Discrete frame work:

We will mostly carryout these models/methods inthier discrete version. This is simpler and importantly allows us to use Linear algebra to calculate.

Again Ω is outcomes of real world, say

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}.$$

Then there are 2^n subsets of Ω, $\mathcal{A}$-(this is our σ-algebra)

Let $\mathbb{P}$ be a probability measure on Ω, $\mathbb{P}: \mathcal{A} \to [0,1]$.

Define $p_i = \mathbb{P}(\omega_i)$.

Let $S(1): \Omega \to [0, \infty)$

ie for each real world outcome ω_i there is an associated price $S(1)$ of the stock.

Ie if ω_i occurs $S(1) = S(1; \omega_i) = S^{(\omega_i)}(1)$.

$$\mathbb{E}(S(1)) = \sum_{i=1}^{n} S^{(\omega_i)}(1)\, p_i = p^T \cdot S(1) = S(1)^T \cdot p.$$

$$= \langle S(1), p \rangle.$$

where $S(1) = \begin{pmatrix} S^{\omega_1}(1) \\ S^{\omega_2}(1) \\ \vdots \\ S^{\omega_n}(1) \end{pmatrix}$, $p = \begin{matrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{matrix}$

$K^{(\omega_i)}(1) = \dfrac{S^{(\omega_i)}(1) - S(0)}{S(0)}$; $\mathbb{E}(K(1)) = (K(1), p)$

EXAMPLE: 2 SECURITIES, 1 time step. EG (1)

Now we consider eg of 2 securities,

Suppose there are 2 securities S_1 + S_2

+ 3 outcomes, $\Omega = \{\omega_1, \omega_2, \omega_3\}$.

Suppose $S_1(0) = S_2(0) = \$100$

> Notation: $S_i(1, \omega_j) = S_i^{\omega_j}(1)$
> Value ~~outcom~~ of ith security
> under outcome ω_j

Let $S_1(0) = S_2(0) = \$100$

Let

$S_1^{\omega_1}(1) = 120$ $\quad S_2^{\omega_1}(1) = 110$ $\quad \mathbb{P}(\omega_1) = 1/4$

$S_1^{\omega_2}(1) = 120$ $\quad S_2^{\omega_2}(1) = 115$ $\quad \mathbb{P}(\omega_2) = 1/4$

$S_1^{\omega_3}(1) = 105$ $\quad S_2^{\omega_3}(1) = 110$ $\quad \mathbb{P}(\omega_3) = 1/2$

Let us organize this info as:

outcome	$S_1(1)$	$S_2(1)$	$\mathbb{P}$
ω_1	120	110	1/4
ω_2	120	115	1/4
ω_3	105	110	1/2.

Let us find the expectation + Variance of S_i + K_i,

$$\mathbb{E}(S_1) = \tfrac{1}{4}\,120 + \tfrac{1}{4}\,120 + \tfrac{1}{2}\,105 = \frac{450}{4} = 112\tfrac{1}{2}$$

$$\mathbb{E}(S_2) = \tfrac{1}{4}\,110 + \tfrac{1}{4}\,115 + \tfrac{1}{2}\,110 = \frac{445}{4} = 111\tfrac{1}{4}$$

$$\mathbb{E}(S_1^2) = \tfrac{1}{4}\,120^2 + \tfrac{1}{4}\,120^2 + \tfrac{1}{2}\,105^2 = 12712.5$$

$$\mathbb{E}(S_2^2) = \tfrac{1}{4}\,110^2 + \tfrac{1}{4}\,115^2 + \tfrac{1}{2}\,110^2 = \frac{49525}{4} = 12381\tfrac{1}{4}$$

$$\mathrm{Var}(S_1) = \mathbb{E}(S_1^2) - (\mathbb{E}S_1)^2 = 12712\tfrac{1}{2} - (112\tfrac{1}{2})^2 = 56\tfrac{1}{4}$$

$$\mathrm{Var}(S_2) = \mathbb{E}(S_2^2) - (\mathbb{E}S_2)^2 = 12381\tfrac{1}{4} - (111\tfrac{1}{4})^2 = 4\tfrac{11}{16}$$
$$= 4.6875$$

$$(7.5)^2 = \mathrm{Var}(S_1) \quad ; \quad (2.17)^2 \approx \mathrm{Var}(S_2)$$

$$\mathbb{E}(K_1) = \frac{\mathbb{E}S_1(1) - S_1(0)}{S_1(0)} = \frac{12\tfrac{1}{2}}{100} = .125$$

$$\mathbb{E}(K_2) = \frac{\mathbb{E}\,S_2(1) - S_2(0)}{S_2(0)} = \frac{111.25}{100} = .1125$$

$$\mathrm{Var}(K_1) = \frac{\mathrm{Var}\,S_1(1)}{S_1(0)} = \frac{56\tfrac{1}{4}}{100} = .5625 \;;\; \mathrm{Var}\,K_2 = \frac{\mathrm{Var}\,S_2(1)}{S_2(0)} = \frac{4.6875}{100}$$
$$= .04688$$

let us consider buying into Port. @ $t=0$.

