Phase Retrieval from Local Measurements in Two Dimensions

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ABSTRACT

The phase retrieval problem has appeared in a multitude of applications for decades. While ad hoc solutions have existed since the early 1970s, recent developments have provided algorithms that offer promising theoretical guarantees under increasingly realistic assumptions. Motivated by ptychographic imaging, we generalize a recent result on phase retrieval of a one dimensional objective vector $\mathbf{x} \in \mathbb{C}^d$ to recover a two dimensional sample $Q \in \mathbb{C}^{d\times d}$ from phaseless measurements, using a tensor product formulation to extend the previous work.

Keywords: Phase Retrieval, Local Measurements, Two Dimensional Imaging, Ptychography

1. INTRODUCTION

Consider the problem of recovering a 2-dimensional image $Q \in \mathbb{C}^{d\times d}$ from measurements of the form

$$y_k = \left| \sum_{i,j=1}^{d} Q_{ij} A_{ij}^{(k)} \right|^2, \tag{1}$$

where $A^{(k)}$ is a collection of known measurement vectors. This is known as the *phase retrieval problem*,[1,2] as the system (1) may be seen as a system of linear equations in the variables Q_{ij} wherein the phases of the measurements y_k have been discarded by the componentwise $|\cdot|^2$ operation. This problem appears in a number of imaging scenarios, for example X-ray crystallography,[3] electron microscopy,[4] and ptychography,[5] which shall be the primary motivation for the technique presented here.

In ptychography, a light source applies a sharply focused beam onto a sample, which scatters the incoming ray onto an array of intensity sensors behind. The light source is then shifted and applied to different parts of the sample to obtain the measurement redundancy necessary to resolve an accurate image from this data. We model this as

$$\left| \left(\mathcal{F}\left[S_{x_0,y_0} a \cdot q \right] \right)(u,v) \right|^2, \quad (u,v) \in \Omega \subset \mathbb{R}^2, \quad (x_0,y_0) \in \mathcal{L} \subset [0,1]^2, \tag{2}$$

where $\mathcal{F}$ denotes the 2 dimensional Fourier transform (arising from the optical diffraction), $a : \mathbb{R}^2 \to \mathbb{C}$ is a known function representing the intensity of the illumination, S_{x_0,y_0} is a shift operator defined by $\left(S_{x_0,y_0} a\right)(x,y) := a(x - x_0, y - y_0)$, Ω is a finite set of sampled frequencies, and $\mathcal{L}$ is a finite set of shifts. As a typical characteristic of ptychography is that the beam is sharply focused, we assume that a is compactly supported within a smaller region $[0, \delta']^2$ for $\delta' \ll 1$.

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Discretizing (2) using periodic boundary conditions yields a finite dimensional problem aimed at recovering an unknown matrix $Q \in \mathbb{C}^{d\times d}$ from phaseless measurements of the form

$$y_{(\ell,\ell',u,v)} = \left| \frac{1}{d^2} \sum_{j=1}^{d} \sum_{k=1}^{d} Q_{j,k} \left(S_\ell A S_{\ell'}^*\right)_{j,k} \mathbb{e}^{\frac{-2\pi \mathrm{i}}{d}(ju+kv)} \right|^2 = \left| \frac{1}{d^2} \sum_{j=1}^{d} \sum_{k=1}^{d} Q_{j,k} \left(W^{-u} S_\ell A S_{\ell'}^* W^{-v}\right)_{j,k} \right|^2. \quad (3)$$

Here $A \in \mathbb{C}^{d\times d}$ is the discretization of our illuminating beam, $S_\ell \in \mathbb{R}^{d\times d}$ is the discrete circular shift operator defined by $(S_\ell \mathbf{x})_j := x_{j-\ell \bmod d}$ for all $\mathbf{x} \in \mathbb{C}^d$ and $j, \ell \in [d] := \{1, \dots, d\}$, and $W = \operatorname{diag}(\mathbb{e}^{\frac{2\pi \mathrm{i} k}{d}})$ is the modulation operator. Also, $(\ell, \ell', u, v) \in \mathcal{L}^2 \times \Omega^2$ where $\mathcal{L}, \Omega \subset [d]$ are the index sets for the shifts and frequencies, respectively. Herein we will make the simplifying assumption that our original illuminating beam function a is not only sharply focused, but also separable. Specifically, we assume that the discretized measurement matrix takes the form

$$\frac{1}{d^2} A := \mathbf{a}\mathbf{b}^*$$

where $\mathbf{a}, \mathbf{b} \in \mathbb{C}^d$ both have $\operatorname{supp}(\mathbf{a}), \operatorname{supp}(\mathbf{b}) \subset [\delta]$ for some $\delta \in \mathbb{Z}^+$ with $\delta \ll d$. We leave the generalization to "non-separable" matrices A to future work.

