Stock price + Portfolio Price.

Return to construction of stock price.

for any t we can write

$$S(t) \cong S(0)\left(1+\frac{r}{N}+\sigma\frac{\epsilon_1}{\sqrt{N}}\right)\cdots\left(1+\frac{r}{N}+\sigma\frac{\epsilon_n}{\sqrt{N}}\right)$$

for $\frac{n}{N} \leq t < \frac{n+1}{N}$.

$P^*(\epsilon_i = 1) = P^*(\epsilon_i = -1) = 1/2$.

$\updownarrow$

$$\log\frac{S(t)}{S(0)} = \sum_{i=1}^{n}\frac{r}{N} - \frac{1}{2}\sigma^2\sum_{i=1}^{n}\frac{\epsilon_i^2}{N} + \sigma\sum_{i=1}^{n}\frac{\epsilon_i}{\sqrt{N}}$$

$$= rt - \frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\sum_{i=1}^{n}\frac{\epsilon_i}{\sqrt{n}}$$

$\uparrow$

$Nt \sim n$

$+ \sum_{i=1}^{n}\frac{\epsilon_i}{\sqrt{n}} \sim W$

for $W \sim N(0,1)$.

Replicating Strategy for option,

Given European Option ~~a~~ w/ expiry T

write replicating portfolio for option,

$$x(t),\ y(t) \quad \text{— for } t = \frac{1}{N}, \frac{2}{N}, \dots, \frac{n}{N} \le T < \frac{n+1}{N}$$

In particular for Euro option, path independent,

$$S^{\pm}(t) = S(t)\left(1 + \frac{r}{N} \pm \sigma\frac{\epsilon}{\sqrt{N}}\right)$$

$$\cancel{V(S(t)(1+\frac{r}{N}}$$

$$\cancel{x_{t+1} =}$$

$$x_{t+1} = \frac{V(S^{+}(t)) - V(S^{-}(t))}{S^{+}(t) - S^{-}(t)}$$

$$y_{t+1} = \frac{1}{1+r}\ \frac{S^{+}(t)\, V(S^{-}(t)) - S^{-}(t)\, V(S^{+}(t))}{S^{+}(t) - S^{-}(t)}.$$

then

$$dV_t = \underset{\substack{\uparrow \\ \text{stock fluctuation}}}{x_{t+1}\, dS_t} + \underset{\substack{\uparrow \\ \text{interest.}}}{y_{t+1}\, dA_t}$$

can we determine dS_t ??

<u>But.</u> $$\mathbb{E}^*\left\{S(0)\left(1+\frac{r}{N}+\sigma\frac{\epsilon_1}{\sqrt{N}}\right)\cdots\left(1+\frac{r}{N}+\sigma\frac{\epsilon_n}{\sqrt{N}}\right)\right\}$$

$$= S(0)\left(1+\frac{r}{N}\right)^n \approx S(0)\,e^{rt}$$

By <u>usual</u> derivation in Binomial model.

$\therefore$ Pass to the limit $\rightarrow$

$$\mathbb{E}^*\{S(t)\} = S(0)\,e^{rt}$$

ie $e^{-rt}S(t)$ is a <u>martingale</u>.

BROWNIAN MOTION.

$W_t \equiv$ B.M.

Properties:

$\frac{W_t}{\sqrt{t}}$ is $N(0,1)$ Random variable.

$$\mathbb{P}\left(a \le \frac{W_t}{\sqrt{t}} \le b\right) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

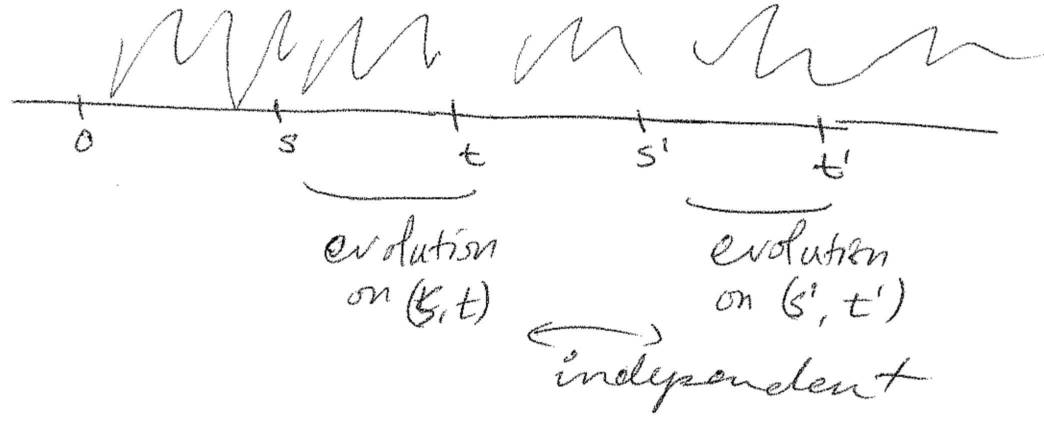
for $s < t < s' < t'$

$W_t - W_s$ is independent of $W_{t'} - W_{s'}$

AND,

$\frac{1}{\sqrt{t-s}}(W_t - W_s)$ is $N(0,1)$ Random Variable,

IE, process 'starts over' at every time and forgets previous path – "Markov property"



