Continuous time Stock Model

Let us consider Stock price model on the time set
$\mathbb{T} = [0,T] \subset \mathbb{R}$.

How should such a model be constructed?

We take the ~~m~~ limit of discrete N step Binomial models.

Thus, $S(0)$ given,

for $n = 0, 1, \ldots, N$, and $t = n\frac{T}{N}$

$$S(t) = S(0)(1+\delta_1)\cdots(1+\delta_n)$$

where $\delta_i = \delta_i^{(N)}$ depends on N.

Let us consider the logarithmic return,

$$\log\frac{S(t)}{S(0)} = \log(1+\delta_1) + \cdots + \log(1+\delta_N)$$

We make assumption of time <u>homogeneity</u> so that all steps δ_i have the same distribution

Moreover assume they are independent.

Thus $k_i^{(N)} = \log\left(1+\delta_i^{(N)}\right)$

We wish to fix variance of the log return independent of N, thus, calculate:

$$\mathrm{Var}\left(k_1^{(N)} + \cdots + k_N^{(N)}\right) = \sigma_T^2$$

$$= N\,\mathrm{Var}\left(k^{(N)}\right) = \sigma_T^2$$

$$\therefore \quad \sigma_{(N)} = \frac{\sigma_T}{\sqrt{N}} \equiv \text{std dev of } k^{(N)}$$

Now assume interest at rate R_N

$$\delta_i^{(N)} = R_N + \varepsilon_i^{(N)} \quad \text{where } \mathbb{P}^*\left(\varepsilon_i^{(N)} = \varepsilon^{(N)}\right) = \mathbb{P}\left(\varepsilon_i^{(N)} = -\varepsilon^{(N)}\right) = \tfrac{1}{2}$$

then notice $\frac{1}{2}a^2 + \frac{1}{2}b^2 - \left(\frac{1}{2}a + \frac{1}{2}b\right)^2 = \frac{1}{4}a^2 + \frac{1}{4}b^2 - \frac{1}{2}ab = \frac{1}{4}(a-b)^2$

$$\text{Thus } \mathrm{Var}\, k^{(N)} = \frac{1}{4}\left(\log(1+R_N+\varepsilon^{(N)}) - \log(1+R_N-\varepsilon^{N})\right)^2$$

$$= \frac{1}{4}\left\{\left[(R+\varepsilon) - \tfrac{1}{2}(R_N+\varepsilon)^2 + \cdots\right] - \left[(R-\varepsilon) - \tfrac{1}{2}(R-\varepsilon)^2 + \cdots\right]\right\}^2$$

$$= \frac{1}{4}\left\{2\,\varepsilon^{(N)} + 2\,R_N\,\varepsilon^{(N)} + \cdots\right\}^2$$

Setting two formulae equal: $\quad R_N \varepsilon^N \to 0$

$$\varepsilon^{(N)} + R_N\varepsilon^N + \cdots = \frac{\sigma_T}{\sqrt{N}} \Rightarrow \varepsilon^{(N)} = \frac{\sigma_T}{\sqrt{N}}.$$

Given assumptions, we can say what σ_t should be,

$$t = n\frac{T}{N}$$

$$\sigma_t^2 = Var\left(k_1^{(N)} + \cdots + k_n^{(N)}\right) = n\, var(k^N) = \frac{n}{N}\sigma_T^2 = \frac{t}{T}\sigma_T^2$$

$$\cancel{\sigma_t} \quad \sigma_t = \sqrt{\frac{t}{T}}\,\sigma_T$$

if $t = 1 \Rightarrow$

$$\sigma_T = \sqrt{T}\,\sigma_1$$

$\therefore$ ~~varing grow~~ std dev grows like $\sqrt{T}$,

Let us write $\delta_i^{(N)} = R_N + \epsilon_i \sigma \sqrt{\frac{T}{N}}$ where $P^*(\epsilon_i = +1) = 1/2$
$P^*(\epsilon_i = -1) = 1/2$.

Then
$$\log\left(1 + R_N + \epsilon_i \sigma \sqrt{\frac{T}{N}}\right)$$
$$= \log(1+R_N) + \log\left(1 + \epsilon_i \frac{\sigma\sqrt{\frac{T}{N}}}{1+R_N}\right)$$

$$\log\left(1+\epsilon_i \frac{\sigma\sqrt{\frac{T}{N}}}{1+R_N}\right) = \epsilon_i \left(\frac{\sigma}{1+R_N}\right)\sqrt{\frac{T}{N}} - \frac{1}{2}\epsilon_i^2\left(\frac{\sigma}{1+R_N}\right)^2\frac{T}{N} + \frac{1}{3}\epsilon_i^3\left(\frac{\sigma}{1+R_N}\right)^3\left(\frac{T}{N}\right)^{3/2}$$

Suming over $i = 1, \ldots, N$ notice the third term drops out - order$\left(\frac{1}{\sqrt{N}}\right)$

$\therefore$

$$\sum \log\left(1+R_N+\epsilon_i\,\sigma\sqrt{\tfrac{T}{N}}\right)$$

$$= N\log(1+R_N) + \frac{\sigma\sqrt{T}}{1+R_N}\,\frac{\epsilon_i}{\sqrt{N}} - \frac{1}{2}\left(\frac{\sigma}{1+R_N}\right)^2 T$$

let us suppose ~~B~~ variance $\to 0$ then stock becomes a bond and we simply have

$$\log\frac{S(T)}{S(0)} = N\log(1+R_N) = rT$$

$$\therefore\ R_N = \frac{rT}{N}$$

$\therefore$

We have, $\underline{1+R_N \to 1}$

$$\log\frac{S(T)}{S(0)} = rT + \sigma\sqrt{T}\sum_i\frac{\epsilon_i}{\sqrt{N}} - \frac{1}{2}\sigma^2 T$$

Of course $\mathrm{Var}(\epsilon_i) = 1$ so $\sum_i \frac{\epsilon_i}{\sqrt{N}} \rightsquigarrow N(0,1)$

$$\sqrt{T}\sum\frac{\epsilon_i}{\sqrt{T}} \rightsquigarrow N(0,T)$$

$\therefore$ letting T vary as t we have,

$$S(t) = e^{rt - \frac{1}{2}\sigma^2 t + \sigma W_t}$$

where W_t is "Brownian Motion"