We can now rewrite the measurements (3) as

$$\begin{aligned} y_{(\ell,\ell',u,v)} &= \left| \frac{1}{d^2} \langle Q, W^{-u} S_\ell A S_{\ell'}^* W^{-v} \rangle_{\mathrm{HS}} \right|^2 && (4) \\ &= \left| \frac{1}{d^2} \mathbb{e}^{\frac{2\pi \mathrm{i}(\ell u + \ell' v)}{d}} \langle Q, S_\ell W^{-u} A W^{-v} S_{\ell'}^* \rangle_{\mathrm{HS}} \right|^2 \\ &= \left| \langle Q, S_\ell W^{-u} \mathbf{a}\mathbf{b}^* W^{-v} S_{\ell'}^* \rangle_{\mathrm{HS}} \right|^2 \\ &= \left| \langle Q, S_\ell \mathbf{a}_u (S_{\ell'} \mathbf{b}_v)^* \rangle_{\mathrm{HS}} \right|^2 && (5) \end{aligned}$$

where $\mathbf{a}_u, \mathbf{b}_v \in \mathbb{C}^d$ are defined by $(a_u)_j := \mathbb{e}^{\frac{-2\pi \mathrm{i} j u}{d}} a_j$ and $(b_v)_k := \mathbb{e}^{\frac{2\pi \mathrm{i} k v}{d}} b_k$ for all $j, k \in [d]$.

Motivated by the above model of ptychographic imaging, we propose a new efficient numerical scheme for solving general discrete phase retrieval problems using measurements of type (5) herein. After a brief discussion of notation, we will outline our proposed method in §2 below, along with a basic analysis guaranteeing the success of the method under appropriate assumptions. A preliminary numerical evaluation of the method is then presented in §3.

Notation and Preliminaries

For any $k \in \mathbb{N}$, we define $[k] := \{1, 2, \dots, k\}$. For $i, j \in \mathbb{N}, e_i$ represents the standard basis vector and $E_{ij} = e_i e_j^*$; the dimensions of such an E_{ij} will always be clear from context. For a matrix $A \in \mathbb{C}^{m\times n}$,

$$\overrightarrow{A} := (a_{11}, a_{21}, \dots, a_{m1}, \dots, a_{mn})^T \in \mathbb{C}^{mn}$$

denotes the column-major vectorization of A. $A \otimes B$ for arbitrary matrices denotes the standard Kronecker product. We remark that $\overrightarrow{\mathbf{a}\mathbf{b}^*} = \overline{\mathbf{b}} \otimes \mathbf{a}$, and in particular

$$\overrightarrow{E_{jk}} \overrightarrow{E_{j'k'}}^* = \overrightarrow{e_j e_k^*} \overrightarrow{e_{j'} e_{k'}^*}^* = (e_k \otimes e_j)(e_{k'} \otimes e_{j'})^* = (e_k e_{k'}^*) \otimes (e_j e_{j'}^*) = E_{kk'} \otimes E_{jj'}. \quad (6)$$

We let $\langle A, B \rangle_{\mathrm{HS}} := \operatorname{Trace}(A^* B) = \langle \overrightarrow{A}, \overrightarrow{B} \rangle$ denote the Hilbert-Schmidt inner product on $\mathbb{C}^{n\times n}$ and remark that, for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$,

$$|\langle \mathbf{x}, \mathbf{y} \rangle|^2 = \langle \mathbf{x}\mathbf{x}^*, \mathbf{y}\mathbf{y}^* \rangle_{\mathrm{HS}}. \quad (7)$$

In addition, indices are always taken modulo d, and for indices we define

$$|i - j| := \min\{k : k \equiv i - j \mod d \text{ or } k \equiv j - i \mod d, k \geq 0\} \quad (8)$$

so that $|i - j| < \ell$ implies that there is some $k, |k| < \ell$ such that $j + k \equiv i \mod d$.

