Notes on Lemma 6, July 26, 2012

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Lemma 6 on page 19 in Section 6 of "Combinatorial Sublinear-Time Fourier Algorithms" contains a couple typos, one of which necessitates a correction.[1] In particular, the calculation performed on line seven of the proof omits a constant factor of $(2\pi)^{2\kappa}$ which, when corrected, has the unfortunate effect of causing the interpolation error considered therein to grow exponentially, instead of shrinking exponentially as desired. This omission is ultimately a consequence of mistakenly considering p_R^x as a polynomial with domain $[0,1]$ instead of $[0,2\pi]$.

It is worth mentioning that the current proof works as written for signals which are oversampled by a factor of ≥ 2. That is, if $\widehat{\mathbf{A}}(\omega) = 0$ for all $|\omega| > N/4$, then Lemma 6 holds essentially as stated. In fact, the neglected constant factor of $(2\pi)^{2\kappa}$ can be entirely erased by considering the signal in question to be oversampled by a factor of 2π (i.e., the oversampling rate directly cancels the neglected constant). This oversampling factor can be reduced to 2 if one is willing to accept a slower rate of exponential decay in interpolation error as κ increases. However, fixing Lemma 6 so that it works for general trigonometric polynomials of degree $N/2$ requires more work.

1 Correction in the General Case

In order to correct Lemma 6 in general we will consider a modified version of the function, $f : [0,2\pi] \rightarrow \mathbb{C}$, defined at the bottom of page 18. Instead, we replace f with the closely related function f_0 defined by

$$f_0(x) := \frac{1}{2\pi} \sum_{\omega = 1 - \left\lceil \frac{N}{2} \right\rceil}^{\left\lfloor \frac{N}{2} \right\rfloor} \widehat{\mathbf{G}}(\omega) \, \widehat{\mathbf{A}}(\omega) \cdot e^{\mathbb{i}\omega \cdot x}, \quad x \in [0, 2\pi]. \tag{1}$$

Here $\widehat{\mathbf{G}}$ is the sequence of Fourier coefficients of a filter function. Thus, f_0 is effectively a filtered version of f. We imagine that we have sampling access to f as defined on page 18, but do our calculations with the filtered version, f_0, instead. We will discuss the discrete and (effectively) band limited "low-pass" filter, $\mathbf{G}$, in more detail below (see Section 2). For now, we simply assume that there exists a constant $C \in \mathbb{R}^+$ such that $\left|\widehat{\mathbf{G}}(\omega)\right| \leq C \cdot e^{-|\omega| \cdot 8\kappa/N}$ for all $\omega \in \mathbb{Z}$.

Correcting the calculation on line seven of the proof of Lemma 6 by inserting the omitted

[1] I would like to thank Jieming Mao for bringing my attention to the error discussed herein.

constant factor in red, we note that:

$$|\mathbb{Re}\left\{f_0(x)\right\} - p^x_R(x)| \;\leq\; \frac{\left|\|f_0^{(2k)}\|_\infty\right|}{(2\kappa)!} \cdot \prod_{m=1}^{\kappa} \left(\frac{m \cdot {\color{red}2\pi}}{N}\right)^2. \tag{2}$$

We must now correct for this additional constant factor, which is accomplished in the general case by replacing f with f_0 in the calculation. Note that

$$\left|\|f_0^{(2k)}\|_\infty\right| \;\leq\; \frac{C}{2\pi} \sum_{\omega=1-\lceil\frac{N}{2}\rceil}^{\lfloor\frac{N}{2}\rfloor} |\omega|^{2\kappa} \cdot e^{-|\omega|\cdot 8\kappa/N} \cdot \left|\widehat{\mathbf{A}}\left(\omega\right)\right| \;\leq\; \frac{C}{2\pi} \cdot \left(\frac{N}{4e}\right)^{2\kappa} \sum_{\omega=1-\lceil\frac{N}{2}\rceil}^{\lfloor\frac{N}{2}\rfloor} \left|\widehat{\mathbf{A}}\left(\omega\right)\right|.$$

