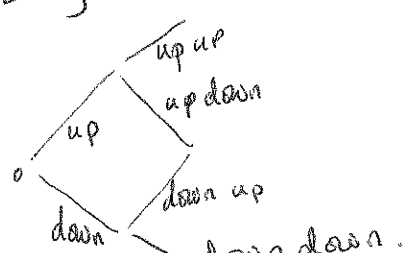
Twostep binomial model.—

Consider binomial model where price of stock ~~increases or dec~~ takes one of two values at $t+1$ given price at time t.

That is for $t = 0, 1$,
there is p_1, p_2 st

$$\mathbb{P}\{S(t+1) = S(t)(1+m_1)\} = p_1$$
$$\mathbb{P}\{S(t+1) = S(t)(1+m_2)\} = p_2 .$$

Pictorally we have



$\therefore$ outcomspace is

$$\Omega = \{ \omega_1 = uu, \ \omega_2 = ud, \ \omega_3 = du, \ \omega_4 = dd \}$$

If we limit info recovered upto time 1 wehave

limited resolution $\longrightarrow$ $\Omega_1 = \{ \tilde{\omega}_1 = u, \ \tilde{\omega}_2 = d \}$

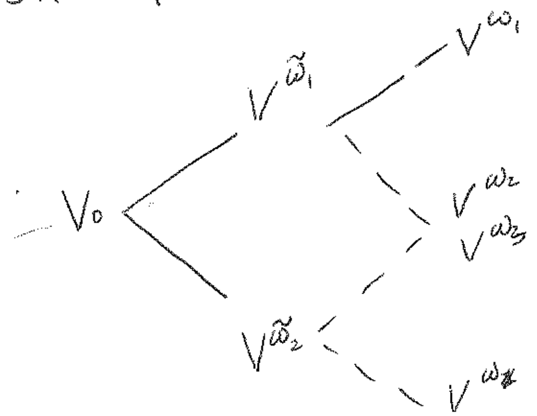
$$= \{ (uu, ud), (du, dd) \}$$

No resolution $\rightarrow$ Info @ time $t=0$: $\Omega_0 = \{ (uu, ud, du, dd) \}$

Consider an option on this model that has payoff V^{ω_i} in event $\omega_i \in \Omega$, the payoff is decided at time $T=2$.

Thus $V(2)$ is a random variable at time $t=0$ but is decided @ time $t=2$.

If we wish to price $V(2)$ notice



We could use the result from the 1 step case to find

$$V(0) = \frac{1}{1+r}\mathbb{E}^*(V(1)) = \frac{1}{1+r}\left(p_1^* V^{\tilde{\omega}_1} + p_2^* V^{\tilde{\omega}_2}\right)$$

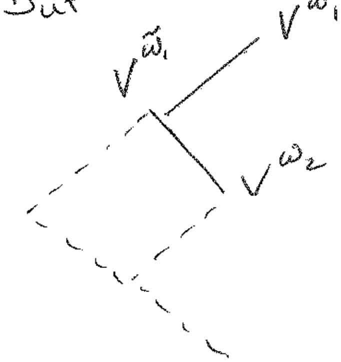
where:

$$p_1^* = \frac{r - m_2}{m_1 - m_2}$$

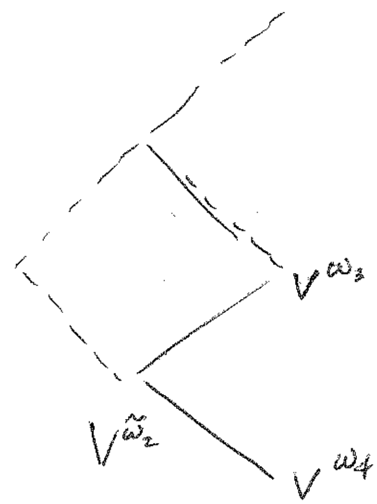
$$p_2^* = \frac{m_1 - r}{m_1 - m_2}.$$

Thus to find $V(0)$ we need only find $V^{\tilde{\omega}_i}$

But



and



that is, the second step is also just collections of sub 1 steps systems.

$$\therefore \quad V^{\tilde{\omega}_1} = \frac{1}{1+r}\,\mathbb{E}^*\left(V(2)\,|\,\tilde{\omega}_1\right) = \frac{1}{1+r}\left(p_1^* V^{\omega_1} + p_2^* V^{\omega_2}\right)$$

$$\text{and} \quad V^{\tilde{\omega}_2} = \frac{1}{1+r}\,\mathbb{E}^*\left(V(2)\,|\,\tilde{\omega}_2\right) = \frac{1}{1+r}\left(p_1^* V^{\omega_3} + p_2^* V^{\omega_4}\right)$$

Thus we have,

$$V(0) = \frac{1}{1+r}\left(p_1^* V^{\tilde{\omega}_1} + p_2^* V^{\tilde{\omega}_2}\right)$$

$$= \frac{1}{(1+r)^2}\left\{p_1^{*2} V^{\omega_1} + p_1^* p_2^* V^{\omega_2} + p_2^* p_1^* V^{\omega_3} + p_2^{*2} V^{\omega_4}\right\}.$$

Letusprice a Euro option inconcretexample -

Consider $S(0) = 80$.

$r = .02$, $m_1 = .04$, $m_2 = .01$

Euro call $T=2$, $X = 83$.