2. AN EFFICIENT METHOD FOR SOLVING THE DISCRETE 2D PHASE RETRIEVAL PROBLEM

Our recovery method, outlined in Algorithm 1, aims to approximate an image $Q \in \mathbb{C}^{d\times d}$ from phaseless measurements of the form (5). In this analysis, we generalize by considering a collection of measurements given by

$$y_{(\ell,\ell',u,v)} := \left|\left\langle Q, S_\ell^* \mathbf{a}_u \left(S_{\ell'}^* \mathbf{b}_v\right)^* \right\rangle_{\mathrm{HS}}\right|^2 \tag{9}$$

for all $(\ell,\ell',u,v) \in [d]^2 \times \Omega^2$ where $\Omega \subset [d]$ has $|\Omega| = 2\delta - 1$. Thus, we collect a total of $D := (2\delta-1)^2 \cdot d^2$ measurements where each measurement is due to a vertical and horizontal shift of a rank one illumination pattern $\mathbf{a}_u \mathbf{b}_v^* \in \mathbb{C}^{d\times d}$. Unlike the example of ptychography, in this analysis we do not require that $\mathbf{a}_u$ and $\mathbf{b}_v$ are modulations of fixed vectors $\mathbf{a}$ and $\mathbf{b}$; rather $\mathbf{a}_u$ and $\mathbf{b}_v$ may be chosen arbitrarily (to allow more general setups). However, we do assume that our measurements are *local* in the sense that $\mathrm{supp}(\mathbf{a}), \mathrm{supp}(\mathbf{b}) \subset [\delta]$. Recall that $\delta \ll d$, so the total number of measurements D is essentially linear in the problem size.

Algorithm 1 consists of first rephrasing the system (9) as a linear system on the space of $d^2 \times d^2$ matrices (following Candes, et al.[6]), and then estimating a projection $\mathcal{P}(\overrightarrow{Q}\overrightarrow{Q}^*)$ of the rank one matrix $\overrightarrow{Q}\overrightarrow{Q}^*$ from this system. This process is described in §2.1. In §§2.2-2.3 we show how the magnitudes of the entries of Q are estimated directly from $\mathcal{P}(\overrightarrow{Q}\overrightarrow{Q}^*)$ and their phases are found from solving an eigenvector problem. Together, the magnitude and phase estimates provide an approximation of Q.

2.1 The Linear Measurement Operator $\mathcal{M}$ and Its Inverse

To produce the linear system of step 1, we observe that

$$\begin{aligned} y_{(\ell,\ell',u,v)} &= \left|\left\langle Q, S_\ell^* \mathbf{a}_u \left(S_{\ell'}^* \mathbf{b}_v\right)^* \right\rangle_{\mathrm{HS}}\right|^2 = |\langle \overrightarrow{Q}, S_{\ell'}^* \overline{\mathbf{b}_u} \otimes S_\ell^* \mathbf{a}_v \rangle_{\mathrm{HS}}|^2 \\ &= \left\langle \overrightarrow{Q}\overrightarrow{Q}^*,\ \left(S_{\ell'}^* \overline{\mathbf{b}_u} \otimes S_\ell^* \mathbf{a}_v\right)\left(S_{\ell'}^* \overline{\mathbf{b}_u} \otimes S_\ell^* \mathbf{a}_v\right)^* \right\rangle_{\mathrm{HS}}, \end{aligned}$$

which allows us to naturally define $\mathcal{M} : \mathbb{C}^{d^2\times d^2} \mapsto \mathbb{R}^D$ as the linear measurement operator given by

$$(\mathcal{M}(Z))_{(\ell,\ell',u,v)} := \left\langle Z,\ \left(S_{\ell'}^* \overline{\mathbf{b}_u} \otimes S_\ell^* \mathbf{a}_v\right)\left(S_{\ell'}^* \overline{\mathbf{b}_u} \otimes S_\ell^* \mathbf{a}_v\right)^* \right\rangle_{\mathrm{HS}} = \left\langle Z,\ \left(S_{\ell'}^* \overline{\mathbf{b}_u \mathbf{b}_u}^* S_{\ell'}\right) \otimes \left(S_\ell^* \mathbf{a}_v \mathbf{a}_v^* S_\ell\right) \right\rangle_{\mathrm{HS}}, \tag{10}$$

so that $\mathbf{y} = \mathcal{M}(\overrightarrow{Q}\overrightarrow{Q}^*)$. This allows us to solve for $\mathcal{P}(\overrightarrow{Q}\overrightarrow{Q}^*)$, the projection of $\overrightarrow{Q}\overrightarrow{Q}^*$ onto the rowspace $\mathcal{P}(\mathbb{C}^{d^2\times d^2})$ of $\mathcal{M}$. For clarity, we will abbreviate $\mathcal{P}(\mathbb{C}^{d^2\times d^2})$ as $\mathcal{P}$, identifying this subspace with its orthogonal projection operator.

