Continuous Compound Interest:

Compound interest such that frequency (m) of compounding events per year goes to ∞.

Recall Compound Interest formula, for $t \in \frac{1}{m}\mathbb{N} = \{\frac{1}{m}, \frac{2}{m}, \dots\}$

$$V_t = P\left(1+\frac{r}{m}\right)^{mt}$$

Let us write this as

$$V_t = P\left\{\left(1+r/m\right)^{m/r}\right\}^{rt}$$

let $x = \frac{m}{r}$ & consider $\lim x = \lim m = \infty$.

Recall from Calculus that

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

Thus @ continuous compounding, the value of the loan is $V_t = Pe^{rt}$

Derivative: $\frac{d}{dt}V_t = rPe^{rt} = rV_t$.

$V(t)$ under continuous compounding rate r

is greater than $V^{(m)}(t) \equiv$ the loan @ rate r
w/ m compounding
events per year
(freq $= m$)

Let us check that

$$V(1) > V^{(m)}(1)$$

(we will show later how to extend to all t).

let $g(x) = 1+x$, $f(x) = e^x$

$$g' = 1 \qquad f'(x) = e^x$$

$$g(0) = 1 = f(0)$$

$$g'(x) < f(x) \text{ for } x > 0$$

$$\therefore \quad g(x) < f(x) \text{ for } x > 0.$$

$$(1+x) < e^x \quad \text{for } x > 0$$

let $x = \frac{r}{m}$

$$\left(1 + \frac{r}{m}\right) < e^{r/m}$$

$$\Longrightarrow$$

$$V^{(m)}(1) = (1 + r/m)^m < e^r = V(1). \quad \#$$

Suppose a loan charges 6% interest.
At what time does the value double?

formula: $V_t = Pe^{rt}$ for the value of the loan @ time t.

$r = .06$ find t st

$$V_t = 2P = Pe^{rt} \rightarrow 2 = e^{(.06)t}$$

$$\ln 2 = (.06)t$$

$$\hookrightarrow t = \frac{\ln 2}{.06} = 11.55 \text{ yrs.}$$

Do the same w/ loan @ 10% interest.

$$2 = e^{t/10}$$

$$t = \frac{\ln 2}{1/10} = 6.931$$

In general, for $V_t = MP$
we have $t = \frac{\ln M}{r}$

As is suggested in the last formula,

It is sometimes useful to consider not the difference of values of an instrument over time but the ratio of values.

In this case, the Logarithmic return is useful,

$$k(s,t) = \ln \frac{V(t)}{V(s)}$$

One thing that is usefull here is additivity: $s<u<t$

$$k(s,t) = \ln \frac{V_t}{V_s} = \ln \frac{V_t}{V_u} \frac{V_u}{V_s}$$

$$= \ln \frac{V_t}{V_u} + \ln \frac{V_u}{V_s}$$

$$= k(u,t) + k(s,u) .$$

For Continuous Interest we have

$$k(s,t) = \ln \frac{V_t}{V_s} = \ln e^{(t-s)r} = (t-s)r$$

Comparing interest rates.

There is a bit of confusion for compound interest rates since deposits occur only at fixed intervals.

We will show that despite this, the value of a loan @ compound interest is continuously growing.

Suppose a contract ~~can~~ pays \$120 in one year and can be purchased today for \$100. $\left(\text{ie } \begin{matrix} m=1 \\ r=.2 \end{matrix}\right)$

Suppose you want to sell it in 6 months, what is its value

option 1

$\frac{1}{2}$ interest

$$V^{(1)}(1/2) = \frac{100 + 120}{2} = 110$$

option 2

Assume continuous compound interest 'behind the scenes'

$$V^{(2)}(1/2) = 100\,(1.2)^{1/2}$$

Let us compare $V^{(1)}_{(1/2)}$ ~~to~~ $V^{(2)}_{(1/2)}$

$$V^{(2)}_{(1/2)} = 100 \times 1.095 = \$109.5$$

ie $\quad V^{(2)}(1/2) < V^{(1)}(1/2)$.

Suppose you believe value is $V^{(2)}$ and someone else believes value is $V^{(1)}$, how can you use this to your advantage?

Ask the other person to buy these contracts @ 6 mo maturity for \$110

Thus, $t=0$:
- Borrow \$ 1000 (10×100) in loan, promising to pay $10 \times 120 = 1200$ @ $t=1$
- Buy 10 contracts for \$1000 @ time 0,

$t = 1/2 = 6$ mo,
- Sell 10 contracts for $10 \times \$110 = \$1100$
- Buy 11 contracts

$t = 1$ year,
- Sell 11 contracts for $\$11 \times 110 = \$1210$
- Pay off original loan @ \$1200.

We pocket \$10 !

Similarly if some one thinks the value is low, say \$109 can you make a similar arguement?

hint: Ask the other person to buy contracts at time $t=0$ & $t = 1/2$ which you will buy off them @ $t = 1/2$ & $t = 1$ resp.

We define the effective rate to be

$$\left(1+\frac{r}{m}\right)^m = 1 + r_e$$

the rate @ frequence $m=1$.

Continuous freq:

$$e^r = 1 + r_e.$$

It is reasonable now to make ext. $V(t) = (1+r_e)^t$

As demo'd in the above example given any loan w/ effective rate r_e we have

$$V(t) = P(1+r_e)^t$$

or if it has freq m: $V(t) = P\left(1+\frac{r}{m}\right)^{mt}$

—— continuous freq: $V(t) = Pe^{rt}$

If one loan has freq m_1 w/ rate r_1 and a second has freq m_2 w/ rate r_2

~~we say~~

~~we say loan~~ and $\bar{r}_1$ and $\bar{r}_2$ are the effective rates we say — loan 1 is prefferable to loan 2 only if $\bar{r}_1 > \bar{r}_2$.