FORWARD CONTRACT

concept: GIVEN SECURITY S worth $S(t)$ at time t

for $t \in \mathbb{T}$, $\mathbb{T} = \begin{cases} 0, 1, \ldots, T \\ [0,T] \end{cases}$

* Assume $S(0)$ is given + $S(t) : \Omega \to (0, \infty)$ for $t \in \mathbb{T} \setminus \{0\}$

We write contract to purchase security at time T

for value $F(0,T)$

In general, a security S might involve transferse

of money ie maintenence/dividend

this is called <u>cost of carry</u>.

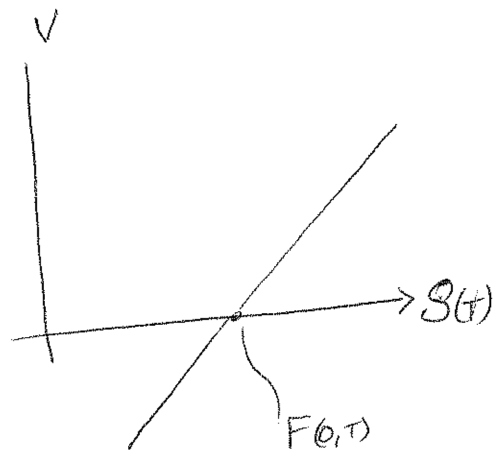
For now let us assume cost of carry $= 0$.

The value of the Future contract (payoff/liability)

Is the difference between the

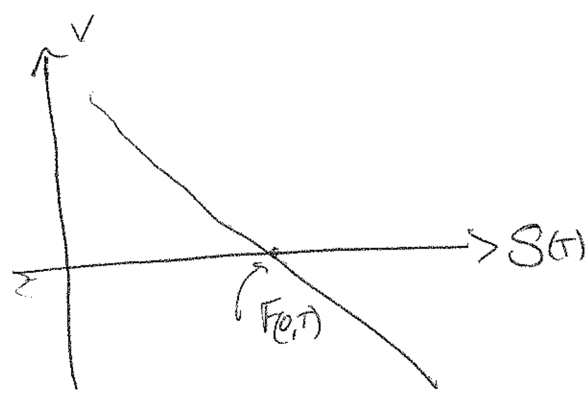
value of the security and the contracted exchange price.

If you hold the Future contract, the payoff is



$V(T) = S(T) - F(0,T)$.

If you write the Future contract the payoff is inverted,



$V(T) = F(0,T) - S(T)$.

To determine the price the security should be traded at, we must include the time change of money.

$\therefore$ Let A be a bond, with value A_t at time $t \in \mathbb{T}$.

Eg $A_t = e^{rt}$.

$\therefore$ Value of time 0 \$ in time T \$ is $\frac{A_T}{A_0}$.

& Value of time T \$ in time 0 \$ is $\frac{A_0}{A_T}$.

Equivalently: discount factor $\beta_T = \frac{A_0}{A_T}$.

$\therefore$ PRICE TO EXCHANGE SECURITY FOR at time T is

$$F(0,T) = S(0)\frac{1}{\beta_T} = S(0)\frac{A_T}{A_0}.$$

Strategy to determine price:

Compare 1. Borrow to purchase stock + clear bond at time T. Short forward.

2. Short contract + invest. Long Forward.

<u>First strategy</u>

Suppose price of exchange offered is $F > S(0)\frac{1}{\beta_T}$

Price of exchange is too <u>high</u> so we need to arrange to <u>sell</u> security @ time T.

Time $t=0$, (i) Borrow $S(0)$

(ii) Buy security for $S(0)$.

(iii) Short forward: Agree to sell at price F @ time T.

$$(\$, S, F) = (-S(0), 1, -1) \leftarrow \text{portfolio.}$$

Time $t=T$. (i) Sell security for F

(ii) close Bond, pay $S(0)\frac{1}{\beta_T}$

$V_T = F - S(0)\frac{1}{\beta_T} > 0$ ↯ No arbitrage.