We observe that the local supports of $\mathbf{a}_u$ and $\mathbf{b}_v$ ensure that $\mathcal{M}(\overrightarrow{E_{j,k}}\overrightarrow{E_{j',k'}}^*) = \mathbf{0}$ whenever either $|j-j'| \geq \delta$ or $|k-k'| \geq \delta$ holds (this is clear from (10) and (6)). As a result we can see that $\mathcal{P} \subset \mathcal{B}$ where

$$\mathcal{B} := \mathrm{span}\{\overrightarrow{E_{j,k}}\overrightarrow{E_{j',k'}}^* \mid |j-j'| < \delta, |k-k'| < \delta\}. \tag{11}$$

Algorithm 1 Two Dimensional Phase Retrieval from Local Measurements

Input: Measurements $\mathbf{y} \in \mathbb{R}^D$ as per (9)
Output: $X \in \mathbb{C}^{d\times d}$ with $X \approx \mathrm{e}^{-\mathrm{i}\theta} Q$ for some $\theta \in [0, 2\pi]$

1: Compute the Hermitian matrix $P = \left(\left(\mathcal{M}|_{\mathcal{P}}\right)^{-1}\mathbf{y}\right)/2 + \left(\left(\mathcal{M}|_{\mathcal{P}}\right)^{-1}\mathbf{y}\right)^*/2 \in \mathcal{P}\left(\mathbb{C}^{d^2\times d^2}\right)$ as an estimate of $\mathcal{P}\left(\overrightarrow{Q}\overrightarrow{Q}^*\right)$. $\mathcal{M}$ and $\mathcal{P}$ are as defined in (10) and §2.1.
2: Form the matrix of phases, $\widetilde{P} \in \mathcal{P}\left(\mathbb{C}^{d^2\times d^2}\right)$, by normalizing the non-zero entries of P.
3: Compute the principal eigenvector of $\widetilde{P}$ and use it to compute $U_{j,k} \approx \mathrm{sgn}\left(Q_{j,k}\right)\ \forall j,k \in [d]$ as per §2.2.
4: Use the diagonal entries of P to compute $M_{j,k} \approx |Q_{j,k}|^2$ for all $j,k \in [d]$ as per §2.3.
5: Set $X_{j,k} = \sqrt{M_{j,k}} \cdot U_{j,k}$ for all $j,k \in [d]$ to form X

In steps 2-4 of Algorithm 1, recovery of Q from $\mathcal{P}(\overrightarrow{Q}\overrightarrow{Q}^*)$ relies on having $\mathcal{P} = \mathcal{B}$ exactly; we say in such a case that $\mathcal{M}|_\mathcal{B}$ is invertible. Clearly, the invertibility of $\mathcal{M}$ over $\mathcal{B}$ will depend on our choice of $\mathbf{a}$ and $\mathbf{b}$. We prove the following proposition, a corollary of which identifies pairs $\mathbf{a}, \mathbf{b}$ which produce an invertible linear system:

PROPOSITION 1. *Let $T_\delta : \mathbb{C}^{d\times d} \to \mathbb{C}^{d\times d}$ be the operator given by*

$$T_\delta(X)_{ij} = \begin{cases} X_{ij}, & |i-j| < \delta \mod d \\ 0, & \textit{otherwise} \end{cases}.$$

If the space $T_\delta(\mathbb{C}^{d\times d})$ is spanned by the collection $\{\mathbf{a}_j\mathbf{a}_j^*\}_{j=1}^K$, then $\mathcal{B}$ is spanned by

$$\{(\mathbf{a}_j \otimes \mathbf{a}_{j'})(\mathbf{a}_j \otimes \mathbf{a}_{j'})^*\}_{(j,j')\in[K]^2} = \{(\mathbf{a}_j\mathbf{a}_j^*) \otimes (\mathbf{a}_{j'}\mathbf{a}_{j'}^*)\}_{(j,j')\in[K]^2}.$$

