Stream of Payments

Consider a contract that pays a value C at fixed sequence of times $\{\tau, 2\tau, \dots, n\tau\}$.

Such a contract is called an <u>Annuity</u>

Given the discounting of future dollars the value of this contract today is

$$V(0) = \beta_\tau C + \beta_{2\tau} C + \dots + \beta_{n\tau} C.$$

If the interest is such that

$$V(\tau) = (1+r)\,V(0).$$

then we have

$$V(0) = \frac{1}{1+r} C + \frac{1}{(1+r)^2} C + \dots + \frac{1}{(1+r)^n} C$$

Let us write Present Anuity value:

$$PA(r,n) = \frac{1}{1+r} + \dots + \frac{1}{(1+r)^n}$$

Notice $PA(r, 1) = \frac{1}{1+r}$

So $\frac{d}{dr} PA(r, 1) = \frac{-1}{(1+r)^2}$

ie the value of an Annuity decreases with increasing interest.

{ - This is a bit counter intuitive since increasing interest makes putting money in bonds more attractive ie higher demand }

Sol^n^ Current buying power of money is low compared to future buying power for large interest.

- Small interest: Buying power of todays money is closer to tomorrows money.

Consider the calculation,

$$(1-q)(1+q+q^2+\cdots+q^m)$$

$$=(1+q+\cdots+q^m)+(-q-q^2-\cdots-q^{m+1})$$

$$=1-q^{m+1}$$

$$\hookrightarrow \quad 1+q+\cdots+q^m = \frac{1-q^{m+1}}{1-q}.$$

Thus

$$PA(r,n) = \frac{1}{1+r}\left(1+\cdots+\frac{1}{(1+r)^{n-1}}\right)$$

$$= \frac{1}{1+r}\left\{\frac{1-\frac{1}{(1+r)^n}}{1-\frac{1}{1+r}}\right\} = \frac{1-\frac{1}{(1+r)^n}}{(1+r)-1} = \frac{1}{r}\left(1-(1+r)^{-n}\right)$$

$$PA(r,n) = r^{-1}\left(1-(1+r)^{-n}\right)$$

Thus the Value of our Annuity is

$$V_0 = C \times PA(r,n) = C\frac{1}{r}\left(1-(1+r)^{-n}\right)$$

let $f(r) = PA(r,n)$ let us prove f is decreasing in r.

$$f(r) = \frac{1}{r}\left(1 - (1+r)^{-n}\right)$$

$$f'(r) = -\frac{1}{r^2}\left(1-(1+r)^{-n}\right) + \frac{1}{r}\left(n(1+r)^{-n-1}\right)$$

$$= -\frac{1}{r^2}\left(\frac{(1+r)^{n+1} - (1+r)}{(1+r)^{n+1}}\right) + \frac{nr}{r^2(1+r)^{n+1}}$$

$$= \frac{1+(n+1)r - (1+r)^{n+1}}{r^2(1+r)^{n+1}}$$

But $(1+r)^{n+1} = 1+(n+1)r + \binom{n+1}{2}r^2 + \cdots > 1+(n+1)r$

So $f'(r) < 0$.

On the other hand $PA(r,n)$ is clearly increasing in n.

$\therefore$ le Present Value of Annuity increases

as # of payments increas

+

Interest rate decreases.

Suppose An annuity paid \$100 every year for the next 10 years (starting in 1 year from today)

What is the value today of the annuity if

* the intrest rate is $r_1 = .05 = 1/20$

* ______________ $r_2 = .1 = 1/10$

r_1: $V_0 = 100 \frac{1}{.05}\left(1-(1+.05)^{-10}\right)$

$= 2{,}000\left(1-\left(\frac{21}{20}\right)^{-10}\right) = 2{,}000\,(1-.6139) = 772.2$

r_2: $V_0 = 100 \frac{1}{.1}\left(1-(1.+.1)^{-10}\right)$

$= 1000\left(1-\left(\frac{10}{11}\right)^{10}\right) = 1{,}000\,(1-.3855) = 614.5$

I.e. as interest increases, the value of the Annuity decreases.

Exercise:

How much can you borrow if interest is 10% make payments of \$10 per year

Clear in 3 years.

Notice: Suppose loan amount P is borrowed, + set aside to pay Annuity of \$10 per year x3 years.

then at time=0 P should be the value of the Annuity. A

Otherwise, if $P > A$ I should borrow many times over

if $P < A$ I should lend many times over (sell annuity contracts + lend out P)

ie $P = A$.

$$P = C \frac{1}{r}\left(1-(1+r)^{-n}\right) = 10 \frac{1}{.1}\left(1-(1.1)^{-3}\right)$$

$$= 100\,(.2487) = 24.87.$$

Alternatively, repeat exercize but find value at <u>end</u> of contract is zero.

time	Loan Value
$t=0$	P
$t=1$	$P(1+r) - C$
$t=2$	$P(1+r)^2 - C(1+r) - C$
$t=3$	$P(1+r)^3 - C(1+r)^2 - C(1+r) - C = 0$

But $\quad (1+r)^2 + (1+r) + 1 = \dfrac{1-(1+r)^3}{1-(1+r)} = -\dfrac{1}{r}\left(1-(1+r)^3\right)$

$\therefore$ $t=3$ value is, $\quad P(1+r)^3 + C\,\dfrac{1}{r}\left(1-(1+r)^3\right) = 0$

$\dfrac{1}{(1+r)^3} \times$

$$P - C\,\frac{1}{r}\left(1-(1+r)^{-3}\right) = 0$$

$$P = C\,\frac{1}{r}\left(1-(1+r)^{-3}\right) \quad \checkmark$$

Same Equation!.

Now consider A contract paying C at fixed intervals forever $\{\tau, 2\tau, 3\tau, \ldots, n\tau, \ldots\}$.

Given discount factor function β_t the value is

$$V_0 = \beta_\tau C + \beta_{\tau 2} C + \cdots + \beta_{\tau n} C + \cdots$$

Such a contract is called a Perpetuity

As you may imagine the value factor may be thought of as

$$PP(r) = \lim_{n \to \infty} PA(r, n)$$

$$= \frac{1}{r} \lim_{n \to \infty} \left(1 - (1+r)^{-n}\right)$$

$$= \frac{1}{r}\left(1 - \lim_{n \to \infty} (1+r)^{-n}\right) = \frac{1}{r}.$$

(Double check..)

$$PP(r) = \frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots = \frac{1}{1+r} \sum_{n=0}^{\infty} \left(\frac{1}{1+r}\right)^n$$

$$= \frac{1}{1+r} \frac{1}{1 - \frac{1}{1+r}} \left(\frac{1+r}{1+r}\right) = \frac{1}{r} \checkmark$$

Again we have $PP(r)$ increases (decreases) as r decreases (increases)

Exercise: Suppose you can buy a perpetuity (w/ first payment in 1 year) w/ $10 payment yearly

What is the effective interest if you can purchase the contract for

(a) $200

(b) $100 ?

(a) $200 = 10 \times PP(r) = 10 \frac{1}{r}$

$r = .05 \longrightarrow 5\%$ interest

(b) $100 = 10 \times PP(r) = 10/r$

$r = .1 \longrightarrow 10\%$. interest.