $\pi' = (s_{i_1}\tau_1, \ldots, s_{i_1}\tau_{j-1}, \tau_j, \ldots, \tau_r; 0, a_1, \ldots, 1)$ is an L-S path of shape λ (it is understood that we omit a_{j-1} if $s_{i_1}\tau_{j-1} = \tau_j$).

It follows that $\pi'(1) - \pi(1)$ is an integral multiple of the simple root α_{i_1}. Let $n_1 \in \mathbb{N}$ be such that $\pi'(1) - \pi(1) = n_1\alpha_{i_1}$. Note that $s_{i_1}\tau_1 = s_{i_2} \ldots s_{i_r}$ is a reduced decomposition, and $s_{i_1}\tau_1 < \tau_1$. Suppose we have already defined $s(\pi') = (n_2, \ldots, n_r)$ (where $s(id; 0, 1)$ is the empty sequence). We define the sequence for π to be the one obtained by adding n_1 to the sequence for π'.

Definition 5.1. We denote by v_π the vector $v_\pi := Y_{i_1}^{(n_1)} Y_{i_2}^{(n_2)} \ldots Y_{i_t}^{(n_t)} v_\lambda \in V_{\mathbb{Z}}(\lambda)$.

Recall that $V_{\mathbb{Z}}(\lambda)_\tau$ can also be described as the subspace obtained from $V_{\mathbb{Z}}(\lambda)_\kappa$ (where $\kappa = s_{i_1}\tau$) by applying Y_{i_1}, i.e., $V_{\mathbb{Z}}(\lambda)_\tau = \sum_{n \geq 0} Y_{i_1}^{(n)} V_{\mathbb{Z}}(\lambda)_\kappa$. It follows hence that $v_\pi \in V_{\mathbb{Z}}(\lambda)_{i(\pi)}$.

Theorem 5.2. *$D(\lambda)_\tau := \{v_\pi \mid i(\pi) \leq \tau\}$ is a basis for $V_{\mathbb{Z}}(\lambda)_\tau$.*

Recall that we can consider $V_R(\lambda)$ as a submodule of $N(\lambda)^{\otimes \overline{\ell}}$. A first step towards the proof of the theorem is the following:

Proposition 5.3. *Suppose $\overline{\ell} a_i \in \mathbb{Z}$ for all i. Then*

$$v_\pi = Y_{i_1}^{(n_1)} Y_{i_2}^{(n_2)} \ldots Y_{i_t}^{(n_t)} v_\lambda = m^\pi + \text{ tensor products } < m^\pi \text{ in the partial order}$$

Proof . The proposition is obviously true for $\pi = (id; 0, 1)$, we proceed by induction on the length of $i(\pi)$. Let $\alpha = \alpha_{i_1}$ and set

$$\pi' = (s_\alpha\tau_1, \ldots, s_\alpha\tau_{j-1}, \tau_j, \ldots, \tau_r; 0, a_1, \ldots, 1),$$

where $j-1$ is maximal such that $s_\alpha\tau_i \leq \tau_i$ for all $1 \leq i \leq j-1$. By assumption, we know that

$$v_{\pi'} = m^{\pi'} + \text{ tensor products } < m^{\pi'} \text{ in the partial order.}$$

Now $v_\pi = Y_\alpha^{(n_1)} v_{\pi'}$, let us first look at the terms we get by calculating $Y_\alpha^{(n_1)} m^{\pi'}$. Up to multiplication by a root of unity, the latter is the sum of terms of the form

$$(F_\gamma^{(h_1)} m_{s_\alpha\tau_1}) \otimes \ldots \otimes (F_\gamma^{(h_{\overline{\ell}})} m_{\tau_r}), \tag{1}$$

where the sum runs over all $\overline{\ell}$-tuples $(h_1, \ldots, h_{\overline{\ell}})$ such that $\sum h_i = \ell_{i_1} n_1$. It is clear that, in the weak ordering, a maximal element must be one such that h_1 is maximal, and then, for the given h_1, the h_2 has to be maximal, etc. Now the maximal h_1 which is possible is $(s_\alpha\tau_1(\lambda), \gamma^\vee)$, and similarly we can calculate the maximal h_2, h_3, etc. By assumption we know that

$$\begin{aligned} n_1 &= \overline{\ell}(a_1(s_\alpha\tau_1(\lambda), \alpha^\vee) + \ldots + (a_{j-1} - a_{j-2})(s_\alpha\tau_{j-1}(\lambda), \alpha^\vee)) \\ &= \ell_{i_1}(a_1(s_\alpha\tau_1(\lambda), \gamma^\vee) + \ldots + (a_{j-1} - a_{j-2})(s_\alpha\tau_{j-1}(\lambda), \alpha^\vee)). \end{aligned}$$

It follows that, up to a scalar factor, the maximal element in the weak ordering is m^π, and this is the only maximal element. A term of the form (1) which is not maximal admits a minimal j such that h_j is not maximal, so the corresponding weight vector $F_\gamma^{h_1} m_{s_\alpha\tau_i} < m_{\tau_i}$ in the strong partial order, and hence m^π is the unique maximal element (with respect to the induced strong lexicographic partial order) in the expression of $Y_\alpha^{(n_1)}m^{\pi'}$ as a linear combination of elements of type (1).

Now suppose $m_{s_\alpha\tau_i} > m_\mu$ in the strong partial order, so $m_\mu \in N_R(\lambda)_{s_\alpha\tau_i}$. Then, for $n>0$, we have $F_\gamma^{(n)}m_\mu \in N_R(\lambda)_{\tau_i}$. In particular, $m_{\tau_i} \geq F_\gamma^{(n)}m_\mu$. But note that, by weight consideration, we can have equality only if $m_{s_\alpha\tau_i} = Cm_\mu$ for some $C\in R$, so $m_{s_\alpha\tau_i} > m_\mu$ implies $m_{\tau_i} > F_\gamma^{(n)}m_\mu$. Combining this with the arguments above, one sees that applying $Y_\alpha^{(n_1)}$ to an arbitrary summand $\neq m^{\pi'}$ in the expression of $v_{\pi'}$ gives only tensors which are smaller in the partial order then m^π. It follows that

$$Y_\alpha^{(n_1)}v_{\pi'} = Cm^\pi + \text{ tensor products } < m^\pi \text{ in the partial order.}$$

To finish the proof of the proposition we have to show that the constant C is equal to 1. Recall that the co-multiplication is given by (see for example [40])

$$\Delta(F_\gamma^{(p)}) = \sum_{p'+p''=p} q^{-d_\gamma p'p''}F_\gamma^{(p')}\otimes K_\gamma^{-p'}F_\gamma^{(p'')}.$$

It follows that the leading term in $Y_\alpha^{(n_1)}v_{\pi'}$ is

$$F_\gamma^{(\ell_{i_1}n_1)}m^{\pi'} = F_\gamma^{(\ell_{i_1}n_1)}(m_{s_\alpha\tau_1}\otimes\ldots\otimes m_{s_\alpha\tau_{j-1}})\otimes K_\gamma^{-(\ell_{i_1}n_1)}(m_{\tau_j}\otimes\ldots\otimes m_{\tau_r})+$$

smaller terms. The weight of the second part in the first tensor product is $\overline{\ell}\mu = \overline{\ell}((a_j-a_{j-1})\tau_j(\lambda)+\ldots+(1-a_{r-1})\tau_r(\lambda))$. By the integrality property for local minima of L-S paths ([35], note that $(\tau_j(\lambda),\alpha^\vee)>0$ by assumption) we know that $(\mu,\alpha^\vee)\in\mathbb{Z}$.

Now $K_\gamma^{-\ell_{i_1}}$ applied to a weight vector of weight $\overline{\ell}\mu$ gives

$$\begin{aligned}K_\gamma^{-\ell_{i_1}}m_{\overline{\ell}\mu} = \mathbf{v}^{-\ell_{i_1}(d/d_{i_1})(\overline{\ell}\mu,\gamma^\vee)}m_{\overline{\ell}\mu} = \mathbf{v}^{-\ell\ell_{i_1}(\mu,\alpha^\vee)}m_{\overline{\ell}\mu} &= \mathbf{v}^{(-2\ell)(\ell_{i_1}/2)(\mu,\alpha^\vee)}m_{\overline{\ell}\mu}\\ &= m_{\overline{\ell}\mu},\end{aligned}$$

because $\ell_{i_1}/2\in\mathbb{Z}$ and $(\mu,\alpha^\vee)\in\mathbb{Z}$. So we see that the leading term is

$$\big(F_\gamma^{(\ell_{i_1}n_1)}(m_{s_\alpha\tau_1}\otimes\ldots\otimes m_{s_\alpha\tau_{j-1}})\big)\otimes m_{\tau_j}\otimes\ldots\otimes m_{\tau_r}.$$

Now it is easy to see that if $n=(\tau(\lambda),\gamma^\vee), m=(\kappa(\lambda),\gamma^\vee)\geq 0$, then $F_\gamma^{(n+m)}(m_\tau\otimes m_\kappa) = (F_\gamma^{(n)}m_\tau)\otimes(F_\gamma^{(m)}m_\kappa)$. By induction one can show that the leading term is hence equal to m^π, so the constant $C=1$. □

Proof of the theorem . Fix ℓ such that for all L-S paths $\pi=(\tau_1,\ldots,\tau_r;0,a_1,\ldots,1)$ and all i we have $\overline{\ell}a_i\in\mathbb{Z}$, and consider the embedding $V_{\mathbb{R}}(\lambda)\hookrightarrow N_R(\lambda)^{\otimes\overline{\ell}}$. The leading term of v_π is the tensor m^π. Since the m^π are obviously linearly independent, the proposition above implies that the v_π are also linearly independent. By the Weyl character formula (for representations and the path model, see section 3), we know hence that the v_π span an R-lattice in $V_R(\lambda)$ of maximal rank. The m^π can be viewed as a subset of an R-basis for $N_R(\lambda)^{\otimes\overline{\ell}}$. Since the coefficient of m^π is 1 the expression for v_π, it follows that the v_π form an R-basis of $V_R(\lambda)$. Since the $v_\pi\in V_{\mathbb{Z}}(\lambda)$ by construction, it follows that the v_π form in fact a $\mathbb{Z}$-basis of $V_{\mathbb{Z}}(\lambda)$.

It remains to prove $D(\lambda)_\tau$ is a basis of $V_{\mathbb{Z}}(\lambda)_\tau$. Denote by V'_τ the $\mathbb{Z}$-submodule spanned by $D(\lambda)_\tau$. We have already pointed out that $V'_\tau\subset V_{\mathbb{Z}}(\lambda)_\tau$. Since the extremal weight vector $v_{(\tau;0,1)}=v_\tau$ is an element of V'_τ, to prove the theorem it suffices to prove that V'_τ is $U^+_{\mathbb{Z}}(\mathfrak{g})$-stable. This is a consequence of the following lemma, which finishes the proof of the theorem. $\square$

Lemma 5.4. *$X^{(n)}_\alpha v_\pi = \sum a_{\pi,\eta} v_\eta$, where $a_{\pi,\eta} \neq 0$ only if $\pi > \eta$.*

Proof . We consider $V_{\mathbb{Z}}(\lambda)$ again as a subspace of $N_R(\lambda)^{\otimes \overline{\ell}}$. We know that $v_\pi = m^\pi$ + terms strictly smaller in the partial order. It is now easy to see that $X^{(n)}_\alpha v_\pi$ is a sum of tensor products of weight vectors which are smaller then m^π in the partial order. In particular, for any maximal η such that $a_{\pi,\eta} \neq 0$, we know that the coefficient of m^η in the expression of $X^{(n)}_\alpha v_\pi$ is not zero, so we have necessarily $\pi > \eta$. $\square$

As an immediate consequence we get by the Demazure type character formula for the L-S paths (section 2):

Corollary 5.5. *(Demazure character formula) $V_{\mathbb{Z}}(\lambda)_\tau$ is a direct summand of $V_{\mathbb{Z}}(\lambda)$, and for any reduced decomposition $\tau = s_{i_1} \dots s_{i_r}$, the character $Char\, V_{\mathbb{Z}}(\lambda)_\tau$ is given by the Demazure character formula $Char\, V_{\mathbb{Z}}(\lambda)_\tau = \Lambda_{i_1} \dots \Lambda_{i_r} e^\lambda$.*

Proof of the Basis Theorem for path vectors . We have obviously $p_\pi(v_\eta) \neq 0$ (where ℓ etc. has been chosen appropriately) only if m^η occurs with non-zero coefficient in the expression of v_η as element of $N(\lambda)^{\otimes \overline{\ell}}$. But this is only possible if $\eta \geq \pi$. We have also seen above that the coefficient of m^π is one in the expression of v_π, so $p_\pi(v_\pi) = 1$. It follows that for any algebraically closed field, the path vectors p_π, π an L-S path of shape λ, form a basis of $V_k(\lambda)^*$. $\square$

The fact that the basis given by the v_π is compatible with the Demazure submodules $V_{\mathbb{Z}}(\lambda)_\tau$ implies:

Corollary 5.6. *The kernel of the restriction map $V_k(\lambda)^* \to V_k(\lambda)^*_\tau$ has as basis the p_π such that $i(\pi) \not\leq \tau$, and the images of the p_π such that $i(\pi) \leq \tau$, form a basis of $V_k(\lambda)^*_\tau$.*

6. Schubert varieties

We apply now the results above to the geometry of Schubert varieties. We show how to obtain from the path basis the normality of Schubert varieties, the vanishing theorems, the reducedness of intersections of unions of Schubert varieties etc. These facts have been proved before, mostly using the machinery of Frobenius splitting (Andersen, Kumar, Mathieu, Mehta, Ramanan, Ramanathan), in some special cases proofs had been given before using standard monomial theory, (Lakshmibai, Musili, Rajeswari, Seshadri), see for example [32], [33], [42], [46], [47] for a description of the development.

Let k be an algebraically closed field, we will omit the subscript k whenever there is no confusion possible. Let G be the simply connected semisimple group corresponding to $\mathfrak{g}$, and, according to the choice of the triangular decomposition of $\mathfrak{g}$, let $B \subset G$ be a Borel subgroup. Fix a dominant weight λ and let $P \supset B$ be the parabolic subgroup of G associated to λ. It is well-known that the space of global sections $\Gamma(G/P, \mathcal{L}_\lambda)$ of the line bundle $\mathcal{L}_\lambda := G \times_P k_{-\lambda}$ is, as a G-representation, isomorphic to $V(\lambda)^*$. Let $\phi : G/P \hookrightarrow \mathbb{P}(V(\lambda))$ be the corresponding embedding.

For $\tau \in W/W_\lambda$ denote by $X(\tau) \subset G/P$ the Schubert variety. Let $Y = \bigcup_{i=1}^r X(\tau_i)$ be a union of Schubert varieties. By abuse of notation, we denote by $\mathcal{L}_\lambda$ and p_π

also the restrictions $\mathcal{L}_\lambda|_Y$ and $p_\pi|_Y$. Recall that the linear span of the affine cone over $X(\tau)$ in $V(\lambda)$ is the submodule $V(\lambda)_\tau$. The restriction map $\Gamma(G/P,\mathcal{L}_\lambda) \to \Gamma(X(\tau),\mathcal{L}_\lambda)$ induces hence an injection $V(\lambda)^*_\tau \hookrightarrow \Gamma(X(\tau),\mathcal{L}_\lambda)$. We call a path vector p_π *standard on* Y if $i(\pi) \leq \tau_i$ for at least one $1 \leq i \leq r$. Denote $\mathbb{B}(\lambda)_Y$ the set of standard path vectors on Y.

Theorem 6.1. a) *$\mathbb{B}(\lambda)_Y$ is a basis of $\Gamma(Y,\mathcal{L}_\lambda)$.*
b) *$p_\pi|_Y \equiv 0$ if and only if $i(\pi) \not\leq \tau_i$ for all $i = 1,\dots,r$.*

Corollary 6.2. *The restriction map $\Gamma(G/P,\mathcal{L}_\lambda) \to \Gamma(Y,\mathcal{L}_\lambda)$ is surjective.*

Further, by the character formula presented in section 2 we get:

Corollary 6.3. *For any reduced decomposition $\tau = s_{i_1}\dots s_{i_r}$, Char $\Gamma(X(\tau),\mathcal{L}_\lambda)^*$ is given by the Demazure character formula Char $\Gamma(X(\tau),\mathcal{L}_\lambda)^* = \Lambda_{i_1}\dots\Lambda_{i_r}e^\lambda$.*

The proof of the theorem is by induction on the dimension and the number of irreducible components of maximal dimension. Let Y, Y_1, Y_2 be unions of Schubert varieties. During the induction procedure we prove in addition:

Theorem 6.4. i) *$H^i(Y,\mathcal{L}_\lambda) = 0$ for $i \geq 1$.*
ii) *$X(\tau)$ is a normal variety.*
iii) *The scheme theoretic intersection $Y_1 \cap Y_2$ is reduced.*

Proof . In the case of Schubert varieties, a proof is given in [39], we will give here only a rough sketch of the proof in this case and concentrate on the generalisation to the case of unions of Schubert varieties. The proof uses the ideas presented in [32], but since the construction of the basis is not anymore part of the induction procedure, these arguments can be applied in a straight forward manner.

The theorems hold obviously if Y is a point. Suppose first that $Y = X(\tau)$ is a Schubert variety of positive dimension, and let α be a simple root such that $\kappa := s_\alpha\tau < \tau$. Denote by $SL_2(\alpha)$ the corresponding subgroup of G with Borel subgroup $B_\alpha = B \cap SL_2(\alpha)$. The canonical map $\Psi : Z_\alpha := SL_2(\alpha) \times_{B_\alpha} X(\kappa) \to X(\tau)$ is birational and has connected fibres. The map induces an injection $\Gamma(X(\tau),\mathcal{L}_\lambda) \hookrightarrow \Gamma(Z_\alpha,\Psi^*\mathcal{L}_\lambda)$.

By induction hypothesis, we know that $H^i(X(\kappa),\mathcal{L}_\lambda) = 0$ for $i \geq 1$. Since the restriction of $\Psi^*\mathcal{L}_\lambda$ to $X(\kappa)$ is again $\mathcal{L}_\lambda$, the bundle map $Z_\alpha \to \mathbb{P}^1 = SL_2(\alpha)/B_\alpha$ induces isomorphisms $H^i(Z_\alpha,\Psi^*\mathcal{L}_\lambda) \to H^i(\mathbb{P}^1,\tilde{\Gamma}(X(\kappa),\mathcal{L}_\lambda))$. (Here $\tilde{\Gamma}(X(\kappa),\mathcal{L}_\lambda)$ denotes the vector bundle associated to the B_α-module $\Gamma(X(\kappa),\mathcal{L}_\lambda)$).

