American Option.

Let exersize have intrinsic value $h(S_t)$.

Value of stockprice $S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t}$

$$dS_t = rS_t dt + \sigma S_t dW_t .$$

Value Replicating portfolio w/ stopping time τ.

$$V_t = \mathbb{E}^* \left\{ e^{-r(\tau - t)} h(S_\tau) \mid \mathcal{F}_t \right\}$$

If Expiry is time T then $\tau \le T$. Otherwise τ not bdd *

* although still want to apply Optional Sampling.

Value is - w/ expiry -

$$V_t = \sup_{\tau < T} \mathbb{E}^* \left(e^{-r(\tau - t)} h(S_\tau) / \mathcal{F}_t \right)$$

w/o expiry

$$V_t = \sup_{\tau} \mathbb{E}^* \left(e^{-r(\tau - t)} h(S_\tau) / \mathcal{F}_t \right)$$

Notice, for American Option w/o expiry, ~ Perpetual.

$\{\text{let } U_t = e^{(r-\frac{1}{2}\sigma^2)t + \sigma W_t}\}$

$$V_0(S_0 = S) = \sup_{\tau} \mathbb{E}^*\{e^{-r\tau} h(S U_\tau)\}$$

and

$$V_t(S_t = S) = \sup_{\tau} \mathbb{E}^*\{e^{-r(\tau-t)} h(S U_{\tau-t})\}$$

$\therefore$ τ "should not" depend on t.

$\therefore$ ~~$V(S_t)$~~ $V(t, S_t) = V(S_t)$ ~ Not a function of time

As it is not a f^n of time τ is def^n as

~~$\tau = \inf\{t:$~~ There exists $A \subset [0, \infty)$

$\tau \equiv \min\{t : S_t \in A\}$

Equivalently, there exits $B \subset [0, \infty)$

$\tau \equiv \min\{t : W_t \in B\}$.

Remark, It is not obvious that

$$V(S) = \sup_L f_L(S)$$

and

$$V(S') = \sup_L f_L(S')$$

are maximized at the same value of L,

but in fact it is true.

This is due to results from Harmonic Theory which we will not cover.

Evaluate $f_L(S)$, Notice $S U_{\tau_L} = L$,

$$\therefore \quad f_L(S) = \mathbb{E}\left\{ e^{-r\tau_L} (K - S U_{\tau_L})^+ \,\middle|\, S_0 = S \right\}$$

$$= (K-L)\, \mathbb{E}\left\{ e^{-r\tau_L} \,\middle|\, S_0 = S \right\}$$

Perpetual American Put,

payoff / Intrinsic value ~ strike price K

$$h(S_t) = (K - S_t)^+$$

Stopping time, when S_t is sufficiently small,

le "Exercise price" $L \in (0, K)$

$$\tau_L := \inf \{ t : S_t \le L \}$$

Then

$$V(S) = \sup_L \mathbb{E}^* \left\{ e^{-r\tau_L} h(S\,U_{\tau_L}) \right\}$$

$\therefore$ for fixed L find formula

$$f_L(S) = \mathbb{E}^* \left\{ e^{-r\tau_L} h(S\,U_{\tau_L}) \right\}$$

$$= \mathbb{E} \left\{ e^{-r\tau_L} (K - S\,U_{\tau_L})^+ \right\}$$

Notice $S U_t = L \Leftrightarrow U_t = L/S$

$$e^{(r-\frac{1}{2}\sigma^2)t + \sigma W_t} = L/S$$

Notice

$$Z_t = U_t e^{-rt} = e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}$$

Is a martingale, Indeed

$$dZ_t = \sigma Z_t dW_t$$ is the solution. (zero intrest security)

We want something more general,

$$Z_t = U_t^\lambda e^{-t(r\lambda - \lambda\sigma^2/2 + \lambda^2\sigma^2/2)} = e^{-\frac{1}{2}\sigma^2\lambda^2 t + \sigma\lambda W_t}$$

Ie "dialate" σ to $\lambda\sigma$.

This is a Mg by the same arguement.

