Recall discrete Martingale Version:
(of stock price + Optionprice)

$$S(t) = \frac{1}{1+r} E\left(S(t+1) \mid \mathcal{F}_t\right).$$

or

$$S(0) = \mathbb{E}\left(\frac{1}{(1+r)^n} S(n)\right).$$

Similarly for option $\mathbb{E}\left(\frac{1}{(1+r)^n} V(t)\right) = V(0)$

$\hookrightarrow$ $V(0) = \mathbb{E}\left(\frac{1}{(1+r)^N} V(N)\right)$

expiry @ time N.

Similar plan for continuous version

let $g(S)$ be payoff of Call @ expiry.

Value @ time t $\quad V(t) = E\left(g(S_T) e^{-r(T-t)} \mid F_t\right)$

then $\tilde{V}(t) = e^{-rt} V(t)$ is Mg.

(More general) say r is process $r = r(S_t, t)$.

$$\beta_{s,t} = e^{-\int_s^t r_s \, ds}$$

$$V(t,x) = \mathbb{E}\left[g(S_t)\beta_{0t} \mid S_0 = x \right]$$

or $$V(\tau, t; x) = \mathbb{E}\left[g(S_t)\beta_{\tau t} \mid S_\tau = x \right]$$

$$\cancel{M_s = E\, g(S_t)\beta_{0,t}}$$

Define (expiry T)

$$M_\tau = E\left[g(S_T)\beta_{0,T} \mid \mathcal{F}_\tau \right]$$

$$= \beta_{0,\tau}\, E\left[g(S_T)\beta_{\tau,T} \mid \underset{S_\tau}{\mathcal{F}_\tau} \right]$$

$$= \beta_{0,\tau}\, V(\tau, T; x).$$

M_τ is a Mg. $u < \tau$

$$\cancel{E\, M_\tau \mid \mathcal{F}_u} \quad E\left[M_\tau \mid \mathcal{F}_u \right] = E\left[E\left[g(S_T)\beta_{0,T} \mid \mathcal{F}_\tau \right] \mathcal{F}_u \right]$$

$$= E\left[g(S_T)\beta_{0,T} \mid \mathcal{F}_u \right] = M_u \checkmark$$

Find diff of M_τ

$$d\beta_{0\tau} = d\, e^{-\int_0^\tau r(s)\, ds} = -r(\tau)\beta_{0\tau}$$

(T is constant)

~~[illegible]~~

Notice we can parametrise V as

$$V(\tau, t; x) = V(t-\tau; x)$$

since time is homogeneous

$$\therefore \quad dV(\tau, T; S_\tau) = \left[+\mathring{V}(\tau, T; S_\tau) + \tfrac{1}{2}\sigma^2 S^2 V''(\tau, T, S_\tau)\right] d\tau$$
$$+ V'(\tau, T, S_\tau)\, dS_\tau$$

$\therefore$

$$dM_\tau = (d\beta_{0,\tau})V(\tau, T, S_\tau) + \beta_{0\tau}\left(dV(\tau, T, S_\tau)\right)$$
$$= \beta_{0,\tau}\left\{-r V(\tau, T, S_\tau) + \mathring{V} + \tfrac{1}{2}\sigma^2 S_\tau^2 V'' + S r V'\right\} d\tau$$
$$+ \beta_{0\tau} V'(\tau, T; S_\tau)\,\sigma S_\tau\, dW_\tau.$$

Deterministic part of M_τ is zero ...

$$\therefore \quad dM_\tau = \beta_{0\tau} V' \sigma S_\tau \, dW_\tau .$$

$\therefore$

$$V = V(t, x), \quad \frac{\partial}{\partial x} V = V', \quad \frac{\partial}{\partial t} V = \mathring{V}$$

$$rV = \mathring{V} + \frac{1}{2}\sigma^2 x^2 V'' + r x V'$$

w/ bdry condition $\quad V(t,0) = 0$
$\qquad\qquad\qquad\qquad V(T,x) = g(x)$.

Equivalent to –

$$V(t,x) = \mathbb{E}\left(g(S_T)\, e^{-r(T-t)} \,\middle|\, S_t = x\right)$$

the Feynman Kac formula.

On the other hand, consider discounted process

$$f(t, W_t) = M_t := \beta_{0t} V(t, T; x)$$

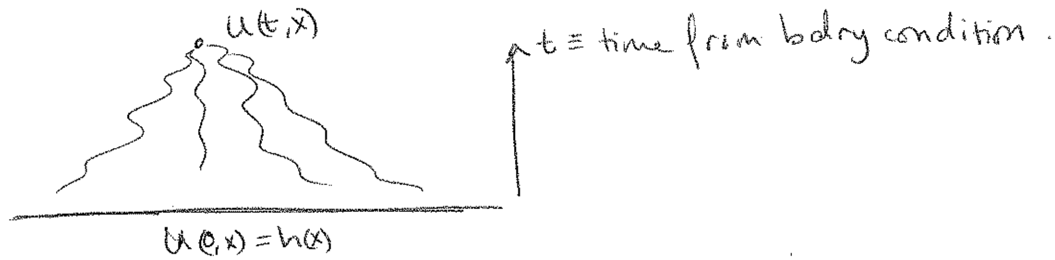
By Ito formula:

$$df = \left(f_t + \frac{1}{2} f_{xx}\right) dt + f_x \, dW_t .$$

$$\therefore \quad f_t + \frac{1}{2} f_{xx} = 0 \qquad \text{w/ } f(T,x) = \beta_{0T}\, g(x)$$
$$f(t,0) = 0 .$$