The short exact sequence $0 \to K \to V(\lambda)^*_\tau \to V(\lambda)^*_\kappa = \Gamma(X(\kappa),\mathcal{L}_\lambda) \to 0$ of B_α-modules induces a long exact sequence in cohomology:

$$\dots \to H^i(\mathbb{P}^1,\tilde{K}) \to H^i(\mathbb{P}^1,\tilde{V}(\lambda)^*_\tau) \to H^i(\mathbb{P}^1,\tilde{\Gamma}(X(\kappa),\mathcal{L}_\lambda)) \to \dots$$

Since $V(\lambda)^*_\tau$ is a $SL_2(\alpha)$-module, the higher cohomology groups vanish for $\tilde{V}(\lambda)^*_\tau$ and hence also for $\tilde{\Gamma}(X(\kappa),\mathcal{L}_\lambda)$. It follows that $H^i(Z_\alpha,\Psi^*\mathcal{L}_\lambda) = 0$ for $i > 0$. Recall that if M is a B_α-module and $\tilde{M}$ the associated vector bundle on $\mathbb{P}^1$, then

$$\Lambda_\alpha \operatorname{Char} M = \operatorname{Char}\Gamma(\mathbb{P}_1,\tilde{M}) - \operatorname{Char} H^1(\mathbb{P}^1,\tilde{M}).$$

Since $H^1(\mathbb{P}^1,\tilde{\Gamma}(X(\kappa),\mathcal{L}_\lambda)) = 0$, it follows that

$$\operatorname{Char}\Gamma(Z_\alpha,\Psi^*\mathcal{L}_\lambda) = \Lambda_\alpha \operatorname{Char}\Gamma(X(\kappa),\mathcal{L}_\lambda).$$

By induction, the character of $\Gamma(Z_\alpha,\Psi^*,\mathcal{L}_\lambda)$ is hence given by the Demazure character formula. Since the same is true for $V(\lambda)^*_\tau$ by the corollary in section 5, the inclusions $V(\lambda)^*_\tau \hookrightarrow \Gamma(X(\tau), \mathcal{L}_\lambda) \hookrightarrow \Gamma(Z_\alpha, \Psi^*\mathcal{L}_\lambda)$ have to be isomorphisms. This proves the theorem for Schubert varieties.

Since $\mathcal{L}_\lambda$ is an arbitrary ample line bundle and Z_α is normal, one concludes easily from the isomorphism $V(\lambda)^*_\tau \simeq \Gamma(X(\tau), \mathcal{L}_\lambda) \simeq \Gamma(Z_\alpha, \Psi^*\mathcal{L}_\lambda)$ that $X(\tau)$ has to be normal. A simple Leray spectral sequence argument shows then that we have in fact $H^i(X(\tau), \mathcal{L}_\lambda) \simeq H^i(Z_\alpha, \Psi^*\mathcal{L}_\lambda)$, which finishes the proof for Schubert varieties because we know already that $H^i(Z_\alpha, \Psi^*\mathcal{L}_\lambda) = 0$ for all $i > 0$.

We show now by induction on the number of irreducible components and on the dimension the corresponding statements for unions of Schubert varieties. Let $b(\lambda)_Y$ be the number of path vectors standard on Y, and denote by $h^0(Y, \mathcal{L}_\lambda)$ the dimension of $H^0(Y, \mathcal{L}_\lambda)$. Note that the path vectors p_π which are standard on Y remain linearly independent: The restriction of a linear dependance relation $\sum a_\pi \pi$ to any maximal irreducible component has to vanish by the results above, which means that all coefficients a_π vanish. As a consequence we get: $h^0(Y, \mathcal{L}_\lambda) \geq b(\lambda)_Y$.

Let Y_1 and Y_2 be unions of Schubert varieties such that $h^0(Y_i, \mathcal{L}_\lambda) = b(\lambda)_{Y_i}$ for all ample line bundles $\mathcal{L}_\lambda$ on G/P. We have the following exact sequences of $\mathcal{O}_{G/P}$-modules:

$$0 \to \mathcal{O}_{Y_1 \cup Y_2} \to \mathcal{O}_{Y_1} \oplus \mathcal{O}_{Y_2} \to \mathcal{O}_{Y_1 \cap Y_2} \to 0,$$

where $Y_1 \cap Y_2$ denotes the scheme theoretic intersection. Let $\mathcal{L}_\lambda$ be an ample line bundle on G/P. If we tensor the sequence above with $\mathcal{L}_{m\lambda}$, then we get for $m \gg 0$ by Serre's vanishing theorem and the long exact sequence in cohomology:

$$h^0(Y_1 \cap Y_2, \mathcal{L}_{m\lambda}) + h^0(Y_1 \cup Y_2, \mathcal{L}_{m\lambda}) = h^0(Y_1, \mathcal{L}_{m\lambda}) + h^0(Y_2, \mathcal{L}_{m\lambda}).$$

It is easy to see that $b(m\lambda)_{(Y_1 \cap Y_2)_{red}} + b(m\lambda)_{Y_1 \cup Y_2} = b(m\lambda)_{Y_1} + b(m\lambda_{Y_2})$. Here $(Y_1 \cap Y_2)_{red}$ is the intersection with the induced reduced structure, i.e., it is the union of all Schubert varieties contained in Y_1 and Y_2. Since $h^0(Y_1 \cup Y_2, \mathcal{L}_{m\lambda}) \geq b(m\lambda)_{Y_1 \cup Y_2}$ and

$$h^0(Y_1 \cap Y_2, \mathcal{L}_{m\lambda}) \geq h^0((Y_1 \cap Y_2)_{red}, \mathcal{L}_{m\lambda}) \geq b(m\lambda)_{Y_1 \cap Y_2},$$

it follows by the assumption $h^0(Y_i, \mathcal{L}_\lambda) = b(\lambda)_{Y_i}$ for $m \gg 0$:

$$b(m\lambda)_{(Y_1 \cap Y_2)_{red}} = h^0((Y_1 \cap Y_2)_{red}, \mathcal{L}_{m\lambda}) = h^0(Y_1 \cap Y_2, \mathcal{L}_{m\lambda}),$$

and $b(m\lambda)_{Y_1 \cup Y_2} = h^0(Y_1 \cup Y_2, \mathcal{L}_{m\lambda})$. The first equality implies that $Y_1 \cap Y_2$ is reduced.

We will now use the reducedness of $Y_1 \cap Y_2$ to prove by induction the basis theorem and the vanishing theorem. Let Y be a union of Schubert varieties, let Y_1 be an irreducible component of maximal dimension and let Y_2 be the union of the other maximal irreducible components. We assume by induction (on the dimension respectively the number of components) that $h^0(Y_i, \mathcal{L}_\lambda) = b(\lambda)_{Y_i}$, $b(\lambda)_{Y_1 \cap Y_2} = h^0(Y_1 \cap Y_2, \mathcal{L}_\lambda)$ and $H^j(Y_i, \mathcal{L}_\lambda) = H^j(Y_1 \cap Y_2, \mathcal{L}_\lambda) = 0$ for $j > 0$ and all ample line bundles $\mathcal{L}_\lambda$ on G/P. The long exact sequence:

$$0 \to H^0(Y, \mathcal{L}_\lambda) \to H^0(Y_1, \mathcal{L}_\lambda) \oplus H^0(Y_2, \mathcal{L}_\lambda) \to H^0(Y_1 \cap Y_2, \mathcal{L}_\lambda) \to H^1(Y, \mathcal{L}_\lambda) \to 0$$

implies that $H^j(Y, \mathcal{L}_\lambda) = 0$ for $j \geq 2$. The basis theorem shows that all global sections on $Y_1 \cap Y_2$ can be lifted to global sections on G/P. But this means that the restriction map $H^0(Y_1, \mathcal{L}_\lambda) \oplus H^0(Y_2, \mathcal{L}_\lambda) \to H^0(Y_1 \cap Y_2, \mathcal{L}_\lambda)$ is surjective and hence $H^1(Y, \mathcal{L}_\lambda) = 0$. It follows that $h^0(Y, \mathcal{L}_\lambda) = h^0(Y_1, \mathcal{L}_\lambda) + h^0(Y_2, \mathcal{L}_\lambda) - h^0(Y_1 \cap Y_2, \mathcal{L}_\lambda)$. So the additivity of $b(\lambda)_{(\cdot)}$ implies again that $h^0(Y, \mathcal{L}_\lambda) = b(\lambda)_Y$, which finishes the proof of the theorems. $\square$

7. Defining ideals, standard monomials and Groebner bases

For $\lambda \in X^+$ let $\pi_1 = (\tau_1^1, \dots, \tau_{r_1}^1; \dots, 1)$, $\dots$, $\pi_s = (\tau_1^s, \dots, \tau_{r_s}^s; \dots, 1)$ be a collection of L-S paths of shape λ, and let $p_{\pi_1}, \dots, p_{\pi_s} \in H^0(G/P, \mathcal{L}_\lambda)$ be the corresponding sections.

Definition 7.1. The monomial $p_{\pi_1} \cdot \ldots \cdot p_{\pi_s} \in H^0(G/P, \mathcal{L}_{s\lambda})$ and the concatenation $\pi_1 * \ldots * \pi_s$ of paths are called *standard monomials* of degree s if

$$\tau_1^1 > \ldots > \tau_{r_1}^1 \geq \tau_1^2 > \ldots \geq \tau_1^s > \ldots > \tau_{r_s}^s.$$

The monomial is called standard on $X(\tau) \subset G/P$ if it is standard and $\tau \geq \tau_1^1$.

Theorem 7.2. *The standard monomials of degree s form a basis of $H^0(G/P, \mathcal{L}_{s\lambda})$. The monomials standard on $X(\tau)$ form a basis of $H^0(X(\tau), \mathcal{L}_{s\lambda})$, and the standard monomials which are not standard on $X(\tau)$ form a basis of* $\ker\big(H^0(G/P, \mathcal{L}_{s\lambda}) \to H^0(X(\tau), \mathcal{L}_{s\lambda})\big)$.

Some remarks on the proof . The idea of the proof is very similar to the proof of the basis theorem for the path vectors. The first step is to prove that the standard monomials $\pi_1 * \ldots * \pi_s$ of degree s are (up to reparametrization) exactly the L-S paths of shape $s\lambda$. The bijection is given by

$$p_{\pi_1} \cdot \ldots \cdot p_{\pi_s} \to (\tau_1^1, \dots, \tau_r^1, \dots, \tau_1^s, \dots, \tau_{r_s}^s; 0, \frac{a_1^1}{s}, \dots, \frac{1}{s}, \frac{1 + a_1^2}{s}, \dots, 1).$$

It is understood that we omit $\tau_{r_i}^i$ if $\tau_{r_i}^i = \tau_1^{i+1}$. For details see [39]. For simplicity we assume in the following that $s = 2$. For $\pi = (\tau_1, \dots; 0, a_1, \dots, 1)$ and $\eta = (\kappa_1, \dots; 0, b_1, \dots, 1)$ let ℓ_1 and ℓ_2 be minimal such that they are divisible by $2d$ and $\overline{\ell}_1 a_i \in \mathbb{Z}$ for all i and $\overline{\ell}_2 b_j \in \mathbb{Z}$ for all j.

Consider the sequence of embeddings of $U_{\tilde{R}}(\mathfrak{g})$-modules:

$$V_{\tilde{R}}(2\lambda) \hookrightarrow V_{\tilde{R}}(\lambda) \otimes V_{\tilde{R}}(\lambda) \hookrightarrow \big(N_{\tilde{R}}(\lambda)^{\otimes \overline{\ell}_1}\big)^{\overline{\ell}_1} \otimes \big(N_{\tilde{R}}(\lambda)^{\otimes \overline{\ell}_2}\big)^{\overline{\ell}_2}$$

The same procedure as in the preceding sections can be used to associate to an L-S paths $\pi * \eta$ of shape 2λ (a standard monimial of degree 2) a vector $v_{\pi*\eta}$, and to prove that, considered as an element of the tensor product above, it can be expressed as $m^\pi \otimes m^\eta$ plus a sum of tensor products of weight vectors which are smaller in the (induced lexicographic) ordering.

It follows for two standard monomials $\pi * \pi'$ and $\eta * \eta'$ that $p_\pi p_{\pi'}(v_{\eta*\eta'}) \neq 0$ only if $\pi * \pi' < \eta * \eta'$ in the ordering, and $p_\pi p_{\pi'}(v_{\pi*\pi'}) = 1$.

So we can use the same arguments as before to deduce that the standard monomials of degree s and standard on $X(\tau)$ form a basis of $H^0(X(\tau), \mathcal{L}_{s\lambda})$, and the standard monomials, not standard on $X(\tau)$, form a basis of the kernel of the restriction map $H^0(G/B, \mathcal{L}_{s\lambda}) \to H^0(X(\tau), \mathcal{L}_{s\lambda})$. □

It remains to consider products of path vectors that are not standard. We associate to a pair of L-S paths (π, π'), $\pi = (\tau_1, \dots, 1)$, $\pi' = (\kappa_1, \dots, 1)$, of shape λ a pair of sequences as follows: Fix a total order "$\geq_t$" on W/W_λ refining the Bruhat order. Then let $\pi \wedge \pi' = (\sigma_1, \dots, \sigma_p; 0, c_1, c_1 + c_2 \dots, \sum_{i=1}^p c_i)$ be defined by: $\{\sigma_1, \dots, \sigma_t\} = \{\tau_1, \dots, \kappa_1, \dots\}$, rewritten such that $\sigma_1 \geq_t \ldots \geq_t \sigma_p$, and c_i is equal to $(a_j - a_{j-1})/2$ if $\sigma_i = \tau_j$, $c_i = (b_j - b_{j-1})/2$ if $\sigma_i = \kappa_j$, respectively $c_i(a_j - a_{j-1} + b_{j'} - b_{j'-1})/2$ if $\sigma_i = \tau_j = \kappa_{j'}$.

Note if $\pi * \pi'$ is standard, then obviously $\pi * \pi' = \pi \wedge \pi'$. More generally, we call a rational λ-path a pair of sequences $(\sigma_1, \dots, \sigma_r; 0, c_1, \dots, 1)$ where $\sigma_i \in W/W_\lambda$, the sequence is linearly ordered with respect to $\geq_t$, and $0 < c_1 < \ldots \leq 1$. We extend the total order on W/W_λ lexicographically to the sequences:

$$(\sigma_1, \ldots, \sigma_r; 0, c_1, \ldots, 1) \geq_t (\kappa_1, \ldots, \kappa_s; 0, d_1, \ldots, 1)$$

if $\sigma_1 >_t \kappa_1$, or $\sigma_1 = \kappa_1$ and $c_1 > d_1$, etc. Similarly, we write "$\geq_t^r$" if we extend the total order reverse lexicographically, i.e., if $\sigma_r >_t \kappa_s$ or $\sigma_r = \kappa_s$ and $1 - c_{r-1} > 1 - d_{s-1}$, or $\sigma_r = \kappa_s$ and $1 - c_{r-1} = 1 - d_{s-1}$ and $\sigma_{r-1} >_t \kappa_{s-1}$, etc.

We define two orderings on pairs of L-S paths of shape λ as follows: $(\pi, \pi') \geq_t (\eta, \eta')$ if $\pi \wedge \pi' \geq_t \eta \wedge \eta'$, and if $\pi \wedge \pi' = \eta \wedge \eta'$, then we define $(\pi, \pi') \geq_t (\eta, \eta')$ if $\pi \geq_t \eta$ respectively $\pi = \eta$ and $\pi' \geq_t \eta'$. We define a reverse version of the ordering by $(\pi, \pi') \geq_t^r (\eta, \eta')$ if $\pi \wedge \pi' \geq_t^r \eta \wedge \eta'$ in the reverse lexocigraphic ordering, and if $\pi \wedge \pi' = \eta \wedge \eta'$, then we define $(\pi, \pi') \geq_t^r (\eta, \eta')$ if $\pi' \geq_t^r \eta'$ respectively $\pi' = \eta'$ and $\pi \geq_t \eta$.

Proposition 7.3. *If π, π' are two L-S paths of shape λ, then $p_\pi p_{\pi'} = \sum a_{\eta,\eta'} p_\eta p_{\eta'}$, where $p_\eta p_{\eta'}$ is standard and $a_{\eta,\eta'} \neq 0$ only if $(\eta, \eta') \geq_t (\pi, \pi') \geq_t^r (\eta, \eta')$.*

Proof . The proposition is obviously correct if either $p_\pi p_{\pi'}$ or $p_{\pi'} p_\pi$ is standard. It remains to consider the case where none of the products are standard.

We can repeat the procedure to construct a basis with a different algorithm. For $\pi = (\tau_1, \ldots, \tau_r; 0, a_1, \ldots, 1)$ let s_{α_1} be such that $s_{\alpha_1}\tau_r > \tau_r$. Let j be minimal such that $s_{\alpha_1}\tau_i \geq \tau_i$ for $i = j, \ldots, r$. It is easy to see that

$$\pi' = (\tau_1, \ldots, \tau_{j-1}, s_{\alpha_1}\tau_j, \ldots, s_{\alpha_1}\tau_r; 0, a_1, \ldots, 1)$$

is again an L-S path. Fix n_1 such that $\pi(1) - \pi'(1) = n_1\alpha_{i_1}$, and let $s(n_1, \ldots, n_t)$ be the sequence obtained from π with respect to a reduced decomposition $w_0 = s_{\alpha_t} \cdots s_{\alpha_1}\tau$ of the longest word in the Weyl group. As in section 5, one shows that

$$u_\pi := X_{\alpha_{i_1}}^{(n_1)} \ldots X_{\alpha_{i_t}}^{(n_t)} v_{w_0} = m^\pi + \sum_{m >^r m^\pi} m \in N_R(\lambda)^{\otimes \overline{\ell}}$$

for an appropriate ℓ. The ordering $>^r$ is defined as follows: $m_\tau \geq m_\kappa$ if $\tau \geq \kappa$ in the Bruhat ordering, and for a weight vector $m_\nu \in N_R(\lambda)$ we write $m_\nu \geq^r m_\kappa$ if $m_\nu \in U_{\mathbf{v}}^-(\mathfrak{g}^t) m_\kappa$. On tensor products we take the induced reverse lexicographic partial order "$\geq^r$".

A first observation to make is that we have defined the path vectors p_π according to a minimal choice of an appropriate ℓ, but, in fact, the definition makes sense for an arbitrary ℓ divisible by $2d$ and with the property that $\overline{\ell} a_i \in \mathbb{Z}$ for all i. Using the proposition describing the embedding of v_π into $N(\lambda)^{\otimes \overline{\ell}}$ in section 5 and the description of u_π above, it is easy to check that such a vector $p_{\pi,\ell}$ has the property $p_{\pi,\ell}(v_\eta) \neq 0$ only if $\eta \geq \pi$ and $p_{\pi,\ell}(u_\eta) \neq 0$ only if $\pi \geq^r \eta$. Since p_π has the same properties it follows that $p_{\pi,\ell}$ can be written as $p_\pi +$ a linear combination of p_η's such that $\eta > \pi >^r \eta$.

The second observation is that if $\eta \geq \pi \geq^r \eta$ and $\eta' \geq \pi' \geq^r \eta'$, then $\eta \wedge \eta' \geq_t \pi \wedge \pi' \geq_t^r \eta \wedge \eta'$, and hence $(\eta, \eta') \geq_t (\pi, \pi') \geq_t^r (\eta, \eta')$.

