Simple Interest Loans –

Loan/Deposit w/ interest accruing only on the Principle.

* The principle is the value of the loan @ time = 0

* Such loans exist in cases w/ strict rules
- Short term
- Repayment schedules – Mortgage.
  – ie Amortizing (having payoff schedule)

Suppose Interest accrues on a daily basis, at a rate r (per year)
then on the nth day the value of the loan is

$$V\left(n\tfrac{1}{365}\right) = V(0)\left(1 + r\tfrac{n}{365}\right) = P\left(1 + r\tfrac{n}{365}\right)$$

where $V(0) = P$ is called the principle.

for $t \in \frac{1}{365} \times \{0, 1, 2, \dots\} = \frac{1}{365}\mathbb{N}_0$

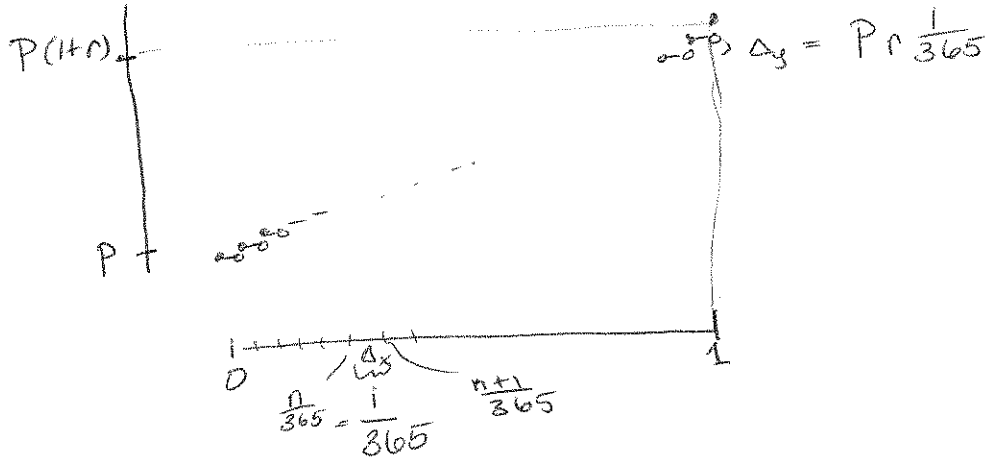
$$V(t) = P\,(1 + rt).$$

for $t \geq 0$

$$V(t) = P\left(1 + r\,\frac{\lfloor t\,365 \rfloor}{365}\right)$$

where $\lfloor n + \alpha \rfloor = n$ for $0 \leq \alpha < 1$. $n \in 0, 1, 2 \dots$

GRAPH OF $V(t)$ ...



why?

$$\Delta_y = P\left(1 + r\,\frac{n+1}{365}\right) - P\left(1 + r\,\frac{n}{365}\right)$$

$$= Pr\,\frac{1}{365} \quad \checkmark$$

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Sometimes it is helpful to interpolate linearly the values at each $\frac{n}{365}$

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The <u>return</u> on a loan is defined as the change in value of the loan over the initial value:

$$K(0,t) = \frac{V(t) - V(0)}{V(0)} = t\,r$$

for ~~continuous~~ interpolated interest

$$K(0,1) = r.$$

Notice If we hold \$100 in our hand

vs invest \$100

the ratio of values becomes:

$$\frac{100}{100(1+r)} = (1+r)^{-1}$$

$\therefore$ $(1+r)^{-1}$ is the <u>discount factor</u>

for the time value of money.

In general $\dfrac{V(0)}{V(t)} = \dfrac{V(0)}{V(0)\,(1+rt)} = \dfrac{1}{1+rt}$ (interpolate

Define (for simple interest) Discount factor

$$\beta_t = \frac{1}{1+rt}.$$

β_t = "factor value of money is discounted by"

$\therefore$ If you have an I.O.U. (Bond) which

pay \$100 in 1 year,

today it is worth

\$ 100β. (in todays dollar

Fundamental reality:

Money Changes Value over Time!

\$100 may be invested to ~~earn~~ become \$100(1+r) in one year

∴ \$100 in todays money is worth

$\geq$ \$100(1+r) in +1 year money

why $\geq$ \$100(1+r)? in +1 year money?

If loan is available I can get <u>at least</u> this much

... what if there is another oportunity to get more?

For now, let us assume some 'ideal & fixed' interest rate $r > 0$.

- All loans @ rate r...

then.

\$ $\frac{100}{1+r}$ invested 1 year ago becomes \$100 today.

Compound Interest - Periodic.

