Trinomial model ~ can we derive Arb free measure?

Let us assume a stock moves on each step as

$$S(t+1) = S(t)\,(1 + M_{t+1})$$

$$\mathbb{P}(M_{t+1} = m_i) = p_i \ .$$

$$p_1 + p_2 + p_3 = 1 \quad . \; 0 < p_i < 1 \ .$$

$$m_1 > m_2 > m_3$$

Interest rate r, as before we must have

$$m_1 > r > m_3 \ .$$

As discussed in prev. Lecture we want

$$\cancel{\frac{1}{1+r}\mathbb{E}^*(S(t+1)) = S(t)}$$

$$\frac{1}{1+r}\mathbb{E}^*(S(t+1) \mid \mathcal{F}_t) = S(t) \ .$$

Equivalently $\frac{1}{1+r}\mathbb{E}^*(1 + M_{t+1}) = 1$.

$$1+r = p_1^*(1+m_1) + p_2^*(1+m_2) + p_3^*(1+m_3)$$

$$r = p_1^* m_1 + p_2^* m_2 + p_3^* m_3$$

$$p_3^* = 1 - p_1^* - p_2^*$$

2 eqn 3 unkn.

$\therefore$ measure must satisfy $\begin{pmatrix} r \\ 1 \end{pmatrix} = \begin{pmatrix} m_1 & m_2 & m_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} p_1^* \\ p_2^* \\ p_3^* \end{pmatrix}$

Equivalently:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ m_1 - r & m_2 - r & m_3 - r \end{pmatrix} \begin{pmatrix} p_1^* \\ p_2^* \\ p_3^* \end{pmatrix}$$

$$\therefore \quad p^* = \begin{pmatrix} p_1^* \\ p_2^* \\ p_3^* \end{pmatrix} \perp \begin{pmatrix} m_1 - r \\ m_2 - r \\ m_3 - r \end{pmatrix} \qquad (\ddagger)$$

What are solutions p^*?

$$* \quad m_1 - r > 0 > m_3 - r$$

$$\hookrightarrow \quad \tilde{p}^* = \begin{pmatrix} r - m_3 \\ 0 \\ m_1 - r \end{pmatrix}$$

But we want $p_i^* > 0$ for $i = 1, 2, 3$.

$$\therefore \text{ "bump" } p^* - \quad p^* = \begin{pmatrix} r - m_3 - \varepsilon \\ \varepsilon + \delta \\ m_1 - r - \delta \end{pmatrix} \rightarrow \quad p^* \cdot \mathbb{1} = 1 \checkmark$$

Need $(\ddagger)$: $\quad -\delta(m_1 - r) + (\delta + \varepsilon)(m_2 - r) + (-\varepsilon)(m_3 - r) = 0$

$$\hookrightarrow \quad \delta = \varepsilon \frac{m_2 - m_3}{m_1 - m_2}$$

$$\therefore \quad p^* = \begin{pmatrix} r - m_3 - \varepsilon \\ \varepsilon \dfrac{m_1 - m_3}{m_1 - m_2} \\ m_1 - r - \varepsilon \dfrac{m_2 - m_3}{m_1 - m_2} \end{pmatrix} ; \quad \begin{array}{l} 0 < \varepsilon < r - m_3 \\ \varepsilon < \dfrac{m_1 - m_2}{m_1 - m_3} \\ \varepsilon < \dfrac{(m_1 - m_2)(m_1 - r)}{m_2 - m_3} \end{array}$$

E.g. $m_1 = .3$

$m_2 = 0$

$m_3 = -.1$

$r = .1$

Arb-free / Risk neutral measures:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} .2 & -.1 & -.2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$a = \tfrac{1}{2} b + c$$

$$1 = \left(\tfrac{1}{2} b + c\right) + b + c = \tfrac{3}{2} b + 2c$$

$$\tfrac{2}{3} - \tfrac{4}{3} c = b$$

$$\therefore \; 1 \geq \tfrac{2}{3} - \tfrac{4}{3} c \geq 0$$

$$\tfrac{1}{2} \geq c$$

$$a = \tfrac{1}{3} - \tfrac{2}{3} c + c = \tfrac{1}{3}(1+c) < 1 \;\; \checkmark$$

$$\therefore \quad p^* = \begin{pmatrix} \tfrac{1}{3}(1+s) \\ \tfrac{2}{3}(1-2s) \\ s \end{pmatrix} \qquad 0 < s < \tfrac{1}{2}.$$

Can we ~~price~~ replicate claims?