Proof. By (6), it suffices to show that

$$(e_k e_{k'}^*) \otimes (e_j e_{j'}^*) \in \operatorname{span}\{(\mathbf{a}_n\mathbf{a}_n^*) \otimes (\mathbf{a}_{n'}\mathbf{a}_{n'}^*)\}_{(n,n')\in[K]^2}$$

for any $|j-j'|, |k-k'| < \delta$. Indeed, we have that $\{E_{jj'} : |j-j'| < \delta \mod d\}$ forms a basis for $T_\delta(\mathbb{C}^{d\times d})$, so $E_{jj'}, E_{kk'} \in \operatorname{span}\{\mathbf{a}_n\mathbf{a}_n^*\}_{n\in[K]}$ and

$$(e_k e_{k'}^*) \otimes (e_j e_{j'}^*) \in \operatorname{span}\{(\mathbf{a}_n\mathbf{a}_n^*) \otimes (\mathbf{a}_{n'}\mathbf{a}_{n'}^*)\}_{(n,n')\in[K]^2}.$$

□

In Theorem 4 of Iwen et al.,[7] an illumination function $\mathbf{a} \in \mathbb{C}^d$ with $\operatorname{supp}(\mathbf{a}) \subset [\delta]$ is offered such that $\{S_\ell \mathbf{a}_u \mathbf{a}_u^* S_\ell^*\}_{(\ell,u)\in[d]^2}$ spans $T_\delta(\mathbb{C}^{d\times d})$. By proposition 1, this gives the following corollary.

COROLLARY 1. *Choose a constant $a \in [4,\infty)$ and let the vectors $\mathbf{a}_\ell$ be defined by $(\mathbf{a}_\ell)_k = \dfrac{\mathbb{e}^{-k/a} \cdot \mathbb{e}^{\frac{2\pi \mathbb{i} k}{2\delta-1}}}{\sqrt[4]{2\delta-1}} \cdot \mathbb{1}_{k\le\delta}$. Then*

$$\{S_\ell^* \overline{\mathbf{a}_u \mathbf{a}_u}^* S_\ell \otimes S_{\ell'}^* \mathbf{a}_v \mathbf{a}_v^* S_{\ell'}\}_{(u,v,\ell,\ell')\in[d]^2\times p[2\delta-1]^2}$$

spans $\mathcal{B}$.

Towards application in ptychography, we remark that the vectors listed in 1 may be achieved as modulations of $\mathbf{a}$ with $\mathbf{a}_k = \mathbb{e}^{-k/a}/\sqrt[4]{2\delta-1}$ if $2\delta-1$ divides d. This condition may be met by zero padding Q as needed. The next corollary provides an example of measurement vectors that span $\mathcal{B}$, but are not produced by modulations. The authors of this work also constructed another collection of vectors (Example 2 of the previous work[10]), yielding another spanning set for T_δ. By the same reasoning as above, we have the following corollary.

COROLLARY 2. *Let $\mathbf{a}_1 = e_1$ and for $k \in [\delta], \mathbf{a}_{2k} = e_1 + e_k, \mathbf{a}_{2k+1} = e_1 + \mathbb{i} e_k$. Then*

$$\operatorname{span}\left(S_\ell \overline{\mathbf{a}_u \mathbf{a}_u^*} S_\ell \otimes S_{\ell'} \mathbf{a}_v \mathbf{a}_v^* S_{\ell'}^*\right) = \mathcal{B}.$$

2.2 Computing the Phases of the Entries of Q after Inverting $\mathcal{M}\big|_\mathcal{B}$

Assuming that $\mathcal{P} = \mathcal{B}$ (as defined in (11); this condition may be satisfied according to Corollaries 1 and 2) so that we can recover $\mathcal{P}(\overrightarrow{Q}\overrightarrow{Q}^*) = \mathcal{B}(\overrightarrow{Q}\overrightarrow{Q}^*)$ from our measurements $\mathbf{y}$, we are still left with the problem of how to recover $\overrightarrow{Q}$ from $\mathcal{B}(\overrightarrow{Q}\overrightarrow{Q}^*)$. Our first step in solving for $\overrightarrow{Q}$ will be to compute all the phases of the entries of $\overrightarrow{Q}$ from $\mathcal{B}(\overrightarrow{Q}\overrightarrow{Q}^*)$. Thankfully, this can be solved as an angular synchronization problem[8] as in BlockPR.[9,10] Let $\mathbb{1} \in \mathbb{C}^{d^2 \times d^2}$ be the matrix of all ones, and $\operatorname{sgn} : \mathbb{C} \mapsto \mathbb{C}$ be

$$\operatorname{sgn}(z) = \begin{cases} \dfrac{z}{|z|}, & z \neq 0 \\ 1, & \text{otherwise} \end{cases}.$$

We now define $\widetilde{Q} \in \mathbb{C}^{d^2 \times d^2}$ by $\widetilde{Q} = \mathcal{B}(\operatorname{sgn}(\overrightarrow{Q}\overrightarrow{Q}^*))$ (i.e. $\widetilde{Q}$ is $\mathcal{B}(\overrightarrow{Q}\overrightarrow{Q}^*)$ with its non-zero entries normalized). As we shall see, the principal eigenvector of $\widetilde{Q}$ will provide us with all of the phases of the entries of $\overrightarrow{Q}$.