It follows that it is sufficient to prove the proposition for the $p_{\pi,\ell}$ for some appropriate ℓ: If the relation above is correct for the $p_{\pi,\ell}$, then we can replace them by the corresponding linear combination $p_\pi + \sum a_\eta p_\eta$. Of course, there may occur now again non-standard products $p_{\eta_i} p_{\eta'_j}$ after replacing the $p_{\eta,\ell}$ by their expression as linear combination of the p_{η_i}. But since $(\eta, \eta') \geq_t (\pi, \pi') \geq_t^r (\eta, \eta')$ and $\eta_i \geq \eta \geq^r \eta_i$, we know that all terms that occur have the property $(\eta_i, \eta'_j) \geq_t (\pi, \pi') \geq_t^r (\eta_i, \eta'_j)$

One may assume by induction that the relation holds for pairs that are $>_t (\pi, \pi')$ and $<_t^r (\pi, \pi')$ in the ordering. So after replacing these non-standard products by the linear combination of standard products provided by induction, we see that we get the desired relation.

It remains to prove that the relation holds for some appropriate ℓ. Let ℓ be such that $2d$ divides ℓ and $\overline{\ell}c_i \in \mathbb{Z}$ for all L-S paths of shape λ. Then $p_{\pi,\ell}p_{\pi',\ell}(v_{\eta*\eta'}) \neq 0$ for a standard monomial $\eta * \eta'$ only if, in the expression of $v_{\eta*\eta'}$ as element of $N(2\lambda)^{\otimes\overline{\ell}} \subset N(\lambda)^{\otimes 2\overline{\ell}}$, the tensor $m^\pi \otimes m^{\pi'}$ occurs with a coefficient different from zero. The tensor product is not symmetric for quantum groups, but for the F's it is symmetric up to multiplication with a root of unity. So to demand that $m^\pi \otimes m^{\pi'}$ occurs with a coefficient different from zero is equivalent to demand that $m^{\pi\wedge\pi'}$ occurs with a non-zero coefficient. Here the definition of $m^{\pi\wedge\pi'}$ is the same as for L-S paths of shape 2λ. The same arguments as before show that such a tensor can occur only if $m^{\eta*\eta'} \geq m^{\pi\wedge\pi'}$. By the definition of the ordering on the tensors this implies $p_{\pi,\ell}p_{\pi',\ell}(v_{\eta*\eta'}) \neq 0$ only if $\eta * \eta' \geq \pi \wedge \pi'$. In terms of the ordering on pairs this implies $(\eta, \eta') \geq_t (\pi, \pi')$ (because we assume that $\pi * \pi'$ is not standard).

The same arguments apply to $u_{\eta*\eta'}$ and show that $p_{\pi,\ell}p_{\pi',\ell}(u_{\eta*\eta'}) \neq 0$ only if $\pi \wedge \pi' \geq^r \eta * \eta'$. Since $\pi * \pi'$ is not standard, this implies $(\pi, \pi') \geq_t^r (\eta, \eta')$. □

There is a case where we can be a little more precise about one coefficient. We say that two L-S paths have the same support if the τ_i and κ_j can be chosen out of one maximal chain in W/W_λ. It is easy to see that in this case the element $\pi \wedge \pi'$ is an L-S path of shape 2λ. So in this case the set of standard monomials $\eta * \eta'$ such that $(\eta, \eta') \geq_t (\pi, \pi')$ admits a unique minimal element: $\pi \wedge \pi'$. The same arguments as above show that the coefficient of $m^{\pi\wedge\pi'}$ in the expression of $v_{\pi\wedge\pi'}$ as element of $N(2\lambda)^{\otimes\overline{\ell}} \subset N(\lambda)^{\otimes 2\overline{\ell}}$ is 1. Let π_1, π_1' be the two L-S paths of shape λ such that $\pi_1 * \pi_1' = \pi \wedge \pi'$, then we get:

Corollary 7.4. *$p_\pi p_{\pi'} = p_{\pi_1}p_{\pi_1'} + \sum a_{\eta,\eta'}p_\eta p_{\eta'}$, where $p_\eta p_{\eta'}$ is standard and $a_{\eta,\eta'} \neq 0$ only if $(\eta, \eta') >_t (\pi, \pi') >_t^r (\eta, \eta')$.*

Now π and π' have obviously the same support if $\pi = \pi'$. If $da_i \in \mathbb{Z}$ for all a_i and $\pi = (\tau_1, \ldots \tau_r; 0, a_1, \ldots, 1)$, then the corollary shows that as smallest term in the expression of p_π^d as a linear combination of standard monomials, with respect to $\geq_t$, we get the product of extremal weight vectors $p_{\tau_1}^{ta_1} \cdots p_{\tau_r}^{t(1-a_{r-1})}$. In that sense we can consider p_π as an approximation of an $\overline{\ell}$-th root of the section $p_{\tau_1}^{\overline{\ell}a_1} \cdots p_{\tau_r}^{\overline{\ell}(1-a_{r-1})}$ in $H^0(G/P, \mathcal{L}_{\overline{\ell}\lambda})$.

The next theorem states that these relations define already the Schubert variety $X(\tau)$ scheme theoretically as a subvariety of $\mathbb{P}(V(\lambda)_\tau)$.

Theorem 7.5. *Denote by S_τ the free associative algebra $k\{x_\pi \mid i(\pi) \leq \tau\}$, and let I be the ideal obtainend as kernel of the canonical surjective map $S_\tau \to \bigoplus_{n\geq 0} H^0(X(\tau), \mathcal{L}_{n\lambda})$, $x_\pi \mapsto p_\pi$. The relations:*

$$\begin{cases} p_\pi p_{\pi'} - p_{\pi'}p_\pi & \text{if } p_{\pi'}p_\pi \text{ is a standard monomial,} \\ p_\pi p_{\pi'} = \sum_{(\eta,\eta')\geq_t(\pi,\pi')\geq_t^r(\eta,\eta')} a_{(\eta,\eta')}p_\eta p_{\eta'} & \text{if } p_{\pi'}p_\pi, p_\pi p_{\pi'} \text{ are not standard,} \end{cases}$$

form a non-commutative reduced Groebner basis for I.

Remark 7.6. The fact that the relations provide a non-commutative Groebner basis for I provides a new proof of the fact that the ring $\bigoplus_{n\geq 0} H^0(X(\tau), \mathcal{L}_{n\lambda})$ is a Koszul ring. This has been proved before for example by S. P. Inamdar and V. Mehta [15]. Using standard arguments from Groebner basis theory one can use the results above to deform the affine cone over $X(\tau)$ into a union of affine spaces. If one takes into account the refined version given by the corollary, then one sees that one can deform the affine cone into a union of toric varieties, where the irreducible components are indexed by maximal chains in $\{\kappa \in W/W_\lambda \mid \kappa \leq \tau\}$.

Proof . To prove that the set is a generating system for the ideal we have to show that we can express any monomial as a linear combination of standard monomials just by using the relations above. Denote by $I' \subset S_\tau$ the ideal generated by the relations above.

As a first step we define an ordering on the n-tuples of L-S paths satisfying $i(\pi) \leq \tau$. We identify in the following n-tuples with monomials of degree n in S_τ. The notion $\pi_1 \wedge \ldots \wedge \pi_r$ can be generalized in the obvious way, and we say $(\pi_1, \ldots, \pi_n) \geq^r_t (\eta_1, \ldots, \eta_n)$ if $\pi_1 \wedge \ldots \wedge \pi_n >^r \eta_1 \wedge \ldots \wedge \eta_n$, and if $\pi_1 \wedge \ldots \wedge \pi_n = \eta_1 \wedge \ldots \wedge \eta_n$, then we say $(\pi_1, \ldots, \pi_n) \geq^r_t (\eta_1, \ldots, \eta_n)$ if this is true in the induced reverse lexicographic ordering on the tuples.

We extend this order to a total order by saying that a monomial of degree n is strictly greater then a monomial of degree m if $n > m$. It is easy to check that this total order is a left and right monomial order.

Note that if we replace a couple (π_i, π_{i+1}) by a couple $(\pi_i, \pi_{i+1}) >^r_t (\pi'_i, \pi'_{i+1})$, then $(\pi_1, \ldots, \pi_i, \pi_{i+1}, \ldots) >^r_t (\pi_1, \ldots, \pi'_i, \pi'_{i+1}, \ldots)$. Recall that, by the definition of a standard monomial, a monomial $\pi_1 * \ldots * \pi_n$ is standard if and only if $\pi_i * \pi_{i+1}$ is standard for all $i = 1, \ldots, n-1$.

We call a monomial $(\eta_1, \ldots, \eta_n) \in S_\tau$ standard if $\eta_1 * \ldots * \eta_n$ is standard. Start with an arbitrary monomial $(\pi_1, \ldots, \pi_n)$ in S_τ and suppose that $\pi_i * \pi_{i+1}$ is not standard, then, using the relations above, we may replace the monomial by a linear combination of monomials in S_τ that are strictly smaller with respect to $\geq^r_t$. Since there are only a finite number of monomials of a given degree, we obtain after a finite number of steps an expression $(\pi_1, \ldots, \pi_n) \equiv$ a sum of standard monomials mod I', where the standard monomials are all strictly smaller then $(\pi_1, \ldots, \pi_n)$ with respect to $\geq^r_t$. It follows that the map $S_\tau/I' \rightarrow \bigoplus_{n \geq 0} H^0(X(\tau), \mathcal{L}_{n\lambda})$ is an isomorphism.

It remains to prove that the generators form a Groebner basis. The leading terms of the generators are the non-standard monomials of the form (π, π'), so the ideal generated by the leading terms are the linear combinations of all non-standard monomials. Suppose f is an element of I', we have to show that its leading term with respect to $\geq^r_t$ is not a standard monomial. Suppose the contrary is true, so $f = s+$ smaller terms. Let f' be the element obtained from f by replacing all non-standard monomials by their corresponding expression as a sum of standard monomials, this gives a nonzero element of S_τ with leading term s. Modulo I', these two elements are equal, so the image of f' is zero in $\bigoplus_{n \geq 0} H^0(X(\tau), \mathcal{L}_{n\lambda})$. On the other hand, f' is a non-zero sum of standard monomials, so the image cannot be equal to zero. It follows that the leading term cannot be a standard monomial.

¿From the description of the generating set it follows imediately that the basis is reduced. $\square$

8. Standard monomials II

Let $\lambda_1, \dots, \lambda_r$ be some dominant weights, set $\lambda = \sum \lambda_i$, and fix $\tau \in W/W_\lambda$. For each i let τ_i be the image of τ in W/W_{λ_i}. A module V_λ (without specifying the underlying ring) is always meant to be the Weyl module of highest weight λ over an algebraically closed field. The inclusion $V_\lambda \hookrightarrow V_{\lambda_1} \otimes \dots \otimes V_{\lambda_r}$ induces a map $V_\lambda(\tau) \hookrightarrow V_{\lambda_1}(\tau_1) \otimes \dots \otimes V_{\lambda_r}(\tau_r)$, and hence in turn a map $V^*_{\lambda_1}(\tau_1) \otimes \dots \otimes V^*_{\lambda_r}(\tau_r) \to V^*_\lambda(\tau)$.

We write π_i and π_λ for the paths $t \mapsto t\lambda_i$ respectively $t \mapsto t\lambda$. Denote by B_i the set of L-S paths of shape λ_i, and by B_λ the set of paths of shape λ. Recall that the associated graph $G(\pi_\lambda)$ has as vertices the set B_λ, and we put an arrow $\eta \xrightarrow{\alpha} \eta'$ with colour a simple root α if $f_\alpha(\eta) = \eta'$.

Denote by $B_1 * \dots * B_r$ the set of concatenations of all paths in $B_1, \dots, B_r$. Remember that the set of paths is stable under the root operators, and the associated graph decomposes into the disjoint union of irreducible components. Denote by $G(\pi_1 * \dots * \pi_r)$ the irreducible component containing $\pi_1 * \dots * \pi_r$. Recall that the map $\pi_1 * \dots * \pi_r \mapsto \pi_\lambda$ extends to an isomorphism of graphs $\phi : G(\pi_1 * \dots * \pi_r) \to G(\pi_\lambda)$. A monomial $\eta_1 * \dots * \eta_r \in B_1 * \dots * B_r$ is called standard if it is in the irreducible component $G(\pi_1 * \dots * \pi_r)$, and in this case we define: $i(\eta_1 * \dots * \eta_r) := i(\phi(\eta_1 * \dots * \eta_r))$.

Definition 8.1. Let $\eta_1, \dots, \eta_r$ be L-S paths of shape $\lambda_1, \dots, \lambda_r$. A monomial of path vectors $p_{\eta_1} \cdots p_{\eta_r}$ is called *standard* if the concatenation $\eta_1 * \dots * \eta_r$ is standard. The standard monomial is called *standard with respect to* τ if $i(\eta_1 * \dots * \eta_r) \leq \tau$.

The proof of the following theorem is very similar to the proof of the corresponding theorem in the previous section. For details see [39].

Theorem 8.2. *The set of standard monomials form a basis of $H^0(G/B, \mathcal{L}_\lambda)$, and the set of monomials, standard with respect to τ, form a basis of $H^0(X(\tau), \mathcal{L}_\lambda)$.*

9. Determination of the singular locus of $X(w)$

Let Sing $X(w)$ denote the singular locus of $X(w)$. In this section, we recall from [22], [23], [29], [31], [34] the description of Sing $X(w)$. We first recall some generalities on G/Q.

Let G be a semisimple and simply connected algebraic group defined over an algebraically closed field k of arbitrary characteristic. As above, let $T \subset G$ be a maximal torus, and $B \supset T$ be a Borel subgroup. Let W be the Weyl group of G. Let R be the root system of G relative to T. Let R^+ (resp. S) be the system of positive (resp. simple) roots of R with respect to B. Let R^- be the corresponding system of negative roots.

9.1. **The set W_Q^{min} of minimal representatives of W/W_Q.** . Let Q be a parabolic subgroup of G containing B, and W_Q be the Weyl group of Q. In each coset wW_Q, there exists a unique element of minimal length (cf. [5]). Let $W_Q^{\min}$ be this set of representatives of W/W_Q. The set $W_Q^{\min}$ is called the *set of minimal representatives of* W/W_Q. We have

$$W_Q^{\min} = \{w \in W \mid l(ww') = l(w) + l(w'),\ \text{for all } w' \in W_Q\}.$$

The set $W_Q^{\min}$ may also be characterized as

$$W_Q^{\min} = \{w \in W \mid w(\alpha) > 0,\ \text{for all } \alpha \in S_Q\}$$

(here by a root β being > 0 we mean $\beta \in R^+$).

In the sequel, given $w \in W$, the minimal representative of wW_Q in W will be denoted by $w_Q^{\min}$.

9.2. **The set W_Q^{max} of maximal representatives of W/W_Q.** In each coset wW_Q there exists a unique element of maximal length. Let $W_Q^{\max}$ be the set of these representatives of W/W_Q. We have

$$W_Q^{\max} = \{w \in W \mid w(\alpha) < 0 \text{ for all } \alpha \in S_Q\}.$$

Further, if we denote by w_Q the element of maximal length in W_Q, then we have

$$W_Q^{\max} = \{ww_Q \mid w \in W_Q^{\min}\}.$$

In the sequel, given $w \in W$, the maximal representative of wW_Q in W will be denoted by $w_Q^{\max}$.

9.3. **The big cell and the opposite big cell.** The B-orbit Be_{w_0} in G/Q (w_0 being the unique element of maximal length in W) is called the *big cell* in G/Q. It is a dense open subset of G/Q, and it gets identified with $R_u(Q)$, the unipotent radical of Q, namely the subgroup of B generated by $\{U_\alpha \mid \alpha \in R^+ \setminus R_Q^+\}$ (cf. [3]). Let B^- be the Borel subgroup of G opposite to B, i.e. the subgroup of G generated by T and $\{U_\alpha \mid \alpha \in R^-\}$. The B^--orbit $B^- e_{\mathrm{id},Q}$ is called the *opposite big cell* in G/Q. This is again a dense open subset of G/Q, and it gets identified with the unipotent subgroup of B^- generated by $\{U_\alpha \mid \alpha \in R^- \setminus R_Q^-\}$. Observe that both the big cell and the opposite big cell can be identified with $\mathbb{A}^{\mathbb{N}_{\mathbb{Q}}}$, where $N_Q = \#\{R^+ \setminus R_Q^+\}$.

For a Schubert variety $X_Q(w) \subset G/Q$, $Y_Q(w) := B^- e_{\mathrm{id}} \cap X_Q(w)$ is called the *opposite cell* in $X_Q(w)$ (by abuse of language). In general, it is not a cell (except for $w = w_0$). It is a nonempty affine open subvariety of $X_Q(w)$, and a closed subvariety of the affine space $B^- e_{\mathrm{id}}$.

9.4. **Equations defining a Schubert variety.** Let L be an ample line bundle on G/Q. Consider the projective embedding $G/Q \hookrightarrow \mathrm{Proj}(H^0(G/Q, L))$. As a consequence of Standard Monomial Theory - abbreviated as SMT in the sequel - we have seen from the previous section that the homogeneous ideal of G/Q for this embedding is generated in degree 2, and any Schubert variety X in G/Q is scheme theoretically (even at the cone level) the intersection of G/Q with all the hyperplanes in $\mathrm{Proj}(H^0(G/Q, L))$ containing X.

For a maximal parabolic subgroup P_i, let us denote the ample generator of $\mathrm{Pic}\,(G/P_i)$ ($\simeq \mathbb{Z}$) by L_i.

Given a parabolic subgroup Q, let us denote $S \setminus S_Q$ by $\{\alpha_1, \ldots, \alpha_t\}$, for some t. Let

$$R = \bigoplus_{\underline{a}} H^0(G/Q, \bigotimes_i L_i^{a_i})$$

$$R_w = \bigoplus_{\underline{a}} H^0(X_Q(w), \bigotimes_i L_i^{a_i}),$$

where $\underline{a} = (a_1, \ldots, a_t) \in \mathbb{Z}_+^t$. We have that the natural map

$$\bigoplus \mathcal{S}^{\dashv_\infty}(\mathcal{H}'(\mathcal{G}/\mathcal{Q}, \mathcal{L}_\infty)) \otimes \cdots \otimes \mathcal{S}^{\dashv_\infty}(\mathcal{H}'(\mathcal{G}/\mathcal{Q}, \mathcal{L}_\sqcup)) \to \mathcal{R}$$

is surjective, and its kernel is generated as an ideal by elements of total degree 2. Further, the restriction map $R \to R_w$ is surjective, and its kernel is generated as an ideal by elements of total degree 1.

9.5. **Sing $X(w)$.** If $X(w)$ is not smooth, then Sing $X(w)$ is a non-empty B-stable closed subvariety of $X(w)$. Given a point $x \in X(w)$, let $T(w, x)$ denote the Zariski tangent space to $X(w)$ at x. To decide if x is a smooth point or not, it suffices (in view of Bruhat decomposition) to determine if the T-fixed point e_τ of the B-orbit through x is a smooth point or not. We shall denote $T(w, e_\tau)$ by just $T(w, \tau)$. Recall that $\mathrm{d}im\ T(w, \tau) \geq \mathrm{d}im\ X(w)$ $(= l(w))$ with equality if and only if e_τ is a smooth point.