$\therefore \quad F \leq S(0)\frac{1}{\beta_T}$

Second case:

Suppose price of exchange offered is $F < S(0) \frac{1}{\beta_T}$.

Price of exchange is too <u>low</u>

so we need to <u>buy</u> security @ this price.

<u>Time $t=0$</u> (i) Borrow security & sell it, gain $S(0)$

(ii) Put $S(0)$ into bond

(iii) Enter into long Forward:

agree to buy at price F @ time T.

$$(\$, S, F) = (S(0), -1, 1)$$

Time $t=T$,

~~(i) buy security for~~

(i) Cash bond, get $S(0) \frac{1}{\beta_T}$

(ii) Buy Bond for F

(iii) Return security to owner.

$$V_T = S(0) \frac{1}{\beta_T} - F > 0$$ ↯ No Arbitrage.

$$\therefore \; S(0) \frac{1}{\beta_T} \leq F$$

Recall notation:

$B(t,T) \equiv$ value of bond w/ (face value \$1 and maturity time T) at time $t<T$

Eg $\quad \underline{\underline{B(t,T) = e^{-r(T-t)}}} \quad ; \quad B(0,T) = \beta_T = e^{-rT}$

Similarly to pricing the $F(0,T)$ exchange price we can price $F(t,T) \equiv$ ~~exch~~ price (to exchange security at) at time T which is agreed at time t.

We find $F(t,T) = \dfrac{S(t)}{B(t,T)}$.

the purpose of allowing variable time t, is to consider ① security w/ cost of carry at given times.

② Buying selling ~~eg~~ Forward agreement.

Value of Forward contract w/o dividend,

At time 0, no money is exchanged to enter into long Forward: agreement to purchase security S @ time T for $F(0,T) = S(0)\frac{1}{B(0,T)}$.

Suppose we decide to ~~~~ liquidate contract at time $0 < t < T$, What is our profit/loss?

At time t a second Forward contract may be created to ~~~~ exchange security at time T for value $F(t,T) = S(t)\frac{1}{B(t,T)}$

$\therefore$ Value of holding contract is time t value of difference of the two Forward prices:

Ie: value @ time t in time T dollars:

$$V^{(T)}_{(t)} = F(t,T) - F(0,T)$$

Now correct to price in time t dollars

$$V(t) = B(t,T)\Big(F(t,T) - F(0,T)\Big) = S(t) - \frac{B(t,T)}{B(0,T)}S(0).$$

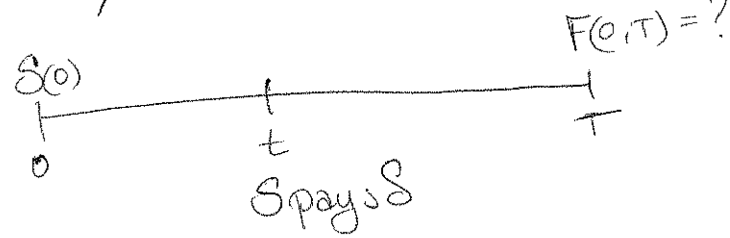
Notice $V(t) = S(t) - e^{+tr}S(0)$ if r is constant, if r is not we have

$$V(t) = S(t) - \frac{e^{-r_t(T-t)}}{e^{-r_0 T}}S(0) = S(t) - e^{(r_0 - r_t)T + r_t t}S(0)$$

Forwards for security w/ dividend payments.

Note with dividends, it is more attractive to hold onto security since then the holder recieves dividend payment.

Let δ be the dividend payment of Security at time t.



IDEA: NOTICE IN PREVIOUS PROOF, holding security would lead payment

ie $F - S(0)\frac{1}{\beta_T}$ + dividend > 0
~~dividend~~ profit !