Indeed, we may note that

$$\widetilde{Q} = \operatorname{diag}\left(\operatorname{sgn}\left(\overrightarrow{Q}\right)\right) \mathcal{B}\left(\mathbb{1}\mathbb{1}^*\right) \operatorname{diag}\left(\overline{\operatorname{sgn}\left(\overrightarrow{Q}\right)}\right) \tag{12}$$

where sgn is applied component-wise to vectors, and where $\operatorname{diag}(\mathbf{x}) \in \mathbb{C}^{d^2 \times d^2}$ is diagonal with $(\operatorname{diag}(\mathbf{x}))_{j,j} := x_j$ for all $\mathbf{x} \in \mathbb{C}^{d^2}$ and $j \in [d^2]$. As $\operatorname{diag}(\operatorname{sgn}(\cdot))$ always produces a unitary diagonal matrix, we can further see that the spectral structure of $\widetilde{Q}$ is determined by $\mathcal{B}\left(\mathbb{1}\mathbb{1}^*\right)$. The following theorem completely characterizes the eigenvalues and eigenvectors of $\mathcal{B}\left(\mathbb{1}\mathbb{1}^*\right)$.

THEOREM 1. *Let $F \in \mathbb{C}^{d \times d}$ be the unitary discrete Fourier transform matrix with $F_{j,k} := \frac{1}{\sqrt{d}} \mathbb{e}^{2\pi \mathbb{i} \frac{(j-1)(k-1)}{d}}$ $\forall j, k \in [d]$, and let $D \in \mathbb{C}^{d \times d}$ be the diagonal matrix with $D_{j,j} = 1 + 2\sum_{k=1}^{\delta-1} \cos\left(\frac{2\pi(j-1)k}{d}\right)$ $\forall j \in [d]$. Then,*

$$\mathcal{B}\left(\mathbb{1}\mathbb{1}^*\right) = \left(F \otimes F\right)\left(D \otimes D\right)\left(F \otimes F\right)^*.$$

In particular, the principal eigenvector of $\mathcal{B}\left(\mathbb{1}\mathbb{1}^*\right)$ is $\mathbb{1}$ and its associated eigenvector is $(2\delta - 1)^2$.

Proof. From the definition of $\mathcal{B}$ we have that

$$\begin{aligned}
\mathcal{B}\left(\mathbb{1}\mathbb{1}^*\right) &= \sum_{j=1}^{d} \sum_{|j-j'|<\delta} \sum_{k=1}^{d} \sum_{|k-k'|<\delta} \overrightarrow{E_{j,k}} \left(\overrightarrow{E_{j',k'}}\right)^* \\
&= \sum_{j=1}^{d} \sum_{|j-j'|<\delta} \sum_{k=1}^{d} \sum_{|k-k'|<\delta} E_{kk'} \otimes E_{jj'} \\
&= \left(\sum_{k=1}^{d} \sum_{|k-k'|<\delta} E_{kk'}\right) \otimes \left(\sum_{j=1}^{d} \sum_{|j-j'|<\delta} E_{jj'}\right) \\
&= T_\delta\left(\mathbb{1}\mathbb{1}^*\right) \otimes T_\delta\left(\mathbb{1}\mathbb{1}^*\right).
\end{aligned}$$

Thankfully the eigenvectors and eigenvalues of $T_\delta(\mathbb{1}\mathbb{1}^*)$ are known (see Lemma 1 of our previous work[10]). Specifically, $T_\delta(\mathbb{1}\mathbb{1}^*) = FDF^*$ which then yields the desired result by Theorem 4.2.12 of Horn and Johnson.[11] $\square$

Theorem 1 in combination with (12) makes it clear that $\operatorname{sgn}(\overrightarrow{Q})$ will be the principal eigenvector of $\widetilde{Q}$. As a result, we can rapidly compute the phases of all the entries of $\overrightarrow{Q}$ by using, e.g., a shifted inverse power method[12] in order to compute the eigenvector of $\widetilde{Q}$ corresponding to the leading eigenvalue $(2\delta - 1)^2$.