Continuing our calculation from Equation 2 we see that

$$|\mathbb{Re}\left\{f_0(x)\right\} - p^x_R(x)| \;\leq\; \frac{1}{(2\kappa)!} \cdot \frac{C \cdot \|\hat{\mathbf{A}}\|_1}{2\pi} \left(\frac{N}{4e}\right)^{2\kappa} \cdot \prod_{m=1}^{\kappa} \left(\frac{m \cdot 2\pi}{N}\right)^2 \;\leq\; \frac{C \cdot \|\hat{\mathbf{A}}\|_1}{2\pi \cdot 2^\kappa} \cdot \frac{\prod_{m=1}^{\kappa} m^2}{(2\kappa)!}.$$

Thus, $|\mathbb{Re}\left\{f_0(x)\right\} - p^x_R(x)| \;\leq\; \frac{C\cdot\|\hat{\mathbf{A}}\|_1}{2\pi\cdot 4^\kappa}$. The remainder of the argument goes through as before, and we obtain the following modified form of Lemma 6.

Lemma 6. *Let* $\boldsymbol{A}$ *be an* N*-length complex valued array and suppose that* $\hat{\mathbf{A}}$ *is* (c, p)*-compressible. Fix* $\tilde{c} \in \mathbb{R}^+$*. Using* $2\kappa = O\left(\log(p \cdot k^p/\tilde{c} \cdot \delta)\right)$ *interpolation points from* $\boldsymbol{A}$ *per* f_0*-evaluation will guarantee that every line 8 DFT entry from Algorithm 2 is calculated to within* $\frac{\tilde{c}\cdot c\delta}{2p-1} \cdot k^{-p}$ *precision.*

We conclude this section by pointing out that this modified version of Lemma 6 can still be used to prove a modified version of Corollary 5 on page 20. This can be done by executing Algorithm 2 $O(\kappa)$-times on $O(\kappa)$ different f_0 variants, instead of executing it on f directly. Given that $\widehat{\mathbf{G}}$ is known and relatively large for all ω with $|\omega| = O(N/\kappa)$, we can recover all energetic frequencies of size $O(N/\kappa)$ from $\hat{\mathbf{A}}$ by using the results of Algorithm 2 on f_0 (see Equation 1 for the definition of f_0). Thus, we can recover all energetic frequencies from $\hat{\mathbf{A}}$ by modulating f $O(\kappa)$-times and then filtering with $\mathbf{G}$. In particular, we may define

$$f_{j'}(x) := \left(\mathbf{G} \star e^{\mathbb{i}\cdot x\cdot\lceil j'N/C'\kappa\rceil} f\right)(x) \approx \frac{1}{2\pi} \sum_{\omega=1-\lceil\frac{N}{2}\rceil}^{\lfloor\frac{N}{2}\rfloor} \widehat{\mathbf{G}}\left(\omega\right) \widehat{\mathbf{A}}\left(\omega - \left\lceil\frac{j'N}{C'\kappa}\right\rceil\right) \cdot e^{\mathbb{i}\omega\cdot x}, \quad x \in [0, 2\pi],$$

for $j' \in [-C''\kappa, C''\kappa] \cap \mathbb{Z}$ and fixed constants $C', C'' \in \mathbb{N}$. Executing Algorithm 2 on each of these $f_{j'}$ will allow one to recover all energetic frequencies from $\hat{\mathbf{A}}$.

2 The filter G

The filter $\mathbf{G}$ must have several properties in order to allow the production of a sublinear-time Fourier algorithm (i.e., in order for a modified version of Corollary 5 on page 20 to hold as discussed above). Most important among these properties are the following: First, the filter array $\mathbf{G} : [1,N] \cap \mathbb{Z} \to \mathbb{C}$ should be zero almost everywhere. This allows $f_{j'} = \mathbf{G} \star \left(e^{\mathbb{i} \cdot x \cdot \lceil j'N/C'\kappa \rceil} f\right)$ to be sampled quickly using only a few samples from f in the process. Of course, it is much more likely that $\mathbf{G}$ will be "almost zero" everywhere, in which case the convolution involved in the definition of $f_{j'}$ can still be (approximately) computed both quickly and accurately using only a few samples from f.