w/ x_i many S_i

Value of Port. is (at time $t=0$)

$$V(0) = x_1 S_1(0) + x_2 S_2(0)$$

Value of Port @ time $t=1$

$$V(1) = x_1 S_1(0)(1+K_1) + x_2 S_1(0)(1+K_2)$$

$$= \underbrace{x_1 S_1(0) + x_2 S_2(0)}_{V(0)} + x_1 S_1(0) K_1 + x_2 S_2(0) K_2$$

Find Return of V, K_V

$$K_V = \frac{V(1) - V(0)}{V(0)} = \frac{x_1 S_1(0) K_1 + x_2 S_2(0) K_2}{V(0)}$$

$$= \left(\frac{x_1 S_1(0)}{V(0)}\right) K_1 + \left(\frac{x_2 S_2(0)}{V(0)}\right) K_2$$

these are weights:

$$W_i = \frac{x_i S_i(0)}{V(0)}.$$

Now return on the portfolio is

$$K_V = w_1 K_1 + w_2 K_2 .$$

a random variable,

Of course

$$\mathbb{E} K_V = w_1 \mathbb{E} K_1 + w_2 \mathbb{E} K_2 .$$

And

$$\begin{aligned}
\operatorname{Var} K_V &= \mathbb{E}\left(K_V^2\right) - \left(\mathbb{E} K_V\right)^2 \\
&= \mathbb{E}\left(w_1^2 K_1^2 + 2 w_1 w_2 K_1 K_2 + w_2 K_2^2\right) - \left\{ \left(w_1 \mathbb{E} K_1\right)^2 + 2 w_1 w_2 \mathbb{E} K_1 \mathbb{E} K_2 + \left(w_2 \mathbb{E} K_2\right)^2 \right\} \\
&= w_1^2\left(\mathbb{E} K_1^2 - \left(\mathbb{E} K_1\right)^2\right) + w_2^2\left(\mathbb{E} K_2^2 - \left(\mathbb{E} K_2\right)^2\right) \\
&\quad + 2 w_1 w_2 \left(\mathbb{E} K_1 K_2 - \mathbb{E} K_1 \mathbb{E} K_2\right) \\
&= w_1^2 \operatorname{Var} K_1 + w_2^2 \operatorname{Var} K_2 + 2 w_1 w_2 \operatorname{cov} K_1 K_2 \\
&= w_1^2 c_1 + w_2^2 c_2 + 2 w_1 w_2 c_{12} .
\end{aligned}$$

$$c_i = \operatorname{Var} K_i$$

$$c_{12} = \operatorname{Cov}(K_1, K_2)$$

Now Alternatively we can find K chart,

$\mathcal{E}a(1)$

outcom	K_1	K_2	$\mathbb{P}$
ω_1	.20	.10	1/4
ω_2	.20	.15	1/4
ω_3	.05	.10	1/2.

And find (as above)

$\mathbb{E} K_1 = \cdots = .125$

$\mathbb{E} K_2 = \cdots = .1125$

$\text{Var } K_1 = \cdots = .5625$

$\text{Var } K_2 = \cdots = .04688$

Working in K is helpful because we are interested in return, we don't care as much about $S_i(0)$.

Now let us return to EG(1)

* if $x_1, x_2 > 0$ then

$$\mathrm{Cov}(x_1 S_1(0), x_2 S_2(0)) > 0$$

But if $x_2 < 0 < x_1$ then

$$\mathrm{Cov}(x_1 S_1(0), x_2 S_2(0)) < 0.$$

. . . .

ie covariance is similar to EG(2)

$\therefore$ Buy stock 1 & short sell stock 2.

Again $\dfrac{x_i S_i(0)}{V(0)} = w_i$ & $w_1 = s$, $w_2 = 1 - s$

let us set $w_1 = -.01$ & $w_2 = 1.01$.

find

σ_V^2 & μ_V.

$$\Sigma = \begin{pmatrix} .5625 & 9.375 \times 10^{-4} \\ 9.375 \times 10^{-4} & .04688 \end{pmatrix}$$

$$\mu = \begin{pmatrix} .125 \\ .1125 \end{pmatrix}$$

But

$$(w_1 \ w_2) \begin{pmatrix} c_1 & c_{12} \\ c_{12} & c_2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$= (w_1 \ w_2) \begin{pmatrix} c_1 w_1 + c_{12} w_2 \\ c_{12} w_1 + c_2 w_2 \end{pmatrix}$$

$$= c_1 w_1^2 + c_{12} w_1 w_2 + c_{12} w_1 + c_2 w_2^2$$

$$= \mathrm{Var}\, K_V$$

Recall $\Sigma = \begin{pmatrix} c_1 & c_{12} \\ c_{12} & c_2 \end{pmatrix}$ is called covariance matrix.

