GREEKS: PARTIAL Derivatives of Option value:

Replicating portfolio: $V \equiv$ value of option w/ payoff $g(S_T)$ @ time T.

$$x(t) = \frac{\partial}{\partial S} V \quad , \quad y = \frac{V_t - S_t\left(\frac{\partial}{\partial S} V_t\right)}{A_t}$$

$$V = x_t S_t + y_t A_t$$

Problem, how do we replicate 'continuously' in time,

~ it is impossible since trades are discrete!

∴ any discrete trading procedure must allow errors to occur on portfolio value approx option value.

Goal: Understand errors, minimize errors.

Let W be portfolio replicating $V \equiv$ value of option.

Delta hedging. $\quad (x_t, y_t) = \left(V', \frac{V - SV'}{A}\right)$

$$W_t = V' S_t + \left(\frac{V - SV'}{A}\right) A_t$$

$$W_{t+\Delta} \cong V' S_t + V'(\Delta S) + \left(\frac{V - SV'}{A}\right) A(1 + r\Delta)$$

$$V_{t+\Delta} \cong V + \Delta V$$

$$W_{t+\Delta} - V_{t+\Delta}$$

$$\approx \underbrace{V'(\Delta S)}_{\sim \text{delta}} + \underbrace{(V - SV')\, r\Delta - \Delta V}_{\sim \text{theta} + \text{error}}$$

$$\approx -\left(\tfrac{1}{2}(\text{gamma})(\Delta S)^2 + (\text{vega})(\Delta \sigma) + (\text{rho})(\Delta r)\right)$$

$$= -\left(\tfrac{1}{2}\gamma_V (\Delta S)^2 + \nu_V (\Delta \sigma) \quad \rho_V (\Delta r)\right).$$

Notice from formula for V,

$$V(t,x) = E\left(g(S_T)\, e^{-r(T-t)} \,\middle|\, S_t = x\right)$$

$$= E\left\{ g\left(x\, e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}\,W}\right) e^{-r(T-t)} \right\}$$

$$= \int_{-\infty}^{\infty} g\left(x\, e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}\,W}\right) e^{-r(T-t)}\, e^{-w^2/2} \frac{dw}{\sqrt{2\pi}}$$

$\therefore$ V depends on $\underline{t,\ S(t),\ \sigma,\ r}$

$$\cancel{\Delta V = (\text{delta})(\Delta S) + \frac{1}{2}}$$

$$\Delta V = \underbrace{\left(\frac{\partial V}{\partial S}\right)}_{\text{delta}}(\Delta S) + \frac{1}{2}\underbrace{\left(\frac{\partial^2 V}{\partial S^2}\right)}_{\text{gamma}}(\Delta S)^2 + \underbrace{\left(\frac{\partial}{\partial t}V\right)}_{\text{theta}}(\Delta t) + \underbrace{\left(\frac{\partial}{\partial \sigma}V\right)}_{\text{vega}}(\Delta\sigma) + \underbrace{\left(\frac{\partial V}{\partial r}\right)}_{\text{rho}}(\Delta r)$$

~~GREEKS: PARTIAL DERIVATIVES OF OPTION VALUE~~

DELTA. :

Change in value of option wrt $S(t)$

$$C_E(t) = S(t)\, N(d_+(t)) - Xe^{-r(T-t)}\, N(d_-(t))$$

$$d_\pm(t) = \frac{\ln\frac{S(t)}{X} + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$\frac{\partial}{\partial(S(t))} C_E(t) = N(d_+) + \left\{ S(t)\frac{\partial}{\partial(S(t))} N(d_+) - Xe^{-r(T-t)}\frac{\partial}{\partial S(t)} N(d_-) \right\}$$

$$= N(d_+) + \left\{ S(t)\, e^{-d_+^2/2} \underline{\frac{\partial}{\partial S(t)}} d_+ - Xe^{-rT} e^{-d_-^2/2} \underline{\frac{\partial}{\partial S(t)} d_-} \right\}$$

$$= N(d_+)$$

Recall, in Replicating portfolio $x(t) = V'$, $y = \frac{V - SV'}{A}$

$\hookrightarrow$

$$x(t) = N(d_+(t))$$

$$y(t) = \frac{-Xe^{-r(T-t)} N(d_-(t))}{A(t)}$$

European Call

Value: (Black-Scholes)

$$C_E(t) = S(t)\, N(d_+(t)) - X e^{-r(T-t)} N(d_-(t))$$

$$d_\pm = \frac{\ln \frac{S(t)}{X} + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

Greeks.

$$\Delta_V = N(d_+)$$

$$\Gamma_V = \frac{1}{S\sigma\sqrt{2\pi(T-t)}}\, e^{-d_+^2/2}$$

$$\theta = -\frac{S\sigma}{2\sqrt{2\pi(T-t)}}\, e^{-d_+^2/2} - rXe^{-rT} N(d_-)$$

$$\nu = \frac{S\sqrt{T-t}}{\sqrt{2\pi}}\, e^{-d_+^2/2}$$

$$\rho = (T-t)\, X e^{-r(T-t)} N(d_-).$$

<u>Delta-Gamma hedge</u>.

Consider two calls w/ strike price X, X'

then consider $\Delta V_X + \Delta V_{X'}$

in particular Γ_X, $\Gamma_{X'}$,

$$\Gamma_X - \Gamma_{X'} = \frac{1}{8\sigma\sqrt{2\pi(T-t)}}\left\{e^{-d_+^2(X)/2} - e^{-d_+^2(X')/2}\right\}$$

$$\cancel{= \frac{1}{8\sigma\sqrt{2\pi(T-t)}} e^{\frac{(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}}}$$

$$= \frac{1}{8\sigma\sqrt{2\pi(T-t)}} e^{-(\ln\frac{S}{X}-b)^2/2\sigma^2(T-t)} - e^{-(\ln\frac{S}{X'}-b)^2/2\sigma^2(T-t)}$$

$$= \frac{1}{8\sigma\sqrt{2\pi(T-t)}} e^{-b^2/2\sigma^2}\left\{e^{-\ln^2\frac{S}{X}} \left(\frac{S}{X}\right)^{2b} - e^{-\ln^2\frac{S}{X'}}\left(\frac{S}{X'}\right)^{2b}\right\}$$

$$= \frac{S^{2b}}{\sigma\sqrt{2\pi(T-t)}} e^{-b^2/2\sigma^2}\left\{\frac{1}{X^{2b}} e^{-\ln^2\frac{S}{X}} - \frac{1}{X'^{2b}} e^{-\ln^2\frac{S}{X'}}\right\}$$

$$\left(\frac{1}{X^{2b}} e^{-\ln^2\frac{S}{X}}\right)' = \frac{1}{X^{2b}} e^{-\ln^2\frac{S}{X}}\left(-2bX + 2\left(\ln\frac{S}{X}\right)\frac{1}{X}\right) \underset{\text{"usually"}}{<} 0$$