Eg Savings account - interest accrued each day or month is added to principle

Initial value of loan/acct. P

Interest rate: r

# of periods in year: m

Eg ~~m = 365~~ - interest added every ~~month~~ day

Deposit Jan 1st value $= P$ $= V(0)$

Jan 2nd value $= P\left(1+\frac{r}{365}\right)$ $= V\left(\frac{1}{365}\right)$

Jan 3rd ——— $P\left(1+\frac{r}{365}\right)^2$ $= V\left(\frac{2}{365}\right)$

$\vdots$

n^{th} day of year $V\left(\frac{n}{365}\right) = P\left(1+\frac{r}{365}\right)^n$

end of year $V(1) = P\left(1+\frac{r}{365}\right)^{365}$.

Again let us 'interpolate' in the time parameter

$$V(t) = P\left(1+\frac{r}{365}\right)^{t\,365}$$

In general, for m periods per year,

$$V(t) = P\left(1+\frac{r}{m}\right)^{tm}.$$

More frequent compounding Increases return.

Define

$$V_m(t) = P\left(1+\frac{r}{m}\right)^{mt}$$

We want to see that if $m > k$ then

$$V_m(t) = P\left(1+\frac{r}{m}\right)^{mt} > P\left(1+\frac{r}{k}\right)^{kt} = V_k(t).$$

Letus show

$$\left(1+\frac{r}{m}\right)^{m} > \left(1+\frac{r}{k}\right)^{k} \text{ for } m > k.$$

~~Defined~~ Define

$$F(x,y) = \left(1+\frac{r}{x}\right)^{y}, \quad x = x(s) = s, \quad y = y(s) = s.$$

Recall

$$\frac{d}{ds}F(x,y) = \frac{d}{dx}F\frac{d}{ds}x + \frac{d}{dy}F\frac{d}{ds}y.$$

$$\frac{dx}{ds} = \frac{dy}{ds} = 1.$$

$\therefore$

$$\frac{d}{ds} F(x,y) = \frac{d}{dx}\left(1+\frac{r}{x}\right)^y + \frac{d}{dy} e^{y\ln(1+\frac{r}{x})}$$

$$= y\left(1+\frac{r}{x}\right)^{y-1}\left(-\frac{r}{x^2}\right) + \ln\left(1+\frac{r}{x}\right) e^{y\ln(1+\frac{r}{x})}$$

$$= \left(-\frac{ry}{x^2}\right)\left(1+\frac{r}{x}\right)^{y-1} + \ln\left(1+\frac{r}{x}\right)\left(1+\frac{r}{x}\right)^y$$

$$= \left[\left(1+\frac{r}{x}\right)\ln\left(1+\frac{r}{x}\right) - \frac{ry}{x^2}\right]\left(1+\frac{r}{x}\right)^{y-1}$$

$y = x = s$

$$= \left[\left(1+\frac{r}{s}\right)\ln\left(1+\frac{r}{s}\right) - \frac{r}{s}\right]\left(1+\frac{r}{s}\right)^{s-1}$$

of course $\left(1+\frac{r}{s}\right)^{s-1} > 0$

so we need only show the first factor is greater than zero, ie

$$\left(1+\frac{r}{s}\right)\ln\left(1+\frac{r}{s}\right) - \frac{r}{s} > 0.$$

Let us write

$$f(s) = \left(1+\frac{r}{s}\right)\ln\left(1+\frac{r}{s}\right) - \frac{r}{s} - 1 + 1.$$

and let $z = 1+\frac{r}{s}$

then

$$f(s) = \tilde{f}(z) = z\ln z - z + 1.$$

when $s \to \infty$ we have $z \to 1$

and when we increase z above 1 s falls to positive real #s.

ie we need to show for $z > 1$ that

$$\tilde{f}(z) > 0. \quad (*)$$

But $\tilde{f}(1) = \underbrace{1\ln 1}_{0} - 1 + 1 = 0.$

+

$$\frac{d}{dz}\tilde{f}(z) = \frac{z}{z} + \ln z - 1 + 0$$

$$= \ln z$$

But $\ln z > 0$ for $z > 1$ so $(*)$ holds.

Discount factor

Recall definition $\beta_t = \dfrac{V(0)}{V(t)}$.

$V(t) = \left(1+\frac{r}{m}\right)^{tm}$

$$\longrightarrow \beta_t = \left(1+\frac{r}{m}\right)^{-tm}$$

Return (on investment)

Defined as

$$K(s,t) = \frac{V(t) - V(s)}{V(s)}$$

$$= \frac{P\left(1+\frac{r}{m}\right)^{tm} - P\left(1+\frac{r}{m}\right)^{sm}}{P\left(1+\frac{r}{m}\right)^{s}}$$

$$= \left(1+\frac{r}{m}\right)^{(t-s)m} - 1.$$

Notice $K(0, \frac{1}{m}) = r/m$.

Also

$$K(s,t) = \frac{V(t) - V(s)}{V(s)} = \frac{V(t)}{V(s)} - 1 = \frac{1}{\beta_{t-s}} - 1$$

or

$$\beta_{t-s} = \frac{1}{1+K(s,t)}$$