Find the value of call at expiry.

$$V^{\omega_1} = V^{uu} = \left(80(1.04)^2 - 83\right)^+ = \underline{3.5}$$

$$V^{\omega_2} = V^{ud} = \left(80(1.04)(1.01) - 83\right)^+ = \underline{1.03}$$

$$V^{\omega_2} = V^{\omega_3} \checkmark$$

$$V^{\omega_4} = \left(80(1.01)^2 - 83\right)^+ = 0$$

$$p_1^* = \frac{.02 - .01}{.04 - .01} = 1/3$$

$$p_2^* = \frac{.04 - .02}{.04 - .01} = 2/3.$$

Value at $t=0$:

$$V(0) = \frac{1}{(1.02)^2}\left\{\left(\frac{1}{3}\right)^2 3.5 + 2\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)(1.03) + \left(\frac{2}{3}\right)(0)\right\}$$

$$= \frac{1}{(1.02)^2}\left\{\frac{7}{18} + .46\right\} = .814$$

Consider asian Call option allowing us to purchase security for $X = 82$ & sell it for average price,

$$V(2) = \max\left(0, \frac{S(1) - S(2)}{2} - X\right)$$

Again find value @ expiry.

$$V^{\omega_1} = \left(\frac{80(1.04)^2 + 80(1.04)}{2} - 82\right)^+ = 2.86$$

$$V^{\omega_2} = \left(\frac{80(1.04)(1.01) + 80(1.04)}{2} - 82\right)^+ = 1.62$$

$$V^{\omega_3} = \left(\frac{80(1.01)(1.04) + 80(1.01)}{2} - 82\right)^+ = .42$$

$$V^{\omega_4} = \left(\frac{80(1.01)^2 + 80(1.01)}{2} - 82\right)^+ = 0.$$

Value @ $t=0$,

$$V(0) = \frac{1}{(1.02)^2}\left\{\left(\frac{1}{3}\right)^2 2.86 + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) 1.62 + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) .42 + \left(\frac{2}{3}\right)^2 0\right\}$$

$$= \frac{1}{(1.02)^2}\left\{.32 + .36 + .09\right\} = .74$$

Although we have priced the two step model, it is instructive to consider forming a portfolio over two steps that replicates value of an option $V^{\omega}(2)$.

Let H^{ω_i}, $H^{\tilde{\omega}_i}$ be value of option under event ω_i, $\tilde{\omega}_i$.

from step 1 model we have,

$$x_0 S(0) = \frac{V^{\tilde{\omega}_1} - V^{\tilde{\omega}_2}}{m_1 - m_2} \equiv \text{amt to invest in stock}$$

$$y_0 = \text{Amt to invest in bond} = \frac{1}{1+r}\left(\frac{V^{\tilde{\omega}_2}(1+m_1) - V^{\tilde{\omega}_1}(1+m_2)}{m_1 - m_2}\right) \quad //$$

time 0 portfolio (x_0, y_0)

We need portfolio in case $\tilde{\omega}_1$ & $\tilde{\omega}_2$.

$\tilde{\omega}_1$:
$$x_1^{\tilde{\omega}_1} S(0)(1+m_1) = \frac{V^{\omega_1} - V^{\omega_2}}{m_1 - m_2}$$

$$y_1^{\tilde{\omega}_1} = \frac{1}{1+r}\left(\frac{V^{\omega_2}(1+m_1) - V^{\omega_1}(1+m_2)}{m_1 - m_2}\right)$$

$\tilde{\omega}_2$:
$$x_1^{\tilde{\omega}_2} S(0)(1+m_2) = \frac{V^{\omega_3} - V^{\omega_4}}{m_1 - m_2}$$

$$y_1^{\tilde{\omega}_2} = \frac{1}{1+r}\left(\frac{V^{\omega_4}(1+m_1) - V^{\omega_3}(1+m_2)}{m_1 - m_2}\right)$$

To check that these portfolios properly price option we need to check that the portfolio at time 1 finances the reallocation of portfolio.

that is check that

$$x_0 S(0)(1+m_1) + y_0(1+r) = x_1^{\tilde{\omega}_1} S(0)(1+m_1) + y_1^{\tilde{\omega}_1}$$

$$\Updownarrow$$

$$\frac{V^{\tilde{\omega}_1} - V^{\tilde{\omega}_2}}{m_1 - m_2}(1+m_1) + \left(\frac{V^{\tilde{\omega}_2}(1+m_1) - V^{\tilde{\omega}_1}(1+m_2)}{m_1 - m_2}\right) = \frac{V^{\omega_1} - V^{\omega_2}}{m_1 - m_2} + \frac{1}{1+r}\left(\frac{V^{\omega_2}(1+m_1) + V^{\omega_2}(1+}{m_1 - m_2}\right.$$

$V^{\tilde{\omega}_i}$ are defined on page 3.

Similar eq. under event $\tilde{\omega}_2$.

This requirement ensures that the portfolio at the end of time step $0 \to 1$ can purchase the portfolio at the begining of time step $1 \to 2$.

In the portfolio discussed above we have
<u>self financing</u>, that is,
to ~~make~~ obtain portfolio (x^ω, y^ω)
described on each step we do not
have to add or subtract funds
~~when~~ at any time step.

The portfolio also has the property that
it is <u>predictable</u> that is
at every time step the holding
decisions are determined by
the available information
(when choosing portfolio at
beginning of $1 \to 2$ timestep
we do not use outcome of $1 \to 2$ step
only ω_1)

Finally the portfolio is <u>admissible</u> since
it always has ~~positive~~ non negative value $V(n) \geq 0$.