Example - solving SDEs.

Consider ODE of the form

$$\frac{dx}{dt} = p(t)\, x(t) + q(t)$$

Integrating factor $g(t) = e^{-\int_0^t p(s)\,ds}$

$$\frac{d}{dt} g = -p(t)\, g(t)$$

Then

$$\frac{d}{dt}(xg) = g(px+q) + x(-pg) = gq$$

Integrate

$$x(t)\, g(t) = x(0)\, g(0) + \int_0^t g(s)\, q(s)\, ds$$

$$g(0) = 1$$

$$\hookrightarrow \quad x(t) = x(0)\, e^{\int_0^t p(r)dr} + \int_0^t e^{-\int_0^s p(r)dr + \int_0^t p(r)dr} q(s) +$$

$$= x(0)\, e^{\int_0^t p(r)dr} + \int_0^t e^{\int_s^t p(r)dr} q(s)\, ds +$$

Similar for SDE:

$$dX = (pX+q)\,dt + h\,dW_t$$

where $p=p(t)$, $q=q(t)$, $h=h(t)$ functions of t.

Integrating factor: $g = e^{-\int_0^t p\,ds}$

$$d(xg) = gq\,dt + gh\,dW_t$$

Integrate.

$$x(t)\,g(t) = x(0)\,g(0) + \int_0^t g(s)\,q(s)\,ds + \int_0^t g(s)\,h(s)\,dW_s$$

Eg 1 Ornstein - Uhlenbeck.

$$dx_t = \rho(\mu - x_t)\,dt + \sigma\, dW_t.$$

Integrating factor $e^{\theta t}$

$$d(x_t e^{\theta t}) = \left[\theta e^{\theta t} x_t + \rho(\mu - x_t) e^{\theta t}\right] dt + e^{\theta t} \sigma\, dW_t$$

let $\theta = \rho$

$\hookrightarrow$

$$d(x_t e^{\rho t}) = \rho \mu e^{\rho t}\, dt + \sigma e^{\rho t}\, dW_t.$$

$$x_t e^{\rho t} = x_0 + \rho\mu \int_0^t e^{\rho s}\, ds + \sigma \int_0^t e^{\rho s}\, dW_s$$

$$= x_0 + \mu\left(e^{\rho t} - 1\right) + \sigma \int_0^t e^{\rho s}\, dW_s$$

$$\cancel{x_t = x_0 e^{-\rho t} \quad e^{-\rho t} +}$$

$$x_t = (x_0 - 1) e^{-\rho t} + \mu + \sigma \int_0^t e^{-\rho(t-s)}\, dW_s.$$

Model of stock w/ value μ and fluctuations in price of std dev σ.

Eg 2

$$dZ_t = \rho t\, Z_t\, dt + \sigma\, dW_t$$

Integrating factor $e^{\vartheta t^2}$

$$d\left(e^{\vartheta t^2} Z_t\right) = \left(2\vartheta t e^{\vartheta t^2} Z_t + e^{\vartheta t^2} Z_t \rho t\right) dt + e^{\vartheta t^2} \sigma\, dW_t$$

let $\vartheta = -\frac{1}{2}\rho$, then :

$$d\left(e^{-\frac{\rho}{2} t^2} Z_t\right) = e^{-\frac{1}{2}\rho t^2} \sigma\, dW_t$$

$$e^{-\rho/2\, t^2} Z_t = Z_0 + \int_0^t e^{-\frac{1}{2}\rho s^2} \sigma\, dW_s.$$

$$Z_t = Z_0 e^{(\rho/2) t^2} + \int_0^t \sigma e^{\frac{1}{2}\rho (t^2 - s^2)}\, dW_s .$$