2.3 Computing the Magnitudes of the Entries of Q after Inverting $\mathcal{M}\big|_{\mathcal{B}}$

Having found the phases of each entry of $\overrightarrow{Q}$ using $\mathcal{B}(\overrightarrow{Q}\overrightarrow{Q}^*)$, it only remains to find each entry's magnitude as well. This is comparably easy to achieve. Note that $\mathcal{B}$ trivially contains $\overrightarrow{E_{j,k}}\overrightarrow{E_{j,k}}^* = e_k e_k^* \otimes e_j e_j^*$ for all $j, k \in [d]$, so $\mathcal{B}(\overrightarrow{Q}\overrightarrow{Q}^*)$ is guaranteed to always provide the diagonal entries of $\overrightarrow{Q}\overrightarrow{Q}^*$, which are exactly the squared magnitudes of the entries of $\overrightarrow{Q}$. Combined with the phase information recovered in step 2 of Algorithm 1, we are finally able to reconstruct every entry of $\overrightarrow{Q}$ up to a global phase as in step 5.

3. NUMERICAL EVALUATION

We will now demonstrate the efficiency and robustness of Algorithm 1. Figures 1b and 1d plot the reconstruction of two 256×256 test images (shown in Figures 1a and 1c respectively) from squared magnitude local measurements of the type described in (3). Here both $\mathbf{a}$ and $\mathbf{b}$ are chosen to be the *deterministic* measurement vectors of Corollary 2, detailed in Example 2 of [10, Section 2], with δ chosen to be 2. The reconstructions were computed using an implementation of Algorithm 1 in Matlab® running on a Laptop computer (Ubuntu Linux 16.04 `x86_64`, Intel® Core™M-5Y10c processor, 8GB RAM, Matlab® R2016b). More specifically, the linear measurement operator $\mathcal{M}|_{\mathcal{P}}$ was constructed by passing the standard basis elements $E_{i,j}, i,j \in [d], |i-j| < \delta \mod d$ through (10). An LU decomposition of this sparse and structured matrix was pre-computed and stored for different values of d, δ for use in the numerical simulations below. We note that an FFT-based implementation of Step 1 of Algorithm 1 is likely to yield improved efficiency; we defer such an implementation to future work. The relative errors, defined by the expression $\frac{\min_\theta \|\mathrm{e}^{\mathrm{i}\theta} X - Q\|_F}{\|Q\|_F}$ (where X denotes the reconstruction (up to a global phase factor) of Q), were 4.288×10^{-16} and 2.857×10^{-16} for the reconstructions in Figures 1b and 1d respectively. The reconstructions were computed in 16.318 and 16.529 seconds respectively.

We next plot the execution time (in seconds, averaged over 50 trials) required to implement Algorithm 1 for different values of d in Fig. 2a. In each case, δ was chosen to be $\lceil \log_2(d) \rceil$, with the same choice of measurement vectors as for the reconstruction in Fig. 1b. The plot confirms that the proposed method is extremely efficient; indeed, the plot reveals an FFT-time empirical computational complexity of $\mathcal{O}(d^2 \log_2(d^2))$.

Finally, Fig. 2b illustrates the robustness of the proposed method to measurement errors (an analysis of noise robustness is omitted from the paper and should follow from an appropriate generalization of the techniques used for the analysis of BlockPR[10] in the one-dimensional case). The figure plots the reconstruction error (averaged over 50 trials) in reconstructing a 64×64 random matrix with i.i.d zero-mean complex Gaussian entries from phaseless measurements (with $\delta = 6$, and with the same measurement construction as with Fig. 1b). An additive noise model with i.i.d. zero-mean Gaussian noise was used to corrupt the measurements. The added noise as well as reconstruction error are reported in decibels, with

$$\text{SNR (dB)} = 10 \log_{10} \left(\frac{\|\mathbf{y}\|_2^2}{D\sigma^2} \right), \qquad \text{Error (dB)} = 10 \log_{10} \left(\frac{\min_\theta \|\mathrm{e}^{\mathrm{i}\theta} X - Q\|_F^2}{\|Q\|_F^2} \right).$$

We observe that the proposed algorithm (indicated by the dashed line) demonstrates robustness across a wide variety of SNRs. Additionally, the results from utilizing an improved (eigenvector-based) magnitude estimation method (detailed in [10, Section 6.1]) in place of Step 4 of Algorithm 1 is plotted using the solid line. In both cases, we observe that the test signals are reconstructed to (almost) the level of added noise.