Now choose λ s.t.

$$Z_t = U_t^{\lambda} e^{-rt}$$

ie

$$r = (r\lambda - \lambda\sigma^2/2 + \lambda^2\sigma^2/2)$$

$$0 = \lambda^2\left(\frac{\sigma^2}{2}\right) + \lambda\left(r - \frac{\sigma^2}{2}\right) - r$$

solutions $\lambda_+ = 1$

$\lambda_- = -2r/\sigma^2$

Let us set $\lambda = -2r/\sigma^2$

Then for $0 \le t \le \tau_L$

$$U_t \ge L/S \Leftrightarrow U_t^{\lambda} \le (L/S)^{\lambda}$$

So

$$0 \le Z_t = U_t^{\lambda} e^{-rt} \le (L/S)^{\lambda}$$

for $0 \le t \le \tau_L$.

Recall ~~optional stopping theorem:~~

optional sampling theorem:

If M_t is martingale and τ is a stopping time $(\text{wrt } \mathcal{F}_t)$ so that $\mathbb{P}(\tau < \infty) = 1$.

and * $\mathbb{E}\{|M_\tau|\} < \infty$

* $\lim_{t \to \infty} \mathbb{E}\{|M_t| \, \mathbb{1}_{\{\tau > t\}}\} = 0$

then $\mathbb{E}\{M_\tau\} = \mathbb{E}\{M_0\}$

ie if M_0 is constant,

$$\mathbb{E}\{M_\tau\} = M_0.$$

Apply OST to Z_t

Check conditions $\{0 \le Z_t \le (L/S)^\lambda\}$.

$\mathbb{P}(\tau < \infty)$ - true Brownian motion hits every point w/ prob 1. ✓

$\mathbb{E}(|Z_\tau|) \le (L/S)^\lambda < \infty$. ✓

$\mathbb{E}\{|Z_t| \, \mathbb{1}_{\{\tau > t\}}\} \le (L/S)^\lambda \, \mathbb{P}(\tau > t) \to 0$. ✓

Now Apply OST to Z_t:

Note: $S u_\tau = L \Leftrightarrow u_\tau^\lambda = (L/S)^\lambda$.

$$\mathbb{E}^*(Z_\tau) = \mathbb{E}^*(Z_0) = 1.$$

But $Z_\tau = u_\tau^\lambda e^{-r\tau}$

$$\therefore \quad \mathbb{E}^*(u_\tau^\lambda e^{-r\tau}) = 1$$

$$\hookrightarrow \quad \mathbb{E}^*(e^{-r\tau}) = (L/S)^{-\lambda}$$

Finally we have

$$f_L(S) = (K-L)\,\mathbb{E}(e^{-r\tau_L} \mid S_0 = S)$$

$$= (K-L)\left(\frac{L}{S}\right)^{-\lambda}$$

$$= (K-L)\left(L/S\right)^{2r/\sigma^2}$$

for $L<S$

from this formula it is clear $\sup_L f_L(S)$ does not depend on S indeed.

$$f(L,S) = \frac{(K-L)\,L^{2r/\sigma^2}}{S^{2r/\sigma^2}} \quad \leftarrow \text{max of numerator.}$$

$$\frac{d}{dL}\left((K-L)L^b\right) = -L^b + (K-L)\,b\,L^{b-1} = 0$$

$$\hookrightarrow L = (K-L)\,b$$

$$\hookrightarrow L = \frac{b}{b+1} K$$

ie max of $f(L,S)$ is @ $L^* = \frac{2r}{2r+\sigma^2} K$.

~~for~~ for $S > L^*$

$$V(S) = \left(\frac{K^{1+\frac{2r}{\sigma^2}}}{S^{2r/\sigma^2}}\right)\left(\frac{\sigma^2}{2r+\sigma^2}\right)\left(\frac{2r}{2r+\sigma^2}\right)^{2r/\sigma^2}$$

for $S \le L^*$

$$V(S) = K - S$$

ODE (Black Scholes "type" no time derivative!)

$V' = \frac{d}{dS} V$

$$-rV + rSV' + \frac{1}{2}\sigma^2 S^2 V'' = \begin{cases} -rK, & 0 < S \le L^* \\ 0, & S > L^* \end{cases}$$

Ito formula:

$$dV = \left(-rV + rSV' + \tfrac{1}{2}\sigma^2 S^2 V''\right)dt + e^{-rt}\sigma S_t V' dW_t .$$