Shorting security requires paying dividend to owner, this payment is taken out of profit:

$$S(0)\frac{1}{\beta_T} - F - \text{div} > 0$$

$\therefore$ we expect F should decrease given dividend.

Pricing future on Security w/ dividend, paying δ @ time $=\tau$.

Repeat strategies:

Case 1 : Time $t=0$
(i) Borrow $S(0)$
(ii) Buy security
(iii) Short forward
(iv) long forward to buy \$ δ worth of bond at time τ.

For (iv): cost of bond w/ maturity @ τ paying δ is $\delta B(0,\tau)$ at time $t=0$

Value at maturity T is then $\delta \dfrac{B(0,\tau)}{B(0,T)}$.

$$(\$, S, F_B, F_S) = (-S(0),\ 1,\ \delta B(0,\tau),\ -1)$$

At time $t=\tau$
(i) Recieve + cash dividend δ
(ii) Buy bond for δ.

At time $t=T$.
(i) cash bonds: $-S(0)\dfrac{1}{B(0,T)} + \delta \dfrac{B(0,\tau)}{B(0,T)}$
(ii) Clear forward on security - sell + collect F.

Value at time T:

$$\underbrace{F - S(0)\frac{1}{B(0,T)} + \delta\frac{B(0,\tau)}{B(0,T)} \le 0}_{\text{No arbitrage.}}$$

Case 2 Time t = 0 (i) Borrow Security & sell it for $S(0)$

(ii) Put $S(0)$ into bond.

(iii) Long forward

(iv) Borrow $\delta B(0,\tau)$

(a) put into Bond w/ maturity τ value δ

(b) amt borrowed has maturity T

we agree to pay $\delta \frac{B(0,\tau)}{B(0,T)}$.

$$(\$, S, F) = \left(\{S(0) - \delta B(0,\tau)\}_{0,T}, \{\delta B(0,\tau)\}_{0,\tau}, -1, 1\right)$$

Time τ (i) Collect Bond value δ

(ii) pay owner of security δ.

$$(\$, S, F) = \left(\{S(0) - \delta B(0,\tau)\}_{0,T}, -1, 1\right).$$

Time T

(i) Collect $\{S(0) - \delta B(0,\tau)\}\frac{1}{B(0,T)}$ in Bond

(ii) Purchase security for F.

(iii) Return security to owner.

Value

$$V_T = \left(S(0) - \delta B(0,\tau)\right)\frac{1}{B(0,T)} - F \underbrace{\leq 0}_{\text{No arbitrage}}.$$

$$\therefore \quad F = \left(S(0) - \delta B(0,\tau)\right)\frac{1}{B(0,T)}.$$

$\therefore$ For Forward @ ~~t~~time t

w/ maturity time T

on security S

which has dividend

payments $\delta_1, \ldots, \delta_n$

at times $\tau_1, \ldots, \tau_n$

has exchange price

$$F(t,T) = \Big(S(t) - \underbrace{\textstyle\sum_i}_{\text{sum over } i \text{ such that } t<\tau_i<T.} B(t,\tau_i)\,\delta_i \Big) \frac{1}{B(t,T)}$$

~~Note~~ Note, if 1 dividend (δ, τ),

& we take $t \nearrow \tau$ & $T \searrow \tau$.

$$\cancel{\lim_{t\nearrow, T\searrow\tau} F(t,T) = \lim \left(S(t) - B(t,\tau)\,\delta\right) \frac{1}{B(t,T)}}$$

$$\cancel{= S(\tau) - \delta.}$$

ie.

$$F(\tau-\varepsilon, \tau+\varepsilon) = \big(S(\tau-\varepsilon) - B(\tau-\varepsilon, \tau)\,\delta \big) \frac{1}{B(\tau-\varepsilon, \tau+\varepsilon)}$$

$$\underset{\varepsilon\sim 0}{\approx} S(\tau-0) - \delta$$

$\therefore$ during dividend payments stock changes by val of dividend: "$S(\tau+0) = S(\tau-0) - \delta$."