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REFERENCES

[1] Gerchberg, R. and Saxton, W., "A practical algorithm for the determination of the phase from image and diffraction plane pictures," *Optik* **35**, 237246 (1972).

[2] Walther, A., "The Question of Phase Retrieval in Optics," *Optica Acta: International Journal of Optics* **10**, 41–49 (1963).

[3] Millane, R., "Phase retrieval in crystallography and optics," *Journal of the Optical Society of America A* **7**(3), 394–411 (1990).

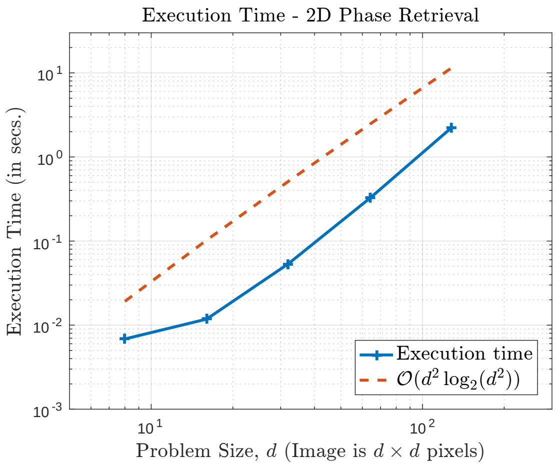
(a) Test Image 1 (256×256 pixels)

(b) Reconstructed Image (Rel. error 4.288×10^{-16})

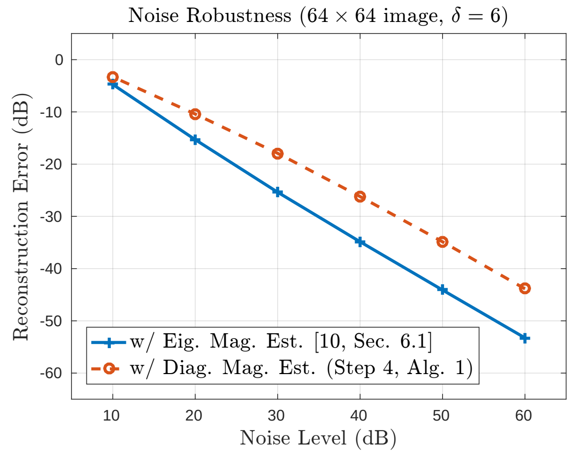
(c) Test Image 2 (256×256 pixels)

(d) Reconstructed Image (Rel. error 2.857×10^{-16})

Figure 1: Two Dimensional Image Reconstruction from Phaseless Local Measurements.



(a) Execution Time vs Problem Size



(b) Noise Robustness of the Proposed Method

Figure 2: Evaluating the Efficiency and Robustness of the Proposed Two Dimensional Phase Retrieval Algorithm.

[4] Drenth, A., Huiser, A., and Ferwerda, H., "The problem of phase retrieval in light and electron microscopy of strong objects," *Optica Acta: International Journal of Optics* **22**(7), 615–628 (1975).

[5] Rodenburg, J., "Ptychography and related diffractive imaging methods," *Advances in Imaging and Electron Physics* **150**, 87–184 (2008).

[6] Candes, E. J., Strohmer, T., and Voroninski, V., "Phaselift: Exact and stable signal recovery from magnitude measurements via convex programming," *Communications on Pure and Applied Mathematics* **66**(8), 1241–1274 (2013).

[7] Iwen, M. A., Viswanathan, A., and Wang, Y., "Fast phase retrieval from local correlation measurements," *SIAM Journal on Imaging Sciences* **9**(4), 1655–1688 (2016).

[8] Singer, A., "Angular synchronization by eigenvectors and semidefinite programming," *Applied and Computational Harmonic Analysis* **30**(1), 20–36 (2011).

[9] Viswanathan, A. and Iwen, M., "Fast angular synchronization for phase retrieval via incomplete information," in [*Proceedings of SPIE*], **9597**, 959718–959718–8 (2015).

[10] Iwen, M. A., Preskitt, B., Saab, R., and Viswanathan, A., "Phase retrieval from local measurements: Improved robustness via eigenvector-based angular synchronization," (2016). preprint arXiv:1612.01182.

[11] Horn, R. A. and Johnson, C. R., [*Topics in Matrix Analysis*], Cambridge University Press (1991).

[12] Trefethen, L. N. and Bau III, D., [*Numerical Linear Algebra*], vol. 50, SIAM (1997).