Since $\frac{1}{\sqrt{t-s}}(W_t - W_s)$ is $N(0,1)$ &
independent of path up to time s,
– ie independent of F_s – info up to time s.

$$E(W_t | F_s) = E(W_t - W_s + W_s | F_s)$$
$$= E(\underbrace{W_t - W_s | F_s}_{0}) + \underbrace{E(W_s | F_s)}_{W_s}$$
$$= W_s.$$

Fact: Gaussian processes (process such that time marginals are Gaussian).
are completely characterized by covariances,

$Cov(W_t, W_s) = min(s,t)$

how do we see this?

Consider the limiting process ~~~~ $s = \frac{n}{N}$, $t = \frac{m}{N}$, $s < t$

$$W_s \approx \sum_{i=1}^{n} \epsilon_i \frac{1}{\sqrt{N}} \quad W_t \approx \sum_{i=1}^{m} \epsilon_i \frac{1}{\sqrt{N}}, \quad \mathbb{P}(\epsilon_i = 1) = 1/2. \quad \mathbb{P}(\epsilon_i = -1) = 1/2.$$

$$\mathbb{E}(W_s W_t) = \mathbb{E} \sum_{i=1}^{n} \sum_{j=1}^{m} \epsilon_i \epsilon_j \frac{1}{N} = \frac{n}{N} = s \checkmark$$

$$\mathbb{E}\,\epsilon_i \epsilon_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

Path properties,

* W_t is continuous,

$$\lim_{s \to t} W_s = W_t$$

* $\sqrt{t}$ fluctuations,

- for $\alpha \geq \frac{1}{2}$ (blow up)

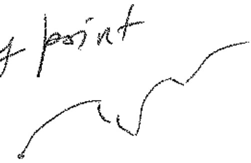
$$\sup_{0 \leq s < t \leq 1} \left| \frac{W_t - W_s}{(t-s)^\alpha} \right| = \infty$$

- for $\alpha < \frac{1}{2}$

$$\sup_{0 \leq s < t \leq 1} \left| \frac{W_t - W_s}{(t-s)^\alpha} \right| < \infty$$

"With probability 1".

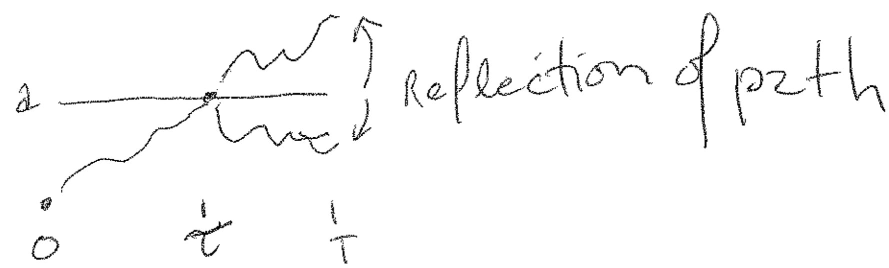
* W_t is <u>nowhere</u> differentiable
- holds since behavior $\sqrt{t}$ at every point



Q: W_t starts @ 0 what is probability W_t hits $a>0$ by time T?

Reflection principle.

let $\tau = \min\{t : W_t \geq a\}$.



let $U_t = \begin{cases} W_t, & t \leq \tau \\ a - W_t, & t > \tau \end{cases}$

Then U_t is again a Brownian Motion -

Since $a - W_t$ still has Gaussian marginals and same covariance fn.

$\therefore \mathbb{P}(U_T > a) = \mathbb{P}(W_T > a)$

* But $U_T > a$ implies $W_T < a$

$\therefore \{U_T > a\} \cup \{W_T > a\} = \{\tau < T\}$.

$$\mathbb{P}(\tau < T) = 2\mathbb{P}(W_T > a)$$

$$\therefore \quad \mathbb{P}(\tau < T) = 2\,\mathbb{P}(W > a/\sqrt{T})$$

for $W \sim N(0,1)$

$$\mathbb{P}(W > a/\sqrt{T}) = \int_{a/\sqrt{T}}^{\infty} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$$

$$f_\tau(T) = \frac{d}{dT}\mathbb{P}(\tau < T) = 2\left(-\frac{e^{-x^2/2}}{\sqrt{2\pi}}\Bigg|_{a/\sqrt{T}}\right)\left(-\frac{a}{2T^{3/2}}\right)$$

$$= \frac{a}{\sqrt{2\pi}\,T^{3/2}} e^{-a^2/2T}$$