9.6. **A canonical affine neighbourhood of a T-fixed point in G/B.** Let $\tau \in W$. Let U_τ^- be the unipotent part of the Borel subgroup B_τ^- , opposite to B_τ $(= \tau B \tau^{-1})$ (it is the subgroup of G generated $\{U_\alpha \mid \alpha \in \tau(R^-)\}$). Then $U_\tau^- e_\tau$ is an affine neighbourhood of e_τ in G/B, and can be identified with $\mathbb{A}^N$, where $N = \#\{R^+\}$. Let us denote it by $\mathcal{O}_\tau^-$.

For $w \in W,\ w \geq \tau$, let us denote $Y(w, \tau) := \mathcal{O}_\tau^- \cap X(w)$. It is a nonempty affine open subvariety of $X(w)$, and a closed subvariety of the affine space $\mathcal{O}_\tau^-$. Let $I(w, \tau)$ be the ideal defining $Y(w, \tau)$ as a closed subvariety of $\mathcal{O}_\tau^-$. As a consequence of SMT, we have

Proposition 9.6.1. *Let $\mathcal{B}^d$ be the basis for $H^0(G/B, L_{\omega_d}),\ 1 \leq d \leq l$ as given by SMT (here, l is the rank of G, and ω_d is the d^{th} fundamental weight). Then $I(w, \tau)$ is generated by $\{u|_{Y(w,\tau)},\ u \in \mathcal{B}^d,\ 1 \leq d \leq l \mid u|_{X(w)} = 0\}$.*

The problem of the determination of the singular locus of a Schubert variety was first solved by the first author (in collaboration with Seshadri (cf. [31]), for G classical. The main idea in [31] is to write down the equations defining $Y(w, \tau)$ as a closed subvariety of the affine space $\mathcal{O}_\tau^-$ (as given by Proposition 9.6.1), and then use the Jacobian criterion for smoothness. Below, we recall the result of [31] for type **A** and we refer the reader to [31], [22], [23] for results for other classical groups.

9.7. **Description of Sing $X(w)$ for type A.**

Theorem 9.7.1. *(cf. [31]) Let $G = SL(n)$. Let $w, \tau \in W, \tau \leq w$. Then*

$$dim\ T(w, \tau) = \#\{\alpha \in R^+ \mid w \geq \tau s_\alpha\}.$$

9.8. **A criterion for smoothness of Schubert varieties for type A in terms of permutations.** Recall that for $G = SL(n),\ W = \mathcal{S}_n$. First consider $G = SL(4)$. In this case $X(3412), X(4231)$ are the only singular Schubert varieties. The situation for a general n turns out to be "nothing more than this" as given by the following theorem.

Theorem 9.8.1. *(cf. [29]). Let $w \in \mathcal{S}_n$, say $w = (a_1, ..., a_n)$. Then $X(w)$ is singular if and only if the following property holds*

$$\left\{ \begin{matrix} there\ \ exist\ \ i, j, k, l,\ \ 1 \leq i < j < k < l \leq n\ \ such\ \ that \\ either\ \ (1)\ \ a_k < a_l < a_i < a_j\ \ or\ \ (2)\ \ a_l < a_j < a_k < a_i. \end{matrix} \right\}$$

9.9. **Determination of the tangent space.** For $\tau \leq w$, let $T(w, e_\tau)$ be the the tangent space to $X(w)$ at e_τ. Let

$$N_{w,\tau} = \{\beta \in \tau(R^+) \mid X_{-\beta} \in T(w, e_\tau)\}.$$

Note that $T(w, e_\tau)$ is spanned by $\{X_{-\beta} \,|\, \beta \in N_{w,\tau}\}$ (since $T(w, e_\tau)$ is a T-stable subspace of $T(w_0, e_\tau) := \oplus_{\ \mathfrak{b} \in \tau(R^+)} \mathfrak{g}_{-\mathfrak{b}}$ (the tangent space to G/B at e_τ)).

9.10. **Description of $N_w (= N_{w,\mathrm{id}})$.** In [23] (see also [24]), the first author has given a description of N_w for G classical as follows.

Theorem 9.10.1. *Let $\beta \in R^+$.*

1. *Let G be of type $\mathbf{A}_n$. Then $\beta \in N_w \iff w \geq s_\beta$.*
2. *Let G be of type $\mathbf{C}_n$.*
   (a) *Let $\beta = \epsilon_i - \epsilon_j$, or $2\epsilon_i$. Then $\beta \in N_w \iff w \geq s_\beta$.*
   (b) *Let $\beta = \epsilon_i + \epsilon_j$. Then $\beta \in N_w \iff w \geq$ either $s_{\epsilon_i + \epsilon_j}$ or $s_{2\epsilon_i}$.*
3. *Let G be of type $\mathbf{B}_n$.*
   (a) *Let $\beta = \epsilon_i - \epsilon_j$, ϵ_n,* or *$\epsilon_i + \epsilon_n$. Then $\beta \in N_w \iff w \geq s_\beta$.*
   (b) *Let $\beta = \epsilon_i, i < n$. Then $\beta \in N_w \iff w \geq$ either s_{ϵ_i} or $s_{\epsilon_i + \epsilon_n}$.*
   (c) *Let $\beta = \epsilon_i + \epsilon_j, j < n$. Then $\beta \in N_w \iff w \geq$ either $s_{\epsilon_i + \epsilon_j}$ or $s_{\epsilon_i} s_{\epsilon_j + \epsilon_n}$.*
4. *Let G be of type $\mathbf{D}_n$.*
   (a) *Let $\beta = \epsilon_k - \epsilon_l$,* or *$\epsilon_i + \epsilon_j$, $j = n-1, n$. Then $\beta \in N_w \iff w \geq s_\beta$.*
   (b) *Let $\beta = \epsilon_i + \epsilon_j$, $j < n-1$. Then $\beta \in N_w \iff w \geq$ either $s_{\epsilon_i + \epsilon_j}$ or $s_{\epsilon_i - \epsilon_n} s_{\epsilon_i + \epsilon_n} s_{\epsilon_j + \epsilon_{n-1}}$.*

9.11. **Description of $N_{w,\tau}$.** Let $\beta \in \tau(R^+)$, say $\beta = \tau(\alpha), \alpha \in R^+$. We denote the positive roots as in [5].

We now state the descriptions for $N_{w,\tau}$, for G classical (cf.[25]).

9.12. **The special linear group.**

Theorem 9.12.1. *(cf.* [31]*). Let G be of type $\mathbf{A}_n$. Then $\beta \in N_{w,\tau} \iff w \geq s_\beta \tau$.*

9.13. **The symplectic group.**

Theorem 9.13.1. *Let G be of type $\mathbf{C}_n$.*

1. *Let $\alpha = \epsilon_i - \epsilon_j$,* or *$2\epsilon_i$. Then $\beta \in N_{w,\tau} \iff w \geq s_\beta \tau$.*
2. *Let $\alpha = \epsilon_i + \epsilon_j$.*
   (a) *If $\tau > s_\beta \tau$, then $\beta \in N_{w,\tau}$ (necessarily).*
   (b) *Let $\tau < s_\beta \tau$. If τ is $>$ either $\tau s_{2\epsilon_i}$, or $\tau s_{2\epsilon_j}$, then $\beta \in N_{w,\tau} \iff w \geq s_\beta \tau$.*
3. *Let $\tau < s_\beta \tau, \tau s_{2\epsilon_i}$, and $\tau s_{2\epsilon_j}$.*
   (a) *If $\tau < \tau s_{\epsilon_i - \epsilon_j}$, then $\beta \in N_{w,\tau} \iff w \geq$ either $s_\beta \tau$* or *$\tau s_{2\epsilon_i}$.*
   (b) *If $\tau > \tau s_{\epsilon_i - \epsilon_j}$, then $\beta \in N_{w,\tau} \iff w \geq s_\beta \tau s_{2\epsilon_j}$.*

Remark 9.13.2. One has similar descriptions of $N_{w,\tau}$ for types $\mathbf{B}$ and $\mathbf{D}$ (see [25] for details).

9.14. **Irreducible components of Sing $X(w)$.** The problem of the determination of the irreducible components of Sing $X(w)$ is open even for type $\mathbf{A}$.

Known results for G/P.

The irreducible components of Sing $X(w)$ have been determined in [34] for $X(w)$ in G/P, for G classical, and P certain parabolic subgroup. We recall this result below.

TYPE A.

Let $G = SL(n)$, and $P = P_d$, the maximal parabolic subgroup (with associated set of simple roots being $S \setminus \{\alpha_d\}$). Then it is well known that G/P gets identified with the Grassmannian variety $G_{d,n}$ = the set of d- dimensional subspaces of k^n. It is well known that W^{P_d}, the set of minimal representatives may be identified as

$$W^{P_d} = \{(a_1, \cdots, a_d) \mid 1 \leq a_1 < a_2 < \cdots < a_d \leq n\}.$$

To $(a_1, \cdots, a_d) \in W^{P_d}$, we associate the partition $\mathbf{a} := (\mathbf{a}_1, \cdots, \mathbf{a}_d)$, where $\mathbf{a}_i = a_{d-i+1} - d - i + 1$. For a partition $\mathbf{a} = (\mathbf{a}_1, \cdots, \mathbf{a}_d)$, we shall denote by $X_{\mathbf{a}}$ the Schubert variety corresponding to $(a_1, \cdots, a_d)$. Then dim $X_{\mathbf{a}} = |\mathbf{a}| = \mathbf{a}_1 + \cdots + \mathbf{a}_d$. It is clear that $\mathbf{a}_i \leq n - d$. Let $\mathbf{a} = (p_1^{q_1}, \cdots, p_r^{q_r}) = (\underbrace{p_1, \cdots, p_1}_{q_1 \text{ times}}, \cdots, \underbrace{p_r, \cdots, p_r}_{q_r \text{ times}})$ (we say that $\mathbf{a}$ consists of r rectangles: $p_1 \times q_1, \cdots, p_r \times q_r$).

Theorem 9.14.1. *(cf. [34]). Let* $\mathbf{a}$ *consist of* r *rectangles. Then Sing* $X_{\mathbf{a}}$ *has* $r-1$ *components* $X_{\mathbf{a}'_1}, \cdots, X_{\mathbf{a}'_{r-1}}$*, where* $\mathbf{a}'_i = (p_1^{q_1}, \cdots, p_{i-1}^{q_{i-1}}, p_i^{q_i - 1}, (p_{i+1}-1)^{q_{i+1}+1}, p_{i+2}^{q_{i+2}}, \cdots, p_r^{q_r})$*, and* $1 \leq i \leq r-1$.

(Note that $\mathbf{a}/\mathbf{a}'_i$, $1 \leq i \leq r-1$ are simply the hooks in the Young diagram $\mathbf{a}$)

Corollary 9.14.2. $X_{\mathbf{a}}$ *is smooth if and only if* $\mathbf{a}$ *consists of one rectangle.*

TYPE C.

Let $G = \mathrm{Sp}(2n)$, and $P = P_n$, the maximal parabolic with associated set of simple roots being $S \setminus \{\alpha_n\}$ (notations being as in [5]). Then G/P can be identified with the isotropic Grassmannian of n spaces in a $2n$-dimensional space with a non-degenerate skew-symmetric bilinear form (,). Then it can be seen easily that $W_G^{P_n}$, the set of minimal representatives of W_G/W_{P_n} can be identified with

$$\left\{ (a_1 \cdots a_n) \left| \begin{array}{ll} (1) & 1 \leq a_1 < a_2 < \cdots < a_n \leq 2n \\ (2) & \text{for } \ 1 \leq i \leq 2n, \ \text{ if } \ i \in \{a_1, ..., a_n\} \\ & \text{then } \ 2n+1-i \notin \{a_1, ..., a_n\} \end{array} \right. \right\}.$$

To $(a_1, \cdots, a_n) \in W^{P_n}$, we associate the partition $\mathbf{a} := (\mathbf{a}_1, \cdots, \mathbf{a}_n)$, where $\mathbf{a}_{n+1-i} = a_i - i$. The conditions on the a_i's imply that the partition $\mathbf{a}$ is a *self-dual* partition contained in an $n \times n$ square. For a partition $\mathbf{a} = (\mathbf{a}_1, \cdots, \mathbf{a}_n)$, we shall denote by $X_{\mathbf{a}}$ the Schubert variety corresponding to $(a_1, \cdots, a_n)$. Thus Schubert varieties in G/P are indexed by self-dual partitions contained in n^n.

Theorem 9.14.3. *(cf. [34]). Let* $\mathbf{a}$ *be a self-dual partition. Then Sing* $X_{\mathbf{a}} = \cup X_{\mathbf{b}}$*, where* $\mathbf{b} \subset \mathbf{a}$*, and either* $\mathbf{a}/\mathbf{b}$ *is a sum of two hooks that are dual to each other, or* $\mathbf{a}/\mathbf{b}$ *is a self-dual hook (different from a box).*

TYPE B.

Let $G = \mathrm{SO}(2n+1)$, and $P = P_n$, the maximal parabolic with associated set of simple roots being $S \setminus \alpha_n$ (notations being as in [5]). Then G/P can be identified with the isotropic Grassmannian of n spaces in a $2n+1$-dimensional space with a non-degenerate symmetric bilinear form (,). Then it can be seen easily that $W_G^{P_n}$, the set of minimal representatives of W_G/W_{P_n} can be identified with

$$\left\{ (a_1 \cdots a_n) \left| \begin{array}{ll} (1) & 1 \leq a_1 < a_2 < \cdots < a_n \leq 2n+1, \ a_i \neq n+1, 1 \leq i \leq n \\ (2) & \text{for } \ 1 \leq i \leq 2n+1, \ \text{ if } \ i \in \{a_1, ..., a_n\} \\ & \text{then } \ 2n+2-i \notin \{a_1, ..., a_n\} \end{array} \right. \right\}.$$

To $(a_1, \cdots, a_n) \in W^{P_n}$, we associate the partition $\mathbf{a} := (\mathbf{a}_1, \cdots, \mathbf{a}_n)$, where $\mathbf{a}_{n+1-i} = a_i - i$, or $a_i - i - 1$ according as $a_i \leq n$ or $> n$. The conditions on the a_i's imply that the partition $\mathbf{a}$ is a *self-dual* partition contained in an $n \times n$ square. For a partition $\mathbf{a} = (\mathbf{a}_1, \cdots, \mathbf{a}_n)$, we shall denote by $X_{\mathbf{a}}$ the Schubert variety corresponding to $(a_1, \cdots, a_n)$. Thus here again Schubert varieties in G/P are indexed by self-dual partitions contained in n^n.

Theorem 9.14.4. *(cf. [34]). Let $\mathbf{a}$ be a self-dual partition. Then Sing $X_{\mathbf{a}} = \cup X_{\mathbf{b}}$, where $\mathbf{b} \subset \mathbf{a}$, and either $\mathbf{a}/\mathbf{b}$ is a disjoint sum of two hooks that are dual to each other, or, $\mathbf{a}/\mathbf{b} = (r+i, r^{r-1}, 1^i) \ / \ ((r-1)^{r-1})$ for some r, i with $i > 0$ (the sum of two hooks dual to each other connected at one box), or $\mathbf{a}/\mathbf{b} = (r^2, 2^{r-2}) \ /(0^r)$ for some $r > 2$ (self-dual double hook).*

TYPE D.

Let $G = \mathrm{SO}(2n)$, and $P = P_n$, the maximal parabolic with associated set of simple roots being $S \setminus \{\alpha_n\}$ (notations being as in [5]). Then G/P can be identified with the isotropic Grassmannian of n spaces in a $2n$-dimensional space with a non-degenerate symmetric bilinear form (,). Then it can be seen easily that W^{P_n} can be identified as

$$\left\{ (a_1 \cdots a_n) \left| \begin{array}{ll} (1) & 1 \leq a_1 < a_2 < \cdots < a_n \leq 2n \\ (2) & \#\{i, 1 \leq i \leq n \mid a_i > n\} \text{ is even} \\ (3) & \text{for } 1 \leq i \leq 2n, \text{ if } i \in \{a_1, ..., a_n\} \\ & \text{then } 2n+1-i \notin \{a_1, ..., a_n\} \end{array} \right. \right\}.$$

Let $P = P_n, Q = P_{n-1}$. Consider the map $\delta : W^P \to W^Q, \delta(a_1, \cdots, a_n) = (b_1, \cdots, b_{n-1})$, where $(b_1, \cdots, b_{n-1})$ is obtained from $(a_1, \cdots, a_n)$ by replacing n by $n'(= n+1)$ (resp. n' by n) if n (resp. n') is present in $\{a_1, \cdots, a_n\}$. Note that if $a_n > n$, then precisely one of $\{n, n'\}$ is present in $(a_1, \cdots, a_{n-1})$; if $a_n = n$, then $(a_1, \cdots, a_n) = (1, \cdots, n)$, and $\delta(a_1, \cdots, a_n) = (1, \cdots, n-1)$. It is easily seen that δ is a bijection preserving the Bruhat order. In fact δ is induced by the isomorphism of the varieties $G/P \to G/Q$.

Let us denote $W' = W(SO(2n-1))$, and define $\theta : W'^{P_{n-1}} \to W^P$ as $\theta(a_1, \cdots, a_{n-1}) = (a_1, \cdots, a_n)$, where $a_n = n$ or n' and the choice is made so that $\#\{i, 1 \leq i \leq n \mid a_i > n\}$ is even (the i' in $(a_1, \cdots, a_{n-1})$ (resp. $\theta(a_1, \cdots, a_{n-1})$) should be understood as $2n - i$ (resp. $2n+1-i$)). Then it is easily seen that θ is a bijection preserving the Bruhat order. In fact θ is induced by the isomorphism of the varieties $SO(2n-1)/P_{n-1} \to SO(2n)/P$.

In view of the isomorphisms θ and δ, we have results for Schubert varieties in $G/P, G/Q, G$ being $SO(2n)$ similar to Theorem 9.14.4.

Remark 9.14.5. For other related results on Sing $X(w)$, we refer the readers to [7], [21] and [45]

10. Applications to other varieties

In this section, we introduce two classes of affine varieties - certain ladder determinantal varieties (cf. §10.15) and certain quiver varieties (cf. §10.19) - and we conclude (cf. [13], [28]) that these varieties are normal, Cohen-Macaulay and have rational singularities by identifying them with $Y_Q(w)$ (cf. §9.3) for suitable Schubert varieties $X_Q(w)$ in suitable $SL(n)/Q$ (note that $Y_Q(w)$ is normal, Cohen-Macaulay and has rational singularities, since $X_Q(w)$ has all these properties).

We first recall some facts on "Opposite cells" in Schubert varieties in $SL(n)/Q$.

10.1. **Opposite cells in Schubert varieties in $SL(n)/B$.** Let $G = SL(n)$, the special linear group of rank $n-1$. Let T be the maximal torus consisting of all the diagonal matrices in G, and B the Borel subgroup consisting of all the upper triangular matrices in G. It is well-known that W can be identified with $\mathcal{S}_\backslash$, the symmetric group on n letters.