Let H be a claim. And $\Omega = (\omega_1, \omega_2, \omega_3)$

ω_i corresponding to m_i

H has value $H(\omega_i)$ in outcome ω_i, $(H : \Omega \to \mathbb{R})$

Solve:

$$x\,S(1) + y(1+r) = H(1)$$

ie

$$\begin{pmatrix} S(0)\begin{pmatrix} 1+m_1 \\ 1+m_2 \\ 1+m_3 \end{pmatrix} & \begin{matrix} 1+r \\ 1+r \\ 1+r \end{matrix} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} H(\omega_1) \\ H(\omega_2) \\ H(\omega_3) \end{pmatrix}$$

3 eq^n & 2 unknowns.

$$\begin{pmatrix} S(0)\begin{pmatrix} 1+m_1 \\ m_1-m_2 \\ m_1-m_3 \end{pmatrix} & \begin{matrix} 1+r \\ 0 \\ 0 \end{matrix} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} H(\omega_1) \\ H(\omega_1)-H(\omega_2) \\ H(\omega_1)-H(\omega_3) \end{pmatrix}$$

$\therefore$ we can only solve when,

$$x = \underbrace{\frac{H(\omega_1)-H(\omega_2)}{S(0)(m_1-m_2)} = \frac{H(\omega_1)-H(\omega_3)}{S(0)(m_1-m_3)}}_{\text{these two are equal.}}$$

Eg Euro call option:

Consider Call option with strike X.

then $\underset{(\omega_i)}{H} = (S(0)(1+m_i) - X)^+$

so require:

$$x = \frac{(S(0)(1+m_1)-X)^+ - (S(0)(1+m_2)-X)^+}{S(0)(m_1-m_2)} = \frac{(S(0)(1+m_1)-X)^+ - (S(0)(1+m_3)-X)^+}{S(0)(m_1-m_3)}.$$

(i) $S(0)(1+m_1) > X \geq S(0)(1+m_2)$

$$x = \frac{(1+m_1) - \frac{X}{S_0}}{m_1-m_2} = \frac{(1+m_1)-\frac{X}{S_0}}{m_1-m_3}$$ $\lightning$ no solution.

(ii) $S(0)(1+m_2) > X > S(0)(1+m_3)$

$$x = \frac{m_1-m_2}{m_1-m_2} = \frac{(1+m_1)-\frac{X}{S_0}}{m_1-m_3}$$ $\lightning$ no soln. (Solving requires $\frac{X}{S_0} = 1+m_3$)

(iii) $X \leq S(0)(1+m_3)$

$$x = \frac{m_1-m_2}{m_1-m_2} = \frac{m_1-m_3}{m_1-m_3}$$ ✓ only case for soln. (or $X > S(0)(1+m_1)$)

Two Securities Trinomial.

$$S_i(t+1) = S(t)\left(1 + M^i_{t+1}\right)$$

$$\mathbb{P}\left(M^1_{t+1} = m^1_i, M^2_{t+1} = m^2_i\right) = p_i$$

(ie both stock move to up mid or down pos.)

Want to have

$$\mathbb{E}^*\left(S_i(t+1) \mid \mathcal{F}_t\right) = S_i(t)$$

$$\therefore$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ m^1_1 - r & m^1_2 - r & m^1_3 - r \\ m^2_1 - r & m^2_1 - r & m^2_3 - r \end{pmatrix} \begin{pmatrix} p_1^* \\ p_2^* \\ p_3^* \end{pmatrix}$$

$\cancel{7}$

Eg 2 trinomial securities 3 outcomes. $r = .02$

Ω	$M^{(1)}$	$M^{(2)}$
ω_1	.04	.01
ω_2	.02	0
ω_3	.01	.05

DOES THERE EXIST AN ABITRAGEFREE MEASURE?

Solve: $$\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\\ .02 & 0 & -.01\\ -.01 & -.02 & .03\end{pmatrix}\begin{pmatrix}p_1^*\\p_2^*\\p_3^*\end{pmatrix}$$

1st security

$$2a - c = 0$$
$$2a = c$$
$$\therefore\ p^* = \begin{pmatrix}s\\1-3s\\2s\end{pmatrix}$$
$$0 < s < \tfrac{1}{3}$$

2nd security

$$-a - 2b + 3c = 0$$
$$3(1-\cancel{(a+b)}) = 3c = a + 2b$$
$$3 = 4a + 5b$$
$$p^* = \begin{pmatrix}t\\(3-4t)/5\\(2-t)/5\end{pmatrix}$$
$$0 < t < 3/4$$

$$\therefore\ \left.\begin{matrix}s & t\\ 1-3s & = (3-4t)/5\\ 2s & (2-t)/5\end{matrix}\right\} \begin{matrix} s=t \\ \to 2-15s = -4t \to (s=t) \to s = 2/11 \\ t + 10s = 2 \to (s=t) \to s = 2/11\end{matrix}$$

✓

$$\therefore\ \text{measure } p^* = \begin{pmatrix}2/11\\5/11\\4/11\end{pmatrix}\ \text{Exists!}$$