Heat Equation,

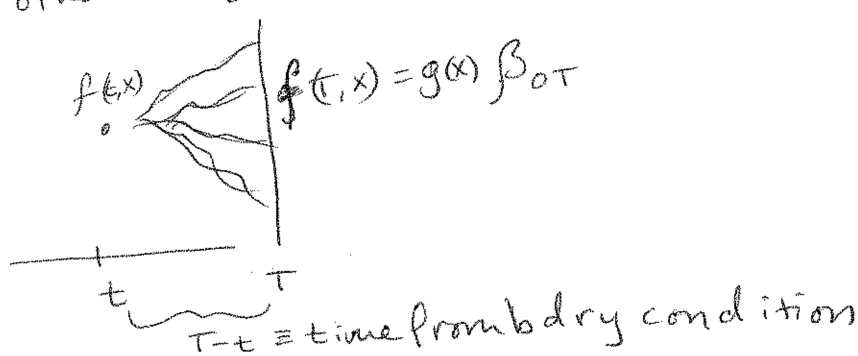
$$u_t = k\, u_{xx}$$



Solution: $u(x,t) = \int \phi(x-y,t)\, g(y)\, dy$

$$\phi(x,t) = \frac{1}{\sqrt{4k\pi t}}\, e^{\left(-x^2/4kt\right)}$$

On the other hand



Solution: value @ time t in time 0 dollars,

$$f(t,x) = \int_{-\infty}^{\infty} \phi(x-y, T-t)\, g(y)\, \beta_{0,T}\, dy$$

$$\phi(x,t) = \frac{1}{\sqrt{2\pi t}}\, e^{-x^2/2t}$$

Value of portfolio: @ time t in time t dollars.

$$V(t,T;x) = \int_{-\infty}^{\infty} \phi((x_t - y), T-t)\, g(y)\, \beta_{t,T}\, dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} \left(e^{-\frac{(x-y)^2}{2(T-t)} - r(T-t)} \right) g(y)\, dy$$

European Call

$$C_E(0) = \mathbb{E}^*\left\{ e^{-rT} \left(S(T) - X \right)^+ \right\}$$

$$= \mathbb{E}^*\left\{ e^{-rT} \left(S(0)\, e^{(r - \frac{1}{2}\sigma^2)T + \sigma W(T)} - X \right)^+ \right\}$$

$W(T) \sim N(0, T)$

$\hookrightarrow W(T) = \sqrt{T}\, W$ for $W \sim N(0,1)$.

$$= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-d}^{\infty} \left(S(0)\, e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T} w} - X \right) e^{-w^2/2}\, dw$$

d solving:

$$S(0)\, e^{(r - \frac{1}{2}\sigma^2)T - \sigma\sqrt{T} d} - X = 0$$

$$\hookrightarrow \quad d = \frac{\ln \frac{S(0)}{X} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$C_E(0) = \frac{S e^{-\frac{1}{2}\sigma^2 T}}{\sqrt{2\pi}} \int_{-d}^{\infty} e^{\sigma\sqrt{T}w - w^2/2}\, dw$$

$$- X \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-d}^{\infty} e^{-w^2/2}\, dw$$

Notice $\sigma\sqrt{T}w - w^2/2 = -\frac{1}{2}(w - \sigma\sqrt{T})^2 + \frac{1}{2}\sigma^2 T$

$w = -d \to \quad d + \sigma\sqrt{T} = \dfrac{\ln\frac{S}{X} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \tilde{d}$

$$C_E(0) = S \int_{-\tilde{d}}^{\infty} e^{-w^2/2} \frac{dw}{\sqrt{2\pi}} - Xe^{-rT} \int_{-d}^{\infty} e^{-w^2/2} \frac{dw}{\sqrt{2\pi}}$$

$$= S \int_{-\infty}^{\tilde{d}} e^{-w^2/2} \frac{dw}{\sqrt{2\pi}} - Xe^{-rT} \int_{-\infty}^{d} e^{-w^2/2} \frac{dw}{\sqrt{2\pi}}$$

$$= S\, N(\tilde{d}) - Xe^{-rT} N(d)$$

GREEKS. "Partial Derivatives of Option value".

Delta:

Change in value of option wrt $S(0)$

$$\frac{\partial}{\partial S(0)} C_E(0) = N(\tilde{d}) + \left\{ S \frac{\partial}{\partial S} N(\tilde{d}) - X e^{-rT} \frac{\partial}{\partial S} N(d) \right\}$$

$$= N(\tilde{d}) + \underbrace{\left\{ S \frac{e^{-\tilde{d}^2/2}}{\sqrt{2\pi}} \frac{\partial}{\partial S} \tilde{d}_* - X e^{-rT} e^{-d^2/2} \frac{\partial}{\partial S} d \right\}}_{0}$$

$$= N(\tilde{d}).$$

notice in ~~port~~ Replicating portfolio $x(t) = V'$.

↳ Replicating portfolio of Call has stock holding

$$x(t) = N(\tilde{d}_t).$$

$$\tilde{d}_t = \frac{\ln \frac{S(t)}{X} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma \sqrt{(T-t)}}$$