Following [5], we denote the simple roots by $\epsilon_i - \epsilon_{i+1}$, $1 \leq i \leq n-1$ (note that $\epsilon_i - \epsilon_{i+1}$ is the character sending $\operatorname{diag}(t_1, \ldots, t_n)$ to $t_i t_{i+1}^{-1}$). Then $R = \{\epsilon_i - \epsilon_j \mid 1 \leq i, j \leq n\}$, and the reflection $s_{\epsilon_i - \epsilon_{i+1}}$ may be identified with the transposition (i, j) in $\mathcal{S}_\backslash$.

For $\alpha = \alpha_i (= \epsilon_i - \epsilon_{i+1})$, we also denote $P_{\hat{\alpha}}$ (resp. $W_{P_{\hat{\alpha}}}^{\min}$) by just P_i (resp. W^i).

10.2. **The partially ordered set $I_{d,n}$.** Let $Q = P_d$. Then

$$Q = \left\{ A \in G \,\middle|\, A = \begin{pmatrix} * & * \\ 0_{(n-d)\times d} & * \end{pmatrix} \right\},$$
$$W_Q = \mathcal{S}_d \times \mathcal{S}_{n-d}.$$

Hence

$$W_Q^{\min} = \{(a_1 \ldots a_n) \in W \mid a_1 < \cdots < a_d, \quad a_{d+1} < \cdots < a_n\}.$$

Thus $W_Q^{\min}$ may be identified with

$$I_{d,n} := \{\underline{i} = (i_1, \ldots, i_d) \mid 1 \leq i_1 < \cdots < i_d \leq n\}.$$

Given $\underline{i}$,, $\underline{j} \in I_{d,n}$, let $X_{\underline{i}}$, $X_{\underline{j}}$ be the associated Schubert varieties in G/P_d. We define $\underline{i} \geq \underline{j} \iff X_{\underline{i}} \supseteq X_{\underline{j}}$ (in other words, the partial order $\geq$ on $I_{d,n}$ is induced by the Chevalley-Bruhat order on the set of Schubert varieties). In particular, we have

$$\underline{i} \geq \underline{j} \iff i_t \geq j_t \text{ for all } 1 \leq t \leq d.$$

10.3. **The Chevalley-Bruhat order on $\mathcal{S}_n$.** For w_1, $w_2 \in W$, we have

$$X(w_1) \subset X(w_2) \iff \pi_d(X(w_1)) \subset \pi_d(X(w_2)), \text{ for all } 1 \leq d \leq n-1,$$

where π_d is the canonical projection $G/B \to G/P_d$. Hence we obtain that for $(a_1 \ldots a_n)$, $(b_1 \ldots b_n) \in \mathcal{S}_n$,

$$(a_1 \ldots a_n) \geq (b_1 \ldots b_n) \iff (a_1 \ldots a_d) \uparrow \geq (b_1 \ldots b_d) \uparrow, \text{ for all } 1 \leq d \leq n-1$$

(here, for a d-tuple $(t_1 \ldots t_d)$ of distinct integers, $(t_1 \ldots t_d) \uparrow$ denotes the ordered d-tuple obtained from $\{t_1, \ldots, t_d\}$ by arranging its elements in ascending order).

10.4. **The partially ordered set $I_{a_1,\ldots,a_k}$.** Let Q be a parabolic subgroup in $SL(n)$. Let $1 \leq a_1 < \cdots < a_k \leq n$, such that $S_Q = S \setminus \{\alpha_{a_1}, \ldots, \alpha_{a_k}\}$ (we follow [5] for indexing the simple roots). Then $Q = P_{a_1} \cap \cdots \cap P_{a_k}$, and $W_Q = \mathcal{S}_{a_1} \times \mathcal{S}_{a_2 - a_1} \times \cdots \times \mathcal{S}_{n-a_k}$. Let

$$I_{a_1,\ldots,a_k} = \{(\underline{i}_1, \ldots, \underline{i}_k) \in I_{a_1,n} \times \cdots \times I_{a_k,n} \mid \underline{i}_t \subset \underline{i}_{t+1} \text{ for all } 1 \leq t \leq k-1\}.$$

Then it is easily seen that $W_Q^{\min}$ may be identified with $I_{a_1,\ldots,a_k}$.

The partial order on the set of Schubert varieties in G/Q (given by inclusion) induces a partial order $\geq$ on $I_{a_1,\ldots,a_k}$, namely, for $\mathbf{i} = (\underline{i}_1, \ldots, \underline{i}_k)$, $\mathbf{j} = (\underline{j}_1, \ldots, \underline{j}_k) \in I_{a_1,\ldots,a_k}$, $\mathbf{i} \geq \mathbf{j} \iff \underline{i}_t \geq \underline{j}_t$ for all $1 \leq t \leq k$.

10.5. **The minimal and maximal representatives as permutations.** Let $w \in W_Q$, and let $\mathbf{i} = (\underline{i}_1, \dots, \underline{i}_k)$ be the element in $I_{a_1,\dots,a_k}$ which corresponds to $w_Q^{\min}$. As a permutation, the element $w_Q^{\min}$ is given by $\underline{i}_1$, followed by $\underline{i}_2 \setminus \underline{i}_1$ arranged in ascending order, and so on, ending with $\{1, \dots, n\} \setminus \underline{i}_k$ arranged in ascending order. Similarly, as a permutation, the element $w_Q^{\max}$ is given by $\underline{i}_1$ arranged in descending order, followed by $\underline{i}_2 \setminus \underline{i}_1$ arranged in descending order, etc..

10.6. **The opposite big cell in G/Q.** Let $Q = \cap_{t=1}^k P_{a_t}$. Let $a = n - a_k$, and Q be the parabolic subgroup consisting of all the elements of G of the form

$$\begin{pmatrix} A_1 & * & * & \cdots & * & * \\ 0 & A_2 & * & \cdots & * & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_k & * \\ 0 & 0 & 0 & \cdots & 0 & A \end{pmatrix},$$

where A_t is a matrix of size $c_t \times c_t$, $c_t = a_t - a_{t-1}$, $1 \le t \le k$ (here $a_0 = 0$), A is a matrix of size $a \times a$, and $x_{ml} = 0$, $m > a_t$, $l \le a_t$, $1 \le t \le k$. Denote by O^- the subgroup of G generated by $\{U_\alpha \mid \alpha \in R^- \setminus R_Q^-\}$. Then O^- consists of the elements of G of the form

$$\begin{pmatrix} I_1 & 0 & 0 & \cdots & 0 & 0 \\ * & I_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ * & * & * & \cdots & I_k & 0 \\ * & * & * & \cdots & * & I_a \end{pmatrix},$$

where I_t is the $c_t \times c_t$ identity matrix, $1 \le t \le k$, I_a is the $a \times a$ identity matrix, and if $x_{ml} \neq 0$, with $m \neq l$, then $m > a_t$, $l \le a_t$ for some t, $1 \le t \le k$. Further, the restriction of the canonical morphism $f : G \to G/Q$ to O^- is an open immersion, and $f(O^-) \simeq B^- e_{\mathrm{id},Q}$. Thus $B^- e_{\mathrm{id},Q} = \mathcal{O}^-$, the opposite big cell in G/Q gets identified with O^-.

10.7. **Plücker coordinates on the Grassmannian.** Let $G_{d,n}$ be the Grassmannian variety, consisting of d-dimensional subspaces of an n-dimensional vector space V. Let us identify V with k^n, and denote the standard basis of k^n by $\{e_i \mid 1 \le i \le n\}$. Consider the Plücker embedding $f_d : G_{d,n} \hookrightarrow \mathbb{P}(\wedge^d V)$, where $\wedge^d V$ is the d-th exterior power of V. For $\underline{i} = (i_1, \dots, i_d) \in I_{d,n}$, let $e_{\underline{i}} = e_{i_1} \wedge \dots \wedge e_{i_d}$. Then the set $\{e_{\underline{i}} \mid \underline{i} \in I_{d,n}\}$ is a basis for $\wedge^d V$. Let us denote the basis of $(\wedge^d V)^*$ (the linear dual of $\wedge^d V$) dual to $\{e_{\underline{i}} \mid \underline{i} \in I_{d,n}\}$ by $\{p_{\underline{j}} \mid \underline{j} \in I_{d,n}\}$. Then $\{p_{\underline{j}} \mid \underline{j} \in I_{d,n}\}$ gives a system of coordinates for $\mathbb{P}(\wedge^d V)$. These are the so-called *Plücker coordinates.*

10.8. **Schubert varieties in the Grassmannian.** Let $Q = P_d$. We have

$$G_{d,n} \simeq G/P_d.$$

Let $\underline{i} = (i_1, \dots, i_d) \in I_{d,n}$. Then the T-fixed point $e_{\underline{i},P_d}$ is simply the d-dimensional span of $\{e_{i_1}, \dots, e_{i_d}\}$. Thus $X_{P_d}(\underline{i})$ is simply the Zariski closure of $B[e_{i_1} \wedge \dots \wedge e_{i_d}]$ in $\mathbb{P}(\wedge^d V)$.

In view of the Bruhat decomposition for $X_{P_d}(\underline{i})$, we have

$$p_{\underline{j}}\big|_{X_{P_d}(\underline{i})} \neq 0 \iff \underline{i} \ge \underline{j}.$$

10.9. **Evaluation of Plücker coordinates on the opposite big cell in G/P_d.** Consider the morphim $\phi_d : G \to \mathbb{P}(\wedge^d V)$, where $\phi_d = f_d \circ \theta_d$, θ_d being the natural projection $G \to G/P_d$. Then $p_{\underline{j}}(\phi_d(g))$ is simply the minor of g consisting of the first d columns and the rows with indices $j_1, \ldots, j_d$. Now, denote by Z_d the unipotent subgroup of G generated by $\{U_\alpha \mid \alpha \in R^- \setminus R^-_{P_d}\}$. We have, as in §10.6

$$Z_d = \left\{ \begin{pmatrix} I_d & 0_{d\times(n-d)} \\ A_{(n-d)\times d} & I_{n-d} \end{pmatrix} \in G \right\}$$

As in §10.6, we identify Z_d with the opposite big cell in G/P_d. Then, given $z \in Z_d$, the Plücker coordinate $p_{\underline{j}}$ evaluated at z is simply a certain minor of A, which may be explicitly described as follows. Let $\underline{j} = (j_1, \ldots, j_d)$, and let j_r be the largest entry $\leq d$. Let $\{k_1, \ldots, k_{d-r}\}$ be the complement of $\{j_1, \ldots, j_r\}$ in $\{1, \ldots, d\}$. Then this minor of A is given by column indices $k_1, \ldots k_{d-r}$, and row indices $j_{r+1}, \ldots, j_d$ (here the rows of A are indexed as $d+1, \ldots, n$). Conversely, given a minor of A, say, with column indices $b_1, \ldots, b_s$, and row indices $i_{d-s+1}, \ldots, i_d$, it is the evaluation of the Plücker coordinate $p_{\underline{i}}$ at z, where $\underline{i} = (i_1, \ldots, i_d)$ may be described as follows: $\{i_1, \ldots, i_{d-s}\}$ is the complement of $\{b_1, \ldots, b_s\}$ in $\{1, \ldots, d\}$, and $i_{d-s+1}, \ldots, i_d$ are simply the row indices (again, the rows of A are indexed as $d+1, \ldots, n$).

10.10. **Evaluation of the Plücker coordinates on the opposite big cell in G/Q.** Consider

$$f : G \to G/Q \hookrightarrow G/P_{a_1} \times \cdots \times G/P_{a_k} \hookrightarrow \mathbf{P}_1 \times \cdots \times \mathbf{P}_k,$$

where $\mathbf{P}_t = \mathbb{P}(\wedge^{a_t} V)$. Denoting the restriction of f to O^- also by just f, we obtain an embedding $f : O^- \hookrightarrow \mathbf{P}_1 \times \cdots \times \mathbf{P}_k$, O^- having been identified with the opposite big cell $\mathcal{O}^-$ in G/Q. For $z \in O^-$, the multi-Plücker coordinates of $f(z)$ are simply all the $a_t \times a_t$ minors of z with column indices $\{1, \ldots, a_t\}$, $1 \leq t \leq k$.

10.11. **Equations defining the cones over Schubert varieties in $G_{d,n}$.** Let $Q = P_d$. Given a d-tuple $\underline{i} = (i_1, \ldots, i_d) \in I_{d,n}$, let us denote the associated element of $W^{\min}_{P_d}$ by $\theta_{\underline{i}}$. For simplicity of notation, let us denote P_d by just P, and $\theta_{\underline{i}}$ by just θ. Then, by §10.8, $X_P(\theta)$ is simply the Zariski closure of $B[e_{i_1} \wedge \ldots \wedge e_{i_d}]$ in $\mathbb{P}(\wedge^d V)$. Now using §9.4, we obtain that the restriction map $R \to R_\theta$ is surjective, and the kernel is generated as an ideal by $\{p_{\underline{j}} \mid \underline{i} \not\geq \underline{j}\}$.

10.12. **Equations defining multicones over Schubert varieties in G/Q.** Let $X_Q(w) \subset G/Q$. Denoting R, R_w as in §9.4, the kernel of the restriction map $R \to R_w$ is generated by the kernel of $R_1 \to (R_w)_1$; but now, in view of §10.11, this kernel is the span of

$$\{p_{\underline{i}} \mid \underline{i} \in I_{d,n}, d \in \{a_1, \ldots, a_k\},\ w^{(d)} \not\geq \underline{i}\},$$

where $w^{(d)}$ is the d-tuple corresponding to the Schubert variety which is the image of $X_Q(w)$ under the projection $G/Q \to G/P_{a_t}$, $1 \leq t \leq k$.

10.13. **Ideal of the opposite cell in $X_Q(w)$.** Let $Y_Q(w) = B^- e_{\mathrm{id},Q} \cap X_Q(w)$. Then as in §10.6, we identify $B^- e_{\mathrm{id},Q}$ with the unipotent subgroup O^- generated by $\{U_\alpha \mid \alpha \in R^- \setminus R^-_Q\}$, and consider $Y_Q(w)$ as a closed subvariety of O^-. In view of §10.12, we obtain that the ideal defining $Y_Q(w)$ in O^- is generated by

$$\{p_{\underline{i}} \mid \underline{i} \in I_{d,n}, d \in \{a_1, \ldots, a_k\},\ w^{(d)} \not\geq \underline{i}\}.$$

10.14. **The classical determinantal variety.** Let $A = (x_{ij}),\ 1 \leq i \leq m,\ 1 \leq j \leq n$ be a $m \times n$ matrix of variables. Let k be a positive integer such that $k \leq \min(m,n)$, and D_k be the determinantal variety defined by the vanishing of all $k{+}1$ - minors of A. Then one knows (see [30] for example) that D_k can be identified with $Y_Q(w)$ (cf. §10.13) for a suitable Schubert variety $X(w)$ in the Grassmannian $G_{n,m+n}$; in particular, one may conclude that D_k is normal, Cohen-Macaulay and has rational singularities.

10.15. **Ladder determinantal varieties.** Let $X = (x_{ba})$, $1 \leq b \leq m$, $1 \leq a \leq n$ be a $m \times n$ matrix of indeterminates.

Given $1 \leq b_1 < \cdots < b_h < m$, $1 < a_1 < \cdots < a_h \leq n$, we consider the subset of X, defined by

$$L = \{x_{ba} \mid \text{ there exists } 1 \leq i \leq h \text{ such that } b_i \leq b \leq m, 1 \leq a \leq a_i\}.$$

We call L an *one-sided ladder* in X, defined by the *outside corners* $\omega_i = x_{b_i a_i}$, $1 \leq i \leq h$. For simplicity of notation, we identify the variable x_{ba} with just (b,a).

Let $\mathbf{s} = (s_1, s_2 \ldots, s_l) \in \mathbb{Z}_+^l$, $\mathbf{t} = (t_1, t_2 \ldots, t_l) \in \mathbb{Z}_+^l$ such that

$$\begin{gathered} b_1 = s_1 < s_2 < \cdots < s_l \leq m, \\ t_1 \geq t_2 \geq \cdots \geq t_l,\ 1 \leq t_i \leq \min\{m - s_i + 1, a_{i^*}\} \text{ for } 1 \leq i \leq l, \text{ and} \\ s_i - s_{i-1} > t_{i-1} - t_i \text{ for } 1 < i \leq l. \end{gathered} \tag{L1}$$

where for $1 \leq i \leq l$, we let i^* be the largest integer such that $b_{i^*} \leq s_i$.

For $1 \leq i \leq l$, let

$$L_i = \{x_{ba} \in L \mid s_i \leq b \leq m\}.$$

Let $k[L]$ denote the polynomial ring $k[x_{ba} \mid x_{ba} \in L]$, and let $\mathbb{A}(L) = \mathbb{A}^{|L|}$ be the associated affine space. Let $I_{\mathbf{s},\mathbf{t}}(L)$ be the ideal in $k[L]$ generated by all the t_i-minors contained in L_i, $1 \leq i \leq l$, and $D_{\mathbf{s},\mathbf{t}}(L) \subset \mathbb{A}(L)$ the variety defined by the ideal $I_{\mathbf{s},\mathbf{t}}(L)$. We call $D_{\mathbf{s},\mathbf{t}}(L)$ a *ladder determinantal variety* (associated to an one-sided ladder).

Let $\Omega = \{\omega_1, \ldots, \omega_h\}$. For each $1 < j \leq l$, let

$$\Omega_j = \{\omega_i \mid 1 \leq i \leq h \text{ such that } s_{j-1} < b_i < s_j \text{ and } s_j - b_i \leq t_{j-1} - t_j\}.$$

Let

$$\Omega' = (\Omega \setminus \bigcup_{j=2}^{l} \Omega_j) \bigcup_{\Omega_j \neq \emptyset} \{(s_j, a_{j^*})\}.$$

Let L' be the one-sided ladder in X defined by the set of outside corners Ω'. Then it is easily seen that $D_{\mathbf{s},\mathbf{t}}(L) \simeq D_{\mathbf{s},\mathbf{t}}(L') \times \mathbb{A}^d$, where $d = |L| - |L'|$.

