Any of the following exercises are fair game for your final oral exam. I suggest that you write up your solutions neatly in your own handwriting to consult during that exam!

Additional Reminder: You have an option of doing a project instead of doing the homework... so keep that in mind (and check the syllabus) if that sounds more appealing.

Coherence, Reach, and Sublinear-time Compressive Sensing

42. Let $T = \{(x,y) \mid x,y \in \mathbb{Z}\} \subset \mathbb{R}^2$. What is the reach of T? Let $\alpha \in \mathbb{R}^+$. What is the reach of $\alpha T := \{\alpha \mathbf{x} \mid \mathbf{x} \in T\}$?

43. Let $r, R \in \mathbb{R}^+$ be such that $r < R$. What is the reach of the d-dimensional annulus

$$\left\{ (x_1, \dots, x_d, 0, \dots, 0) \;\middle|\; r^2 \leq \sum_{j=1}^{d} x_j^2 \leq R^2 \right\} \subset \mathbb{R}^D?$$

44. Prove that a set $T \subset \mathbb{R}^N$ has infinite reach if and only if it is closed and convex.

45. Let $A \in \mathbb{C}^{m \times N}$ have ℓ_2-normalized columns. What is the largest possible value of coherence $\mu(A)$ that it can have? Prove that your answer is correct.

46. Do homework exercise 6.1.1 on page 197 of the notes (`https://math.msu.edu/~iwenmark/Notes_Fall2020_Iwen_Classes.pdf`). Note that it has 4 parts.

47. Suppose you know that a 2π-periodic function $f : [-\pi, \pi] \to \mathbb{C}$ you want to learn about is composed of exactly one frequency component. That is, that $f(x) = A\mathbb{e}^{\mathbb{i}\omega x}$ for unknown parameters $A \in \mathbb{C}$ and $\omega \in \mathbb{Z} \cap [-127627, 127627]$. Use **ALIASING THM** from class (see below) together with the Chinese Remainder Theorem in order to show that you can learn both A and ω by sampling f at just 51 different points $\in [-\pi, \pi]$.

    (a) **ALIASING THM:** Let $\tilde{c}_n := \frac{(-1)^n}{N} \sum_{k=0}^{N-1} f\left(-\pi + k \cdot \frac{2\pi}{N}\right) \mathbb{e}^{\frac{-2\pi \mathbb{i} n k}{N}}$. Then,

    $$\tilde{c}_n = \sum_{q=-\infty}^{\infty} (-1)^q \, c_{n+Nq} = \sum_{m \equiv n \mod N} (-1)^{\frac{m-n}{N}} c_m,$$

    where c_n is the n^{th} Fourier series coefficient of $f : [-\pi, \pi] \to \mathbb{C}$.

    HINT: You will want to use 6 sets of equally spaced samples in $[-\pi, \pi]$, each associated with a different prime number.