Again, we can determine Time value of Forward Contract:

Enter Forward at time 0 $\quad F(0,T)$

then value at time t in time T dollars is

$$V^{(T)}(t) = F(t,T) - F(0,T)$$

say $B(t,\tau_i) = 0$ for $t > \tau_i$

$$= \left(S(t) - \sum_i B(t,\tau_i)\,\delta_i\right)\frac{1}{B(t,T)}$$

$$- \left(S(0,T) - \sum B(0,\tau_i)\,\delta_i\right)\frac{1}{B(0,T)}$$

$$\cancel{S(t)\tfrac{1}{B}}$$

$$V(t) = S(t) - S(0,T)\,\frac{B(t,T)}{B(0,T)} + \sum \delta_i\left(B(0,\tau_i)\,\frac{B(t,T)}{B(0,T)} - B(t,\tau_i)\right)$$

Suppose one were to agree to purchase a security S at future date T for a given price P. What should one pay for this at date of contract writing?

Again this is just difference of exchange values corrected back to time t=0.

$$\therefore \quad V^{(T)}(0) = V_P^{(T)}(0) - V_F^{(T)}(0)$$

$$= \{-P + S(T)\} - \{S(T) - F(0,T)\}$$

$$= F(0,T) - P$$

$$\therefore \quad V(0) = B(0,T)\{F(0,T) - P\}$$

$$= S(0) - B(0,T)P$$

Eg $S(0) = 100$ $\quad r_e = .05$ Suppose you agree to purchase S at $t=1$ for \$ 90 today

you should pay

$$V(0) = 100 - \frac{1}{1.05} 90$$

Continuous dividend yields:

In some cases it is reasonable to assume security pays dividend at rate of its value

$\therefore$ say m div payments per year,

~~\$~~ dividen payment at time t is, $t \in \frac{1}{m}\mathbb{N}$.

$$\frac{r_\delta}{m} S(t)$$

If dividend is reinvested in stock:

then ~~\$~~ if ~~X~~ shares is owned ~~per~~ previously $x S(t)$ in stock

we have $x \frac{r_\delta}{m} S(t)$ collected we buy stock

then we then have $x\left(1+\frac{r_\delta}{m}\right) S(t)$ in stocks

which is $x\left(1+\frac{r_\delta}{m}\right)$ shares.

Ie price of S does not change # of shares we purchase

$\therefore$ after 1 year: if we start with 1 share

we have $\left(1+\frac{r_\delta}{m}\right)^m$ shares at end of year.

as $m \to \infty$ we find we own e^{r_δ} shares if money is continously reinvested.

Price Forward ~~for~~ w/ continuous dividend:

Time $t=0$ * Short ~~fwd~~ forward - promise to sell 1 share

* Borrow $S(0)\,e^{-r_\delta T}$ buy $e^{-r_\delta T}$ shares.

Time $0<t<T$ * continuously reinvest - @ time T you have 1 share.

Time T   * Sell share for F.

* Pay $\dfrac{S(0)\,e^{-r_\delta T}}{B(0,T)}$ to clear loan.

Finally: value $V_T = F - S_0 \dfrac{e^{-r_\delta T}}{B(0,T)} \underset{\text{no arb.}}{\leq 0}$

opposite:

Time $t=0$ * Long forward -

* Short stock $e^{-r_\delta T}$ shares.

Time $0<t<T$ * continuously short stock ~~to~~ to pay dividends, at ~~end of~~ time T 1 stock is shorted.

Time T,   * Collect bond $S(0)\,e^{-r_\delta T}\dfrac{1}{B(0,T)}$

* Buy share w/ forward + return to owner

$$V_T = S(0)\,\frac{e^{-r_\delta T}}{B(0,T)} - F \leq 0 .$$