Let $\omega'_k = (b'_k, a'_k) \in \Omega'$, for some k, $1 \leq k \leq h'$, where $h' = |\Omega'|$. If $b'_k \notin \{s_1, \ldots, s_l\}$, then $b'_k = b_i$ for some i, $1 \leq i \leq h$, and we define $s_{j^-} = b_i$, $t_{j^-} = t_{j-1}$, $s_{j^+} = s_j$, $t_{j^+} = t_j$, where j is the unique integer such that $s_j < b_i < s_{j+1}$. Let $\mathbf{s}'$ (resp. $\mathbf{t}'$) be the sequence obtained from $\mathbf{s}$ (resp. $\mathbf{t}$) by replacing s_j (resp. t_j) with s_{j^-} and s_{j^+} (resp. t_{j^-} and t_{j^+}) for all k such that $b'_k \notin \{s_1, \ldots, s_l\}$, j being the unique integer such that $s_{j-1} < b_i < s_j$, and i being given by $b'_k = b_i$. Let $l' = |\mathbf{s}'|$. Then $\mathbf{s}'$ and $\mathbf{t}'$ satisfy (L1), and in addition we have $\{b'_1, \ldots, b'_{h'}\} \subset \{s'_1, \ldots, s'_{l'}\}$. It is easily seen that $D_{\mathbf{s},\mathbf{t}}(L') = D_{\mathbf{s}',\mathbf{t}'}(L')$, and hence

$$D_{\mathbf{s},\mathbf{t}}(L) \simeq D_{\mathbf{s}',\mathbf{t}'}(L') \times \mathbb{A}^d.$$

Therefore it is enough to study $D_{\mathbf{s},\mathbf{t}}(L)$ with $\mathbf{s},\mathbf{t} \in \mathbb{Z}_+^l$ such that

$$\{s_1,\ldots,s_l\} \supset \{b_1,\ldots,b_h\}. \tag{L2}$$

Without loss of generality, we can also assume that

$$t_l \geq 2, \text{ and } t_{i-1} > t_i \text{ if } s_i \notin \{b_1,\ldots,b_h\}, 1 < i \leq l. \tag{L3}$$

For $1 \leq i \leq l$, let

$$L(i) = \{x_{ba} \mid s_i \leq b \leq m, 1 \leq a \leq a_{i^*}\}.$$

Note that the ideal $I_{\mathbf{s},\mathbf{t}}(L)$ is generated by the t_i-minors of X contained in $L(i)$, $1 \leq i \leq l$.

The ladder determinantal varieties (associated to one-sided ladders) get related to Schubert varieties(cf. [13]). We describe below the main results of [13].

10.16. **The varieties Z and $X_Q(w)$.** Let $G = SL(n)$, $Q = P_{a_1} \cap \cdots \cap P_{a_h}$. Let $\mathcal{O}^-$ be the opposite big cell in G/Q (cf. §10.6). Let H be the one-sided ladder defined by the outside corners $(a_i + 1, a_i)$, $1 \leq i \leq h$. Let $\mathbf{s},\mathbf{t} \in \mathbb{Z}_+^l$ satisfy (L1), (L2), (L3) above. For each $1 \leq i \leq l$, let $L(i) = \{x_{ba} \mid s_j \leq b \leq n, 1 \leq a \leq a_{i^*}\}$. Let Z be the variety in $\mathbb{A}(H) \simeq \mathcal{O}^-$ defined by the vanishing of the t_i-minors in $L(i)$, $1 \leq i \leq l$. Note that $Z \simeq D_{\mathbf{s},\mathbf{t}}(L) \times \mathbb{A}(H \setminus L) \simeq D_{\mathbf{s},\mathbf{t}}(L) \times \mathbb{A}^r$, where $r = \dim SL(n)/Q - |L|$.

We shall now define an element $w \in W_Q^{\min}$, such that the variety Z identifies with the opposite cell in the Schubert variety $X_Q(w)$ in G/Q. We define $w \in W_Q^{\min}$ by specifying $w^{(a_i)} \in W^{a_i}$ $1 \leq i \leq h$, where $\pi_i(X(w)) = X(w^{(a_i)})$ under the projection $\pi_i : G/Q \to G/P_{a_i}$.

Define $w^{(a_i)}$, $1 \leq i \leq h$, inductively, as the (unique) maximal element in W^{a_i} such that

(1) $w^{(a_i)}(a_i - t_j + 1) = s_j - 1$ for all $j \in \{1,\ldots,l\}$ such that $s_j \geq b_i$, and $t_j \neq t_{j-1}$ if $j > 1$.

(2) if $i > 1$, then $w^{(a_{i-1})} \subset w^{(a_i)}$.

Note that $w^{(a_i)}$, $1 \leq i \leq h$, is well defined in W^i, and w is well defined as an element in $W_Q^{\min}$.

Theorem 10.16.1. *(cf.* [13]*) The variety Z $(= D_{\mathbf{s},\mathbf{t}}(L) \times \mathbb{A}^r)$ identifies with the opposite cell in $X_Q(w)$, i.e. $Z = X_Q(w) \cap \mathcal{O}^-$ (scheme theoretically).*

The above theorem is proved using §10.13. As a consequence of the above Theorem, we obtain (cf. [13])

Theorem 10.16.2. *The variety $D_{\mathbf{s},\mathbf{t}}(L)$ is irreducible, normal, Cohen-Macaulay, and has rational singularities.*

10.17. **The varieties V_i, $1 \leq i \leq l$.** Let V_i, $1 \leq i \leq l$ be the subvariety of $D_{\mathbf{s},\mathbf{t}}(L)$ defined by the vanishing of all $(t_i - 1)$-minors in $L(i)$, where $L(i)$ is as in §10.16.

In [13] the singular locus of $D_{\mathbf{s},\mathbf{t}}(L)$ has also been determined, as described below.

Theorem 10.17.1. *$Sing\, D_{\mathbf{s},\mathbf{t}}(L) = \cup_{i=1}^l V_i$.*

10.18. **The varieties Z_j, $X_Q(\theta_j)$, $1 \leq j \leq l$.** Let us fix $j \in \{1,\ldots,l\}$, and let $Z_j = V_j \times \mathbb{A}(H \setminus L)$. We shall now define $\theta_j \in W_Q^{\min}$ such that the variety Z_j identifies with the opposite cell in the Schubert variety $X_Q(\theta_j)$ in G/Q.

Note that $w^{(a_r)}(a_r - t_j + 1) = s_j - 1$, and $s_j - 1$ is the end of a block of consecutive integers in $w^{(a_r)}$, where $r = j^*$ is the largest integer such that $b_r \leq s_j$. Also, the beginning of this block is ≥ 2 (if the block started with 1, we would have

$a_r - t_j + 1 = s_j - 1 \geq b_r - 1 \geq a_r$, which is not possible, since $t_j \geq 2$). Let $u_j + 1$ be the beginning of this block, where $u_j \geq 1$. Then it is easily seen that if $s_j - 1$ is the end of a block in $w^{(a_i)}$, $1 \leq i \leq h$, then the beginning of the block is $u_j + 1$. For each i, $1 \leq i \leq h$, such that $u_j \notin w^{(a_i)}$, let v_i be the smallest entry in $w^{(a_i)}$ which is bigger than $s_j - 1$. Note that $v_i = w^{(a_i)}(a_k - t_j + 2)$, where $k \in \{1, \ldots, i\}$ is the largest such that $b_k \leq s_j$.

Define ${\theta_j}^{(a_i)}$, $1 \leq i \leq h$, as follows.

If $s_j - 1 \notin w^{(a_i)}$ (which is equivalent to $j > 1$, $t_{j-1} = t_j$ and $i < r$), let $\theta_j^{(a_i)} = w^{(a_i)} \setminus \{v_i\} \cup \{s_j - 1\}$.

If $s_j - 1 \in w^{(a_i)}$ and $u_j \notin w^{(a_i)}$, then $\theta_j^{(a_i)} = w^{(a_i)} \setminus \{v_i\} \cup \{u_j\}$.

If $s_j - 1$ and $u_j \in w^{(a_i)}$, then $\theta_j^{(a_i)} = w^{(a_i)}$ (note that in this case $i > r$).

Note that θ_j is well defined as an element in $W_Q^{\min}$, and $\theta_j \leq w$.

Remark 10.18.1. An equivalent description of θ_j is the following. Let $t_{i_k} < t_j \leq t_{i_{k-1}}$.

(I) If $j \notin \{i_1, \ldots, i_m\}$ (i.e. $j > 1$ and $t_{j-1} = t_j$), then

for $i < r$, $\theta_j^{(a_i)} = w_j^{(a_i)} \setminus \{e_{i_k}\} \cup \{s_j - 1\}$;

for $i = r$, $\theta_j^{(a_r)} = w_j^{(a_r)} \setminus \{e_{i_k}\} \cup \{u_j\}$, where u_j is the largest entry in $\{1, \ldots, s_j - 1\} \setminus w^{(a_r)}$;

for $i > r$ and $u_j \in w^{(a_i)}$, $\theta_j^{(a_i)} = w_j^{(a_i)}$;

for $i > r$ and $u_j \notin w^{(a_i)}$, $\theta_j^{(a_i)} = w_j^{(a_i)} \setminus \{v_i\} \cup \{u_j\}$, where v_i is the smallest entry in $w^{(a_i)} \setminus \theta_j^{(a_{i-1})}$.

(II) If $j \in \{i_1, \ldots, i_m\}$, (i.e. $t_{j-1} > t_j$ if $j > 1$), then

for $i \leq r$, $\theta_j^{(a_i)} = w_j^{(a_i)} \setminus \{e_{i_k}\} \cup \{u_j\}$, where u_j is the largest entry in $\{1, \ldots, s_j - 1\} \setminus w^{(a_r)}$;

for $i > r$ and $u_j \in w^{(a_i)}$, $\theta_j^{(a_i)} = w_j^{(a_i)}$;

for $i > r$ and $u_j \notin w^{(a_i)}$, $\theta_j^{(a_i)} = w_j^{(a_i)} \setminus \{v_i\} \cup \{u_j\}$, where v_i is the smallest entry in $w^{(a_i)} \setminus \theta_j^{(a_{i-1})}$.

Theorem 10.18.2. *(cf. [13]) The subvariety $Z_j \subset Z$ identifies with the opposite cell in $X_Q(\theta_j)$, i.e. $Z_j = X_Q(\theta_j) \cap \mathcal{O}^-$ (scheme theoretically).*

As a consequence of the above theorem, we obtain (cf. [13])

Theorem 10.18.3. *The irreducible components of $Sing\, D_{\mathbf{s},\mathbf{t}}(L)$ are precisely the V_j's, $1 \leq j \leq l$.*

Let $X(w^{\max})$ (resp. $X(\theta_j^{\max})$, $1 \leq j \leq l$) be the pull-back in $SL(n)/B$ of $X_Q(w)$ (resp. $X_Q(\theta_j)$, $1 \leq j \leq l$) under the canonical projection $\pi : SL(n)/B \to SL(n)/Q$ (here B is a Borel subgroup of $SL(n)$ such that $B \subset Q$). Then using Theorems 10.16.1, 10.18.2 and 10.18.3 above, we obtain (cf. [13])

Theorem 10.18.4. *The irreducible components of $Sing\, X(w^{max})$ are precisely $X(\theta_j^{max})$, $1 \leq j \leq l$.*

In [13], it is also shown that the conjecture of [29] on the irreducible components of Sing $X(\theta)$, $\theta \in W$ holds for $X(w^{\max})$.

Remark 10.18.5. Ladder determinantal varieties were first introduced by Abyankar (cf. [2]).

Remark 10.18.6. A similar identification as in Theorem 10.16.1 for the case $t_1 = \cdots = t_l$ has also been obtained by Mulay (cf. [43]).

Remark 10.18.7. In [13], the theory of Schubert varieties and the theory of ladder determinantal varieties are complementing each other. To be more precise, geometric properties such as normality, Cohen-Macaulayness, etc., for ladder determinantal varieties are concluded by relating these varieties to Schubert varieties. The components of singular loci of Schubert varieties are determined by first determining them for ladder determinantal varieties, and then using the above mentioned relationship between ladder determinantal varieties and Schubert varieties.

10.19. **Quiver varieties.** Fulton [10] and Buch-Fulton [6] have recently given a theory of "universal degeneracy loci", characteristic classes associated to maps among vector bundles, in which the role of Schubert varieties is taken by certain degeneracy schemes. The underlying varieties of these schemes arise in the theory of quivers: they are the closures of orbits in the space of representations of the equioriented quiver A_h. Many other classical varieties also appear as quiver varieties, such as determinantal varieties and the variety of complexes (see [8], [14], [44].)

In [28], the quiver varieties (corresponding to the equioriented type A quiver) are shown to be normal and Cohen-Macaulay (in arbitrary characteristic) by identifying them with $Y_Q(w)$ (cf. §9.3) for suitable Schubert varieties $X_Q(w)$ in suitable $SL(n)/Q$.

Fix an h-tuple of non-negative integers $\mathbf{n} = (\mathbf{n_1}, \ldots, \mathbf{n_h})$ and a list of vector spaces $V_1, \ldots, V_h$ over an arbitrary field $\mathbf{k}$ with respective dimensions $n_1, \ldots, n_h$. Define Z, the *variety of quiver representations* (of dimension $\mathbf{n}$, of the equioriented quiver of type A_h) to be the affine space of all $(h-1)$-tuples of linear maps $(f_1, \ldots, f_{h-1})$:

$$V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{h-2}} V_{h-1} \xrightarrow{f_{h-1}} V_h \ .$$

If we endow each V_i with a basis, we get $V_i \cong \mathbf{k}^{\mathbf{n_i}}$ and

$$Z \cong M(n_2 \times n_1) \times \cdots \times M(n_h \times n_{h-1}),$$

where $M(l \times m)$ denotes the affine space of matrices over $\mathbf{k}$ with l rows and m columns. The group

$$G_{\mathbf{n}} = GL(n_1) \times \cdots \times GL(n_h)$$

acts on Z by

$$(g_1, g_2, \cdots, g_h) \cdot (f_1, f_2, \cdots, f_{h-1}) = (g_2 f_1 g_1^{-1}, g_3 f_2 g_2^{-1}, \cdots, g_h f_{h-1} g_{h-1}^{-1}),$$

corresponding to change of basis in the V_i.

Now, let $\mathbf{r} = (r_{ij})_{1 \le i \le j \le h}$ be an array of non-negative integers with $r_{ii} = n_i$, and define $r_{ij} = 0$ for any indices other than $1 \le i \le j \le h$. Define the set

$$Z^\circ(\mathbf{r}) = \{(f_1, \cdots, f_{h-1}) \in Z \ \mid \ \forall\, i < j,\ \operatorname{rank}(f_{j-1} \cdots f_i : V_i \to V_j) = r_{ij}\}.$$

(This set might be empty for a bad choice of $\mathbf{r}$.)

Proposition 10.19.1. *(cf.* [11]*) The $G_{\mathbf{n}}$-orbits of Z are exactly the sets $Z^\circ(\mathbf{r})$ for $\mathbf{r} = (r_{ij})$ with*

$$r_{ij} - r_{i,j+1} - r_{i-1,j} + r_{i-1,j+1} \ge 0, \quad \forall\ 1 \le i < j \le h.$$

Definition 10.19.2. We define the *quiver variety* as the algebraic set

$$Z(\mathbf{r}) = \{(f_1, \cdots, f_{h-1}) \in Z \mid \forall i, j,\ \operatorname{rank}(f_{j-1} \cdots f_i : V_i \to V_j) \le r_{ij}\}.$$

Remark 10.19.3. The variety $Z(\mathbf{r})$ is simply the Zariski closure of $Z^\circ(\mathbf{r})$ (cf. [1], [28]).

10.20. **The Schubert varieties.** Given $\mathbf{n} = (\mathbf{n_1}, \cdots, \mathbf{n_h})$, for $1 \le i \le h$ let

$$a_i = n_1 + n_2 + \cdots + n_i, \qquad a_0 = 0, \qquad \text{and} \qquad n = n_1 + \cdots + n_h \ .$$

For positive integers $i \le j$, we shall frequently use the notations

$$[i, j] = \{i, i+1, \ldots, j\}, \qquad [i] = [1, i], \qquad [0] = \{\}\ .$$

Let $\mathbf{k^n} \cong \mathbf{V_1} \oplus \cdots \oplus \mathbf{V_h}$ have basis $e_1, \ldots, e_n$ compatible with the V_i. Consider its general linear group $GL(n)$, the subgroup B of upper-triangular matrices, and the parabolic subgroup Q of block upper-triangular matrices

$$Q = \{(a_{ij}) \in GL(n) \mid a_{ij} = 0 \text{ whenever } j \le a_k < i \text{ for some } k\}\ .$$

In this section, we look at G/Q as the space of partial flags as follows: a *partial flag of type* $(a_1 < a_2 < \cdots < a_h = n)$ (or simply a *flag*) is a sequence of subspaces $U_\bullet = (U_1 \subset U_2 \subset \cdots \subset U_h = \mathbf{k^n})$ with $\dim U_i = a_i$. Let $E_i = V_1 \oplus \cdots \oplus V_i = \langle e_1, \ldots, e_{a_i}\rangle$, and $E'_i = V_{i+1} \oplus \cdots \oplus V_h = \langle e_{a_i+1}, \ldots, e_n\rangle$, so that $E_i \oplus E'_i = \mathbf{k^n}$. The *flag variety* Fl is the set of all flags $U_\bullet$ as above. Fl has a transitive $GL(n)$-action induced from $\mathbf{k^n}$, and $Q = \operatorname{Stab}_{GL(n)}(E_\bullet)$, so we have the identification $\mathrm{Fl} \cong GL(n)/Q,\ g{\cdot}E_\bullet \leftrightarrow gQ$. The *Schubert varieties* are the closures of B-orbits on Fl. Such orbits are usually indexed by certain permutations of $[n]$, but we prefer to use *flags of subsets* of $[n]$, of the form

$$\tau = (\tau_1 \subset \tau_2 \subset \cdots \subset \tau_h = [n]), \qquad \#\tau_i = a_i\ .$$

A permutation $w : [n] \to [n]$ corresponds to the subset-flag with

$$\tau_i = w[a_i] = \{w(1), w(2), \ldots, w(a_i)\}.$$

This gives a one-to-one correspondence between cosets of the symmetric group $W = \mathcal{S}_n$ modulo the Young subgroup $W_{\mathbf{n}} = \mathcal{S}_{n_1} \times \cdots \times \mathcal{S}_{n_h}$, and subset-flags.

Given such τ, let $E_i(\tau) = \langle e_j \mid j \in \tau_i\rangle$ be a coordinate subspace of $\mathbf{k^n}$, and $E_\bullet(\tau) = (E_1(\tau) \subset E_2(\tau) \subset \cdots) \in \mathrm{Fl}$. Then we may define the *Schubert cell*

$$\begin{aligned} X_Q^\circ(\tau) &= B \cdot E(\tau) \\ &= \left\{ (U_1 \subset U_2 \subset \cdots) \in \mathrm{Fl} \;\middle|\; \begin{array}{c} \dim U_i \cap \mathbf{k^j} = \#\,\tau_\mathbf{i} \cap [\mathbf{j}] \\ 1 \le i \le h,\ 1 \le j \le n \end{array} \right\} \end{aligned}$$

and the *Schubert variety*

$$\begin{aligned} X_Q(\tau) &= \overline{X_Q^\circ(\tau)} \\ &= \left\{ (U_1 \subset U_2 \subset \cdots) \in \mathrm{Fl} \;\middle|\; \begin{array}{c} \dim U_i \cap \mathbf{k^j} \ge \#\,\tau_\mathbf{i} \cap [\mathbf{j}] \\ 1 \le i \le h,\ 1 \le j \le n \end{array} \right\} \end{aligned}$$

where $\mathbf{k^j} = \langle \mathbf{e_1}, \ldots, \mathbf{e_j}\rangle \subset \mathbf{k^n}$.

Under the identification of G/Q with Fl, the opposite cell $\mathcal{O}^-$ in G/Q gets identified with the set of flags in general position with respect to the spaces $E'_1 \supset \cdots \supset E'_{h-1}$:

$$\mathcal{O}^- = \{(U_1 \subset U_2 \subset \cdots) \in \mathrm{Fl} \;\mid\; U_i \cap E'_i = 0\}.$$

Let $Y_Q(\tau) = X_Q(\tau) \cap \mathcal{O}^-$, the opposite cell of $X(\tau)$.