Second, the Fourier transform of the filter, $\widehat{\mathbf{G}}$, should have both the properties alluded to in Section 1 above. Mainly, $\widehat{\mathbf{G}}$ should exhibit exponential decay for larger frequencies, i.e. $|\widehat{\mathbf{G}}(\omega)|$ should be $O\left(e^{-|\omega| \cdot 8\kappa/N}\right)$. However, $|\widehat{\mathbf{G}}(\omega)|$ should not decay *too quickly.* That is, $\mathbf{G}$ should serve as a decent low-pass filter. In particular, we require that $|\widehat{\mathbf{G}}(\omega)|$ be relatively large (e.g., larger than 1/10) for all ω with $|\omega| \leq N/2\kappa$.

Gaussian filters generally fulfill the required properties listed above. For example, one can take

$$\widehat{\mathbf{G}}(\omega) = \begin{cases} e^{\frac{-3 \cdot \kappa^2 \cdot \omega^2}{N^2}} & \text{if } \omega \in \left(\left\lceil \frac{N}{2} \right\rceil, \left\lfloor \frac{N}{2} \right\rfloor\right] \cap \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} . \tag{3}$$

The filter $\mathbf{G}$ can then be taken as the inverse discrete Fourier transform of $\widehat{\mathbf{G}}$.[2] In this case, $\mathbf{G}$ will also "look Guassian", and therefore be "almost zero everywhere" as desired. See Figure 1 for graphs of this Gaussian filter when $N = 200,001$ and $\kappa = 7$. Note that the desired properties we have discussed in this section are indeed achieved in this example.

[2]Creating $\mathbf{G}$ in this fashion will result in a one-time computational cost of $O(N \log N)$. This one-time cost can be avoided however – see, e.g., section 7 of "Nearly Optimal Sparse Fourier Transform" by Hassanieh, Indyk, Katabi, and Price.

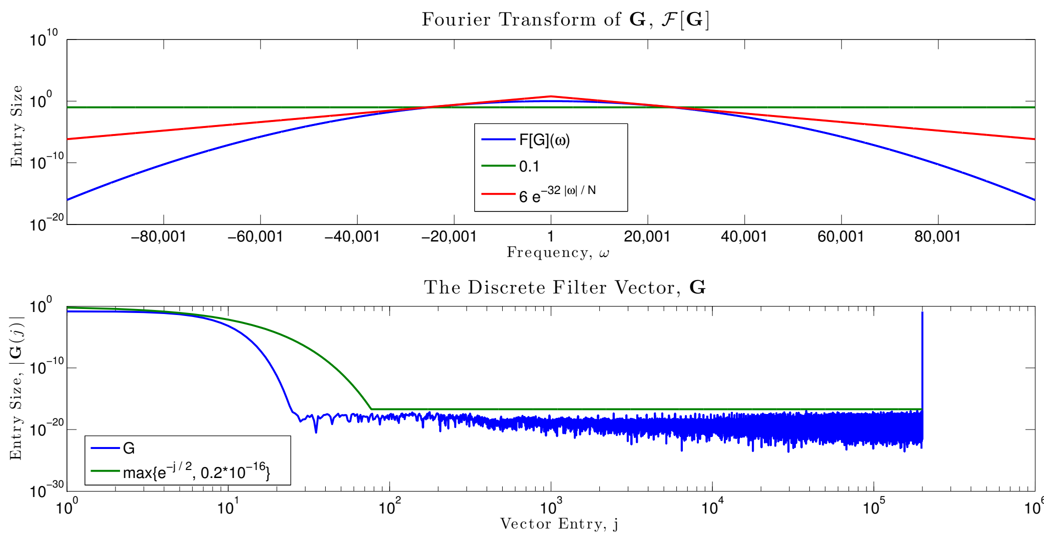

Figure 1: The example filter, $\mathbf{G}$, in Equation 3 with length $N = 200,001$ and $\kappa = 7$. The top graph demonstrates that $|F[\mathbf{G}]| = |\widehat{\mathbf{G}}|$ decays exponentially in accordance with the assumption made in Section 1 above. Furthermore, $F[\mathbf{G}] = \widehat{\mathbf{G}}$ is shown to be relatively large in magnitude (e.g., larger than 0.1) for all frequencies ω with $|\omega| \leq 20,000$. Hence, the filter effectively passes one fifth of the lowest magnitude frequencies. The bottom graph demonstrates the exponential decay of the entries of $\mathbf{G}$ in magnitude. Hence, convolutions with $\mathbf{G}$ can be approximatly sampled both quickly and accurately.