We define a special subset-flag $\tau^{\max} = (\tau_1^{\max} \subset \cdots \subset \tau_h^{\max} = [n])$ corresponding to $\mathbf{n} = (\mathbf{n_1}, \dots, \mathbf{n_h})$. We want each $\tau_i^{\max}$ to contain numbers as large as possible given the constraints $[a_{j-1}] \subset \tau_j^{\max}$ for all j. Namely, we define $\tau_i^{\max}$ recursively by

$$\tau_h^{\max} = [n]; \quad \tau_i^{\max} = [a_{i-1}] \cup \{\text{largest } n_i \text{ elements of } \tau_{i+1}^{\max}\}.$$

Furthermore, given $\mathbf{r} = (r_{ij})_{1 \le i \le j \le h}$ indexing a quiver variety, define a subset-flag $\tau^{\mathbf{r}}$ to contain numbers as large as possible given the constraints

$$\#\, \tau_i^{\mathbf{r}} \cap [a_j] = \begin{cases} a_i - r_{i,j+1} & \text{for } i \le j \\ a_j & \text{for } i > j \end{cases}$$

Namely,

$$\tau_i^{\mathbf{r}} = \{ \underbrace{1 \dots a_{i-1}}_{a_{i-1}} \ \underbrace{\dots\dots a_i}_{r_{ii} - r_{i,i+1}} \ \underbrace{\dots\dots a_{i+1}}_{r_{i,i+1} - r_{i,i+2}} \ \underbrace{\dots\dots a_{i+2}}_{r_{i,i+2} - r_{i,i+3}} \ \cdots \ \underbrace{\dots\dots n}_{r_{i,h}} \}$$

where we use the visual notation

$$\underbrace{\cdots\cdots a}_{b} = [a - b + 1, a].$$

Recall that $a_j = a_{j-1} + n_j$ and $0 \le r_{ij} - r_{i,j+1} \le n_j$, so that each $\tau_i^{\mathbf{r}}$ is an increasing list of integers. Also $r_{ij} - r_{i,j+1} \le r_{i+1,j} - r_{i+1,j+1}$, so that $\tau_i^{\mathbf{r}} \subset \tau_{i+1}^{\mathbf{r}}$. Thus, $\tau^{\mathbf{r}}$ are indeed subset-flags.

10.21. **Examples.** We give below four examples.

Example 1 *A small generic case.*
Let $h = 4$, $\mathbf{n} = (\mathbf{2}, \mathbf{3}, \mathbf{2}, \mathbf{2})$,

$$\mathbf{r} = \boxed{\begin{matrix} 2 & 2 & 0 & 0 \\ & 3 & 1 & 1 \\ & & 2 & 2 \\ & & & 2 \end{matrix}}$$

where r_{ij} are written in the usual matrix positions.

Then we get $(a_1, a_2, a_3, a_4) = (2, 5, 7, 9)$, $n = 9$, and

$$\tau^{\max} = (89 \subset 12589 \subset 1234589 \subset [9]), \qquad \tau^{\mathbf{r}} = (45 \subset 12459 \subset 1234589 \subset [9]),$$

which correspond to the cosets in $W/W_{\mathbf{n}}$

$$w^{\max} = 89|125|34|67, \qquad w^{\mathbf{r}} = 45|129|38|67.$$

(The minimal-length representatives of these cosets are the permutations as written; the other elements are obtained by permuting numbers within each block.) The partial flag variety is $\mathrm{Fl} = \{U_1 \subset U_2 \subset U_3 \subset \mathbf{k^9} \mid \dim \mathbf{U_i} = \mathbf{a_i}\}$, and the Schubert varieties are:

$$X_Q(\tau^{\max}) = \left\{ U_\bullet \;\middle|\; \begin{matrix} \mathbf{k^2} \subset \mathbf{U_2} \\ \mathbf{k^5} \subset \mathbf{U_3} \end{matrix} \right\}, \quad X_Q(\tau^{\mathbf{r}}) = \left\{ U_\bullet \;\middle|\; \begin{matrix} U_1 \subset \mathbf{k^5} \subset \mathbf{U_3},\ \mathbf{k^2} \subset \mathbf{U_2} \\ \dim U_2 \cap \mathbf{k^5} \ge \mathbf{4} \end{matrix} \right\}.$$

The opposite cells $Y_Q(\tau)$ are defined by the extra conditions $U_i \cap E_i' = 0$.

Example 2. *Fulton's universal degeneracy schemes* (cf. [10]).
Given $m > 0$, let Z be the affine space associated to the quiver data $h = 2m$, $\mathbf{n} = (\mathbf{1}, \mathbf{2}, \dots, \mathbf{m}, \mathbf{m}, \dots, \mathbf{2}, \mathbf{1})$. For each $w \in \mathcal{S}_{m+1}$, Fulton defines a "degeneracy

scheme" $\Omega_w = Z(\mathbf{r})$ as follows. Denote $\bar{i} = 2m+1-i$, and define $\mathbf{r} = \mathbf{r}(w) = (r_{ij})$ by:

$$r_{ij} = r_{\overline{j}\overline{i}} = i$$
$$r_{i\overline{j}} = \#\,[i] \cap w[j]$$

for $1 \leq i, j \leq m$. The associated Schubert varieties $X_Q(\tau^{\mathbf{r}})$ are given by $\tau^{\mathbf{r}} = (\tau^{\mathbf{r}}_1 \subset \cdots \subset \tau^{\mathbf{r}}_{\bar{1}})$ or by cosets $\widetilde{w} = \widetilde{w}_1 | \cdots | \widetilde{w}_{\bar{1}} \in W/W_{\mathbf{n}}$

$$\tau^{\mathbf{r}}_i = [a_{i-1}] \cup \{a_{\overline{w^{-1}(1)}}, a_{\overline{w^{-1}(2)}}, \ldots, a_{\overline{w^{-1}(i)}}\}, \qquad \tau^{\mathbf{r}}_{\bar{i}} = [a_{\bar{i}} - 1] \cup \{a_{\bar{1}}, a_{\bar{2}}, \ldots, a_{\overline{m}}\}$$

$$\widetilde{w}_i = [a_{i-2}+1, a_{i-1}] \cup \{a_{\overline{w^{-1}(i)}}\}, \qquad \widetilde{w}_{\overline{m}} = [a_{m-1}+1, a_m - 1] \cup \{a_{\overline{w^{-1}(m+1)}}\}, \qquad \widetilde{w}_{\bar{j}} = [a_{\overline{j}-2}+1, a_{\overline{j}-1}]$$

for $1 \leq i \leq m$, $1 \leq j \leq m-1$. Furthermore $\tau^{\max} = \tau^{\mathbf{r}(w)}$ and $\widetilde{w}^{\max} = \widetilde{w}^{\mathbf{r}(w)}$ for $w = e \in \mathcal{S}_{m+1}$, the identity permutation.

Example 3. *The variety of complexes.*
For a given h and $\mathbf{n}$, the *variety of complexes* is defined as the union $\mathcal{C} = \cup_{\mathbf{r}} Z(\mathbf{r})$ over all $\mathbf{r} = (r_{ij})$ with $r_{i,i+2} = 0$ for all i. The subvarieties $Z(\mathbf{r})$ correspond to the multiplicity matrices $\mathbf{m} = (m_{ij})$ with $m_{ij} = 0$ for all $i+2 \leq j$, and $m_{ii} + m_{i-1,i} + m_{i,i+1} = n_i$ for all i. In [44], Musili-Seshadri have shown that each component of $\mathcal{C}$, is isomorphic to the opposite cell in a Schubert variety.

Example 4. *The classical determinantal variety.*
The classical *determinantal variety* of $k \times l$ matrices of rank $\leq t$ is $\mathcal{D} = Z(r)$ for $\mathbf{r} = \begin{pmatrix} l & m \\ 0 & k \end{pmatrix}$ and $\mathbf{m} = \begin{pmatrix} l-m & m \\ 0 & k-m \end{pmatrix}$ where $m = \min(t+1, k, l)$. Also $n = k + l$,

$$\tau^{\max} = ([k+1, k+l] \subset [n]), \quad \tau^{\mathbf{r}} = ([m+1, l] \cup [k+l-m+1, k+l] \subset [n])$$
$$X(\tau^{\max}) = \mathrm{Fl} = \mathrm{Gr}(l, \mathbf{k}^{\mathbf{n}}), \quad \mathbf{X}(\tau^{\mathbf{r}}) \cong \{\mathbf{U} \in \mathrm{Gr}(\mathbf{l}, \mathbf{k}^{\mathbf{n}}) \mid \mathbf{U} \cap \mathbf{k}^{\mathbf{l}} = \mathbf{l} - \mathbf{m}\},$$
$$\mathcal{D} = Z(\mathbf{r}) \cong Y(\tau^{\mathbf{r}}) = \{U \in \mathrm{Gr}(l, \mathbf{k}^{\mathbf{n}}) \mid \mathbf{U} \cap \mathbf{k}^{\mathbf{l}} = \mathbf{l} - \mathbf{m},\ \mathbf{U} \cap \mathbf{E}' = \mathbf{0}\},$$

where $E' = \langle e_{l+1}, e_{l+2}, \ldots, e_n \rangle$.

Denote a generic element of the quiver space $Z = M(n_2 \times n_1) \times \cdots \times M(n_h \times n_{h-1})$ by $(A_1, \ldots, A_{h-1})$, so that the coordinate ring of Z is the polynomial ring in the entries of all the matrices A_i. Let $\mathbf{r} = (r_{ij})$ index the quiver variety $Z(\mathbf{r}) = \{(A_1, \ldots, A_{h-1}) \mid \operatorname{rank} A_{j-1} \cdots A_i \leq r_{ij}\}$.

Let $\mathcal{J}(\mathbf{r}) \subset \mathbf{k}[\mathbf{Z}]$ be the ideal generated by the determinantal conditions implied by the definition of $Z(\mathbf{r})$:

$$\mathcal{J}(\mathbf{r}) = \left\langle \det(A_{j-1} A_{j-2} \cdots A_i)_{\lambda \times \mu} \;\middle|\; \begin{array}{c} j > i,\ \lambda \subset [n_j],\ \mu \subset [n_i] \\ \#\lambda = \#\mu = r_{ij} + 1 \end{array} \right\rangle .$$

Clearly $\mathcal{J}(\mathbf{r})$ defines $Z(\mathbf{r})$ set-theoretically.

Theorem 10.21.1. *(cf.* [28]*) $\mathcal{J}(\mathbf{r})$ is a prime ideal and is the vanishing ideal of $Z(\mathbf{r}) \subset Z$. There are isomorphisms of reduced schemes*

$$Z(\mathbf{r}) = \mathit{Spec}(\mathbf{k}[\mathbf{Z}] \,/\, \mathcal{J}(\mathbf{r})) \cong \mathit{Spec}(\mathbf{k}[\mathcal{O}^-] \,/\, \mathcal{I}(\tau^{\mathbf{r}})) = \mathbf{Y_Q}(\tau^{\mathbf{r}}).$$

That is, the quiver scheme $Z(\mathbf{r})$ defined by $\mathcal{J}(\mathbf{r})$ is isomorphic to the (reduced) variety $Y_Q(\tau^{\mathbf{r}})$, the opposite cell of a Schubert variety.

In proving the above theorem again, one uses the standard monomial theory for Schubert varieties.

Remark 10.21.2. Over a field of characteristic 0, the normality and Cohen-Macaulayness of $Z(\mathbf{r})$ also follow from [1].

11. Bott-Samelson varieties

Throughout this section, we once again take G to be a simply connected semisimple algebraic group over an algebraically closed field k.

11.1. **Geometry.** The Bott-Samelson varieties are an important tool in the representation theory of the group G and the geometry of the flag variety G/B. First defined in [4] as a desingularization of the Schubert varieties in G/B, they were exploited by Demazure [9] to analyze the singular cohomology or Chow ring $H^{\cdot}(G/B)$ (the Schubert calculus), and the projective coordinate ring $k[G/B]$. Since the irreducible representations of G are embedded in the coordinate ring, Demazure was able to obtain a new iterative character formula for these representations.

Bott-Samelson varieties are so useful because they "factor" the flag variety into a "product" of projective lines. More precisely, they are iterated $\mathbb{P}^1$-fibrations and they each have a natural, birational map to G/B. The Schubert subvarieties themselves lift birationally to iterated $\mathbb{P}^1$-fibrations under this map (hence the desingularization). The combinatorics of Weyl groups enters because a given G/B can be "factored" in many ways, indexed by sequences $\mathbf{i} = (i_1, i_2, \dots, i_N)$ such that $w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}$ is a reduced decomposition of the longest Weyl group element w_0 into simple reflections.

More generally, we may define a Bott-Samelson variety $Z_{\mathbf{i}}$ for an arbitrary reduced or non-reduced sequence of indices $\mathbf{i} = (i_1, i_2, \dots, i_N)$. Let $P_k \supset B$ be the minimal parabolic associated to the simple reflection s_k so that $P_i/B \cong \mathbb{P}^1$, the projective line. Then

$$Z_{\mathbf{i}} = P_{i_1} \times \cdots \times P_{i_N}/B^N,$$

where B^N acts on the right of the product via:

$$(p_1, p_2, \dots, p_N) \cdot (b_1, b_2, \dots, b_N) = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{N-1}^{-1} p_N b_N).$$

Furthermore, B acts on the left of $Z_{\mathbf{i}}$ by multiplication of the first factor.

Although we will not use it here, a key structure in analyzing the geometry of $Z_{\mathbf{i}}$ (and hence G/B) is the *opposite big cell*

$$\begin{array}{ccc} k^N & \to & Z_{\mathbf{i}} \\ (t_1, \dots, t_N) & \mapsto & (\exp(t_1 F_{i_1}), \dots, \exp(t_N F_{i_N})), \end{array}$$

where $t \mapsto \exp(tF_i)$ is the exponential map onto the one-parameter unipotent subgroup corresponding to the negative simple root α_i. The image of k^N is a dense open cell in $Z_{\mathbf{i}}$.

We may embed $Z_{\mathbf{i}}$ in a product of flag varieties by the iterated multiplication map:

$$\begin{array}{cccc} \mu : & Z_{\mathbf{i}} & \to & (G/B)^{N+1} \\ & (p_1, \dots, p_N) & \mapsto & (eB, p_1 B, p_1 p_2 B, \cdots, p_1 \cdots p_N B). \end{array}$$

The embedding is compatible with the B-action on $Z_{\mathbf{i}}$ and the diagonal B-action on $(G/B)^{N+1}$. The image of this embedding is a dual version of $Z_{\mathbf{i}}$, a fiber product:

$$\mu(Z_{\mathbf{i}}) = eB \times_{G/P_{i_1}} G/B \times_{G/P_{i_2}} \cdots \times_{G/P_{i_N}} G/B \subset (G/B)^{N+1}.$$

By composing μ with various projections of $(G/B)^{N+1}$, we obtain maps from $Z_{\mathbf{i}}$. For example, the canonical map to the flag variety is

$$\begin{array}{ccc} Z_{\mathbf{i}} & \to & G/B \\ (p_1, \dots, p_N) & \mapsto & p_1 p_2 \cdots p_N B. \end{array}$$

which is a birational morphism exactly when $\mathbf{i}$ is a reduced decomposition of the longest element of W. For general $\mathbf{i}$ the image is the Schubert variety $X(s_{i_1} \cdots s_{i_N})$.

Let $\mathrm{Gr}(\mathbf{i}) = G/\widehat{P}_{i_1} \times \cdots \times G/\widehat{P}_{i_N}$, where $\widehat{P}_i$ is the *maximal* parabolic subgroup associated to all the simple reflections *except* s_i. If we compose μ with the projection of $(G/B)^{N+1}$ to $\mathrm{Gr}(\mathbf{i})$, the result is still an embedding of $Z_{\mathbf{i}}$:

$$\begin{array}{cccc} \bar{\mu}: & Z_{\mathbf{i}} & \to & \mathrm{Gr}(\mathbf{i}) \\ & (p_1, \dots, p_N) & \mapsto & (p_1 \widehat{P}_{i_1}, p_1 p_2 \widehat{P}_{i_2}, \cdots, p_1 \cdots p_N \widehat{P}_{i_N}). \end{array}$$

That is, $Z_{\mathbf{i}} \cong \mu(Z_{\mathbf{i}}) \cong \bar{\mu}(Z_{\mathbf{i}})$. This gives an embedding of $Z_{\mathbf{i}}$ in a conveniently small variety.

Finally, if we project $\mathrm{Gr}(\mathbf{i})$ to any product of $G/\widehat{P}_i$ with some of the $G/\widehat{P}_{i_j}$ factors missing, the image of $\bar{\mu}(Z_{\mathbf{i}})$ is no longer isomorphic to $Z_{\mathbf{i}}$: we call this image a *configuration variety.*

Line bundles on $Z_{\mathbf{i}}$ are indexed by sequences of integers $\mathbf{m} = (m_1, \dots, m_N)$. Define the line bundle

$$\mathcal{L}_{\mathbf{m}} = (P_{i_1} \times \cdots \times P_{i_N}) \times_{B^N} (k_{-m_1 \varpi_{i_1}} \otimes \cdots \otimes k_{-m_N \varpi_{i_N}})$$

associated to the character $e^{-m_1 \varpi_1} \otimes \cdots \otimes e^{-m_N \varpi_{i_N}} : B^N \to k^\times$, where ϖ_i denotes the i-th fundamental weight of G. We can also define $\mathcal{L}_{\mathbf{m}}$ in terms of the embedding $\bar{\mu}$. Let $\mathcal{O}(1) = G \times^{\widehat{P}_i} k_{-\varpi_i}$ denote the unique minimal ample line bundle on $G/\widehat{P}_i$. Then $\mathcal{L}_{\mathbf{m}}$ is the pullback via $\bar{\mu}$ of the bundle $\mathcal{O}(\mathbf{m}) = \mathcal{O}(1)^{\otimes m_1} \otimes \cdots \otimes \mathcal{O}(1)^{\otimes m_N}$ over $\mathrm{Gr}(\mathbf{i})$.

Our substitutes for Weyl modules and Demazure modules will be the spaces of global sections

$$V(\mathbf{i}, \mathbf{m})^* = \Gamma(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{m}}).$$

For appropriately chosen $\mathbf{i}$ and $\mathbf{m}$, the B-representations $V(\mathbf{i}, \mathbf{m})^*$ are isomorphic to the dual Weyl modules $V(\lambda)^*$ and the Demazure modules $V(\lambda)^*_\tau$ considered previously. The vanishing theorem of Mathieu [41] and Kumar [20] implies:

Theorem 11.1.1. *(i) The restriction map* $\Gamma(\mathrm{Gr}(\mathbf{i}), \mathcal{O}(\mathbf{m})) \to \Gamma(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{m}})$ *is surjective.*

(ii) The character of $\Gamma(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{m}})^*$ *is given by the Demazure formula:*

$$\mathit{Char}\, \Gamma(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{m}})^* = \Lambda_{i_1}(e^{m_1 \varpi_{i_1}} \Lambda_{i_2}(e^{m_2 \varpi_{i_2}} \dots (\Lambda_{i_N} e^{m_N \varpi_{i_N}}) \dots)).$$

It should be possible to prove this theorem by the same methods used above in the case of Schubert varieties. ¿From the theorem, we see that $V(\mathbf{i}, \mathbf{m})^*$ is a quotient of the tensor product

$$V(m_1 \varpi_{i_1})^* \otimes \cdots \otimes V(m_N \varpi_{i_N})^* = \Gamma(\mathrm{Gr}(\mathbf{i}), \mathcal{O}(\mathbf{m})).$$

Example. Let $G = SL(n)$. Then $G/\widehat{P}_i \cong \mathrm{Gr}(i, k^n)$, the Grassmannian of i-planes in linear n-space. Let $E_i \in \mathrm{Gr}(i, k^n)$ be the span of the first i standard coordinate vectors in k^n. Then we may identify $\bar{\mu}(Z_{\mathbf{i}}) \subset \mathrm{Gr}(\mathbf{i}) = \mathrm{Gr}(i_1, k^n) \times \cdots \times \mathrm{Gr}(i_1, k^n)$ as the variety of N-tuples of subspaces $(V_1, \dots, V_N) \in \mathrm{Gr}(\mathbf{i})$ with $\dim V_j = i_j$, and subject to the following inclusions: if $i_p = i_q + 1$, and $i_r \neq i_p, i_q$ for every r between

p and q, then $V_p \subset V_q$; and if $i_q \neq i_p - 1$ for $q < p$, then $E_{i_p-1} \subset V_p$; and if $i_q \neq i_p + 1$ for $q < p$, then $V_p \subset E_{i_p+1}$.

Letting $G = SL(4)$ and $\mathbf{i} = (1,3,2,1,2)$, we have that $(V_1, \ldots, V_5) \in \bar{\mu}(Z_{\mathbf{i}})$ precisely if:

$$\begin{tikzcd}
 & E_1 \arrow[r] & E_2 \arrow[r] \arrow[dr] & E_3 \arrow[dr] & \\
0 \arrow[r] \arrow[ur] \arrow[dr] & V_1 \arrow[r] \arrow[ur] & V_3 \arrow[r] & V_2 \arrow[r] & k^4 \\
 & V_4 \arrow[r] \arrow[ur] & V_5 \arrow[ur] & &
\end{tikzcd}$$

where the arrows indicate codimension one inclusions of subspaces. Furthermore we have the opposite big open cell $k^5 \subset Z_{\mathbf{i}}$ given by the coordinates:

$$(t_1, t_2, t_3, t_4, t_5) \in k^5 \mapsto (V_1, \ldots, V_5) =$$

$$\begin{pmatrix} 1 \\ t_1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ t_1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & t_2 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ t_1 & 1 \\ 0 & t_3 \\ 0 & t_2 t_3 \end{pmatrix} \times \begin{pmatrix} 1 \\ t_1 + t_4 \\ t_3 t_4 \\ t_2 t_3 t_4 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ t_1 + t_4 & 1 \\ t_3 t_4 & t_3 + t_5 \\ t_2 t_3 t_4 & t_2(t_3 + t_5) \end{pmatrix},$$

where the spaces $V_1, V_2, \ldots$ are spanned by the column vectors of the matrices. Letting $\mathbf{m} = (0,0,1,0,2)$, the space $V(\mathbf{i},\mathbf{m})^* = \Gamma(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{m}})$ is spanned by restrictions of sections in $\Gamma(\mathrm{Gr}(\mathbf{i}), \mathcal{O}(\mathbf{m}))$. These latter sections are products of Plücker coordinates, minors in the homogeneous coordinates on the $\mathrm{Gr}(i_j)$. A typical section is

$$\phi(V_1, \ldots, V_5) = \det_{ab}(V_3) \det_{cd}(V_5) \det_{ef}(V_5)$$

where $\det_{pq}$ indicates the 2×2 minor in rows p, q of the matrix of basis vectors of a two-dimensional subspace of k^4. Restricting these sections on $\mathrm{Gr}(\mathbf{i})$ to the big cell in $Z_{\mathbf{i}}$, we obtain polynomials in t_j:

$$\bar{\mu}^* \phi = \det_{ab} \begin{pmatrix} 1 & 0 \\ t_1 & 1 \\ 0 & t_3 \\ 0 & t_2 t_3 \end{pmatrix} \det_{cd} \begin{pmatrix} 1 & 0 \\ t_1 + t_4 & 1 \\ t_3 t_4 & t_3 + t_5 \\ t_2 t_3 t_4 & t_2(t_3 + t_5) \end{pmatrix} . \det_{ef} \begin{pmatrix} 1 & 0 \\ t_1 + t_4 & 1 \\ t_3 t_4 & t_3 + t_5 \\ t_2 t_3 t_4 & t_2(t_3 + t_5) \end{pmatrix}.$$

This gives a total of $6^3 = 216$ spanning vectors for $V(\mathbf{i},\mathbf{m})^*$, of which 54 are linearly independent over k, as we may check by the Demazure character formula. In the following section, we will show how to extract a *standard basis* of $V(\mathbf{i},\mathbf{m})^*$ from the spanning set.

11.2. **Path model and indexing system for bases.** To find bases for our B-representations $V(\mathbf{i},\mathbf{m})^* = \Gamma(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{m}})$, we formulate an analog of the path model for a highly non-standard "root system" associated to $Z_{\mathbf{i}}$. We define this *pseudo root system* in terms of the usual root system of the group G. To avoid confusion, we use the usual notation α, f_α, etc., for objects of the usual root system, and write their pseudo counterparts with a tilde: $\tilde{\alpha}$, $\tilde{f}_{\tilde{\alpha}}$, etc.

For $\mathbf{i} = (i_1, \ldots, i_N)$, define the pseudo Cartan matrix $\tilde{A}(\mathbf{i}) = (\tilde{a}_{jk})$ of size $N \times N$ by

$$\tilde{a}_{jk} = \langle \alpha_{i_j}, \alpha_{i_k}^\vee \rangle,$$

which is a Cartan integer for the usual root system of G. However, we have $\tilde{a}_{jk} = 2$ whenever $i_j = i_k$, which violates a basic condition of generalized Cartan matrices.

Nevertheless we can define many of the usual notions as in [17]. We have the pseudo weight lattice and its dual,

$$\tilde{X} = \mathbb{Z}^N = \langle \tilde{e}_1, \dots, \tilde{e}_N \rangle \qquad \tilde{X}^\vee = (\mathbb{Z}^N)^* = \langle \tilde{e}_1^*, \dots, \tilde{e}_N^* \rangle,$$

as well as the real version $\tilde{X}_\mathbb{R} = \tilde{X} \otimes_\mathbb{Z} \mathbb{R}$. The pseudo simple roots and coroots are

$$\tilde{\alpha}_j = \sum_{k=1}^{N} \tilde{a}_{jk} \tilde{e}_k \ \in \ \tilde{X} \qquad \tilde{\alpha}_j^\vee = \tilde{e}_j^* \ \in \ \tilde{X}^\vee.$$

Note that $\tilde{\alpha}_j = \tilde{\alpha}_k$ if $i_j = i_k$, but $\tilde{\alpha}_1^\vee, \dots, \tilde{\alpha}_N^\vee$ are linearly independent. Then we clearly have

$$\langle \tilde{\alpha}_j, \tilde{\alpha}_k^\vee \rangle = \langle \alpha_{i_j}, \alpha_{i_k}^\vee \rangle = \tilde{a}_{jk}.$$

A pseudo simple reflection is

$$\begin{array}{cccc} \tilde{s}_j : & \tilde{X}_\mathbb{R} & \to & \tilde{X}_\mathbb{R} \\ & x & \mapsto & x - \langle x, \tilde{\alpha}_j^\vee \rangle \tilde{\alpha}_j \end{array}$$

and these generate a pseudo Weyl group $\tilde{W}$. Also define certain analogs of fundamental weights

$$\tilde{\delta}_j = \sum_{\substack{k \le j \\ i_k = i_j}} \tilde{e}_k \ \in \ \tilde{X},$$

which form a basis of $\tilde{X}$, but not the dual basis of $\{\tilde{\alpha}_j^\vee\}$. We consider the linear map $\mathrm{proj} : \tilde{X} \to X$ defined by $\mathrm{proj}(\tilde{\delta}_j) = \varpi_{i_j}$, where ϖ_i is the ith fundamental weight of the ordinary root system. Then we have $\mathrm{proj}(\tilde{\alpha}_j) = \alpha_{i_j}$, but in general $\mathrm{proj}(\tilde{s_j}\tilde{\lambda}) \neq s_{i_j} \mathrm{proj}(\tilde{\lambda})$.

Example. For our running example $G = SL(4)$, $\mathbf{i} = (1, 3, 2, 1, 2)$, we have the ordinary and pseudo Cartan matrices,

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad \text{and} \quad \tilde{A}(\mathbf{i}) = \begin{pmatrix} 2 & 0 & -1 & 2 & -1 \\ 0 & 2 & -1 & 0 & -1 \\ -1 & -1 & 2 & -1 & 2 \\ 2 & 0 & -1 & 2 & -1 \\ -1 & -1 & 2 & -1 & 2 \end{pmatrix};$$

the pseudo simple coroots and roots

$$\tilde{\alpha}_1^\vee = \tilde{e}_1^*, \quad \cdots \quad \tilde{\alpha}_5^\vee = \tilde{e}_5^*,$$

$$\tilde{\alpha}_1 = \tilde{\alpha}_4 = 2\tilde{e}_1 - \tilde{e}_3 - 2\tilde{e}_4 - \tilde{e}_5, \qquad \tilde{\alpha}_2 = 2\tilde{e}_2 - \tilde{e}_3 - \tilde{e}_5,$$

$$\tilde{\alpha}_3 = \tilde{\alpha}_5 = -\tilde{e}_1 - \tilde{e}_2 + 2\tilde{e}_3 - \tilde{e}_4 + 2\tilde{e}_5;$$

and the analogs of fundamental weights

$$\tilde{\delta}_1 = \tilde{e}_1, \quad \tilde{\delta}_2 = \tilde{e}_2, \quad \tilde{\delta}_3 = \tilde{e}_3, \quad \tilde{\delta}_4 = \tilde{e}_1 + \tilde{e}_4, \quad \tilde{\delta}_5 = \tilde{e}_3 + \tilde{e}_5.$$

Now consider rational piecewise linear paths $\tilde{\pi} : [0, 1] \to \tilde{X}_\mathbb{R}$. For any pseudo simple root $\tilde{\alpha}$, we may define the analogs $\tilde{f}_{\tilde{\alpha}}$, $\tilde{e}_{\tilde{\alpha}}$ of the lowering and raising operators exactly as for the ordinary root system, but using the pseudo roots and coroots, etc. These operators have the same properties as those for the usual root system.

Let $\tilde{\Pi}^+$ be the set of all *dominant paths*, those $\tilde{\pi}$ with $\langle \tilde{\pi}(t), \tilde{\alpha}_j^\vee \rangle \geq 0$ for all t and j. For $\tilde{\pi} \in \tilde{\Pi}^+$, let $B(\tilde{\pi})_{\mathbf{i}}$ be the set of paths generated from $\tilde{\pi}$ by applying the lowering operators to $\tilde{\pi}$ in the fixed order given by $\mathbf{i}$:

$$B(\tilde{\pi})_{\mathbf{i}} = \{\tilde{f}_1^{n_1} \cdots \tilde{f}_N^{n_N} \tilde{\pi} \mid n_1, \ldots, n_N \geq 0\},$$

where $\tilde{f}_j$ is the lowering operator associated to $\tilde{\alpha}_j$. The character of a set B of paths is again the formal sum of the endpoints of the paths, projected to the ordinary weight lattice X: $\mathrm{Char}\, B = \sum_{\tilde{\pi} \in B} e^{\mathrm{proj}\, \tilde{\pi}(1)}$.

Theorem 11.2.1. *Let $\tilde{\pi} \in \tilde{\Pi}^+$ be a dominant path with $\tilde{\pi}(1) = \tilde{\delta} := m_1\tilde{\delta}_1 + \cdots + m_N\tilde{\delta}_N$. Then the character $CharB(\tilde{\pi})_{\mathbf{i}}$ is equal to the character of $V(\mathbf{i}, \mathbf{m})$, the dual B-representation to $V(\mathbf{i}, \mathbf{m})^*$.*

We may define *L-S paths* for $(\mathbf{i}, \mathbf{m})$ as the set $LS(\mathbf{i}, \mathbf{m}) = B(\tilde{\pi}_{\tilde{\delta}})_{\mathbf{i}}$, where $\tilde{\pi}_{\tilde{\delta}} : t \mapsto t\tilde{\delta}$ is the straight-line path from 0 to $\tilde{\delta}$ in $\tilde{X}_{\mathbb{R}}$. The *extremal paths* of $LS(\mathbf{i}, \mathbf{m})$ are by definition the straight-line paths, which are all of the form $\tilde{\pi}_{\tilde{w}\tilde{\delta}}$ for some $\tilde{w} \in \tilde{W}$. Any path in $LS(\mathbf{i}, \mathbf{m})$ is a sequence of straight-line steps in extremal path directions, and so may be described like a usual L-S path by a sequence $\underline{\tau}$ of extremal weights $\tau_j = \tilde{w}_j\tilde{\delta}$ and a sequence $\underline{a}$ of increasing rational numbers between 0 and 1 encoding the lengths of the steps.

The L-S paths for $(\mathbf{i}, \mathbf{m})$ are closely related to the following geometric partially ordered set. A Bott-Samelson subvariety Y of $Z_{\mathbf{i}}$ is a product $P_{i_1} \times \cdots \times B \times \cdots \times P_{i_N}/B^N$, where we have replaced some of the factors P_{i_j} with B. Now consider the projection $\eta : \mathrm{Gr}(\mathbf{i}) \to \prod_{j \,:\, m_j > 0} G/\widehat{P}_{i_j}$, where we drop all factors with $m_j = 0$. Then consider the set of all images $\{\eta\bar{\mu}(Y) \mid\ Y \subset Z_{\mathbf{i}}$ a Bott-Samelson subvariety $\}$, and order these varieties by inclusion. The resulting poset bears a relationship to $LS(\mathbf{i}, \mathbf{m})$ similar to that of the usual Bruhat order to usual L-S paths.

Now we consider a set of paths which will allow us to construct a basis in the framework of the previous section. Let $\tilde{\mu} = \tilde{\mu}(\mathbf{i}, \mathbf{m}) \in \tilde{\Pi}^+$ be the piecewise-linear path defined as a concatenation of N straight line paths

$$\tilde{\nu} = \tilde{\pi}_{m_1\tilde{\delta}_1} * \cdots * \tilde{\pi}_{m_N\tilde{\delta}_N}.$$

so that $\tilde{\nu}(1) = \tilde{\delta}$. Define the set of *pseudo standard tableaux* as the paths $\tilde{ST}(\mathbf{i}, \mathbf{m}) = B(\tilde{\nu})_{\mathbf{i}}$. Now, for each path $\tilde{\pi}$ in $\tilde{X}_{\mathbb{R}}$, consider its projection $\mathrm{proj}\, \tilde{\pi}(t)$ to $X_{\mathbb{R}}$, and define the set of *standard tableaux* as $ST(\mathbf{i}, \mathbf{m}) = \mathrm{proj}\, \tilde{ST}(\mathbf{i}, \mathbf{m})$, the projection of the pseudo standard tableaux. There is an obvious inclusion

$$\tilde{ST}(\mathbf{i}, \mathbf{m}) = B(\tilde{\nu})_{\mathbf{i}} \subset B(m_1\tilde{\delta}_1)_{\mathbf{i}} * \cdots * B(m_N\tilde{\delta}_N)_{\mathbf{i}}$$

which projects to the inclusion

$$ST(\mathbf{i}, \mathbf{m}) \subset B(\varpi_{m_1 i_1}) * \cdots * B(m_N\varpi_{i_N}).$$

We may also construct the standard tableaux $ST(\mathbf{i}, \mathbf{m})$ using only the usual lowering operators f_i in $X_{\mathbb{R}}$ and another path version of Demazure's character formula:

$$ST(\mathbf{i}, \mathbf{m}) = \{\, f_{i_1}^{n_1}(\pi_{m_1\varpi_{i_1}} * f_{i_2}^{n_2}(\pi_{m_1\varpi_{i_2}} * \ldots (f_{i_N}^{n_N}(\pi_{m_N\varpi_{i_N}}) \ldots))) \mid n_1, \ldots, n_N \geq 0 \,\}.$$

Finally, we can characterize the paths in $ST(\mathbf{i}, \mathbf{m})$ by certain standardness conditions (the factors must decrease in an appropriate analog of the Bruhat order). See [26], [27].

Now we construct our basis for $V(\mathbf{i},\mathbf{m})^*$. Recall the path basis $\mathbb{B}(\lambda) = \{p_\pi \mid \pi \in B(\lambda)\}$ for each G-representation $V(\lambda)^*$ with lowest weight $-\lambda \in X$. Now, for each standard tableau

$$\nu = \pi_1 * \cdots * \pi_N \in ST(\mathbf{i},\mathbf{m}) \subset B(m_1\varpi_{i_1}) * \cdots * B(m_N\varpi_{i_N})$$

we may define

$$p_\nu = p_{\pi_1} \cdots p_{\pi_N} \in V(m_1\varpi_{i_1})^* \otimes \cdots \otimes V(m_N\varpi_{i_N})^*.$$

Let $\mathbb{B}(\mathbf{i},\mathbf{m}) = \{p_\nu \in ST(\mathbf{i},\mathbf{m})\}$.

Theorem 11.2.2. *The set* $\mathbb{B}(\mathbf{i},\mathbf{m})$ *restricts to a basis of* $V(\mathbf{i},\mathbf{m})^*$.

Example. Again taking $G = GL(4)$, $\mathbf{i} = 13212$, $\mathbf{m} = (0,0,1,0,2)$, let us denote an extremal weight $w(\varpi_i)$ by a subset of i elements in $\{1,2,3,4\}$, and use the same symbol to denote the straight-line path $\pi_{w(\varpi_i)}$ in $X_\mathbb{R}$. Thus a path in $ST(\mathbf{i},\mathbf{m})$ is of the form $ab * cd * ef$, where $1 \le a,b,c,d,e,f \le 4$ and $a < b$, $c < d$, $e < f$; but not all such paths are standard tableaux. A typical standard tableau is $\pi = 24 * 23 * 13$, which is generated by our Demazure formula as

$$\begin{aligned}\pi &= f_1^{n_1} f_3^{n_2} f_2^{n_3}(12 * f_1^{n_4} f_2^{n_5}(12 * 12)) \\ &= f_1^2 f_3^1 f_2^3 (12 * f_1^0 f_2^0 (12 * 12)) = f_1 f_3 f_2^2 (12 * f_1 f_2 (12 * 12)).\end{aligned}$$

By taking all 54 such tableaux one obtains the standard basis as indicated in the example of the previous section.

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