It is diagonalizable & semipositive definite (~~pos~~ non neg eigenvalues

We know it ~~is~~ has non neg eigenvalues because: if $(\bar{w}_1 \ \bar{w}_2)$ is an eigen vector then

$$(\bar{w}_1 \ \bar{w}_2) \begin{pmatrix} c_1 & c_{12} \\ c_{12} & c_2 \end{pmatrix} \begin{pmatrix} \bar{w}_1 \\ \bar{w}_2 \end{pmatrix} = \bar{\lambda} (\bar{w}_1, \bar{w}_2) \begin{pmatrix} \bar{w}_1 \\ \bar{w}_2 \end{pmatrix} = \bar{\lambda} \left| \begin{matrix} \bar{w}_1 \\ \bar{w}_2 \end{matrix} \right|^2 = \mathrm{Var}\, K_V \geq 0$$

$$\therefore \bar{\lambda} \geq 0 .$$

Moreover $\mathrm{Var}\, K_V > 0$ if "each rv has some independent randomness" - K_2 is not fn of K_1.

We will generally make the assumption that
Σ is positive definite ie $\bar{\lambda} > 0$
ie $\mathrm{Var}\, K_V > 0 \ \forall (w_1\ w_2)$ ie K_2 not a fn of K_1.

Let us calculate $\mathrm{cov}(K_1, K_2)$.

Recall:

Ω	K_1	K_2	$\mathbb{P}$
ω_1	.2	.1	1/4
ω_2	.2	.15	1/4
ω_3	.05	.1	1/2

$\mu_1 = \mathbb{E} K_1 = .125$; $\mu_2 = \mathbb{E} K_2 = .1125$.

$\mathrm{cov} = \mathbb{E} K_1 K_2 - \mathbb{E} K_1 \mathbb{E} K_2 = \mathbb{E}\{(K_1 - \mu_1)(K_2 - \mu_2)\}$.

$\mathbb{E} K_1 K_2 = \frac{1}{4}(.2)(.1) + \frac{1}{4}(.2)(.15) + \frac{1}{2}(.05)(.1) = .015$

$\mathbb{E} K_1 \mathbb{E} K_2 = .01406$

$\mathrm{cov}(K_1, K_2) = .0009375 = 9.375 \times 10^{-4}$

Also let us recall the correlation coefficient

$$\rho_{12} = \frac{\mathrm{cov}(K_1, K_2)}{\sigma_1 \sigma_2} = \frac{\mathrm{cov}(K_1, K_2)}{\sqrt{\mathrm{Var} K_1 \, \mathrm{Var} K_2}} = .005773.$$

Now we can find risk on any portfolio $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$\{$ let $w_1 = s$ $\xrightarrow{+}$ $w_1 + w_2 = 1 \Rightarrow w_2 = 1 - s \}$.

covariance matrix $\Sigma_{K_1 K_2} = \begin{pmatrix} .5625 & 9.375 \times 10^{-4} \\ 9.375 \times 10^{-4} & .04688 \end{pmatrix}$

Let us take $w_1 = w_2 = \frac{1}{2}$.

$$\sigma_v^2 = \text{Var } K_v = \left(\tfrac{1}{2} \ \tfrac{1}{2}\right) \Sigma \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \frac{1}{4} C_1 + \frac{1}{4} 2 \cdot C_{12} + \frac{1}{4} C_2$$

$$= .1528$$

$$\mu_v = \mathbb{E}(K_v) = \tfrac{1}{2}(.125) + \tfrac{1}{2}(.1125) = .1188.$$

Thus we have constructed a portfolio w so that

$$C_2 < \sigma_v^2 < C_1 \quad ; \quad \mu_2 < \mu_v < \mu_1$$

This may be usefull if the ris of S_1 is too high + return of S_2 is too low.

Note here, if $V(0) = \$100$ we have bought ½ of each stock @ \$100/stock at time 0. $V(0) = \$100$; $V(1) = V(0)(1 + K_v)$

Consider an Alternative system... $\mathcal{EG}(2)$

$S_1(0) = S_2(0) = 100.$

Ω	$S_1(1)$	$S_2(1)$	$\mathbb{P}$
ω_1	125	105	1/4
ω_2	110	110	1/2
ω_3	105	115	1/4

$\xrightarrow{1^{st}}$

Ω	K_1	K_2	$\mathbb{P}$
ω_1	.25	.05	1/4
ω_2	.10	.10	1/2
ω_3	.05	.15	1/4

$\mu_1 = .125 \; ; \; \mu_2 = .10$

$c_1 = \text{Var } K_1 = \mathbb{E} K_1^2 - (\mathbb{E} K_1)^2 = 5.625 \times 10^{-3}$

$\mathbb{E} K_1^2 = .02125$

$c_2 = \text{Var } K_2 = 1.125 \times 10^{-3}$

$\mathbb{E} K_2^2 = .01125$

$c_{12} = \text{cov } K_1 K_2 = .01 - .0125 = -.0025 = -2.5 \times 10^{-3}$

$\mathbb{E} K_1 K_2 = \frac{1}{4}(.25)(.05) + \frac{1}{2}(.1)(.1) + \frac{1}{4}(.05)(.15)$
$= .01$

$$\rho_{12} = \frac{-2.5 \times 10^{-3}}{\sqrt{(5.625 \times 10^{-3})(1.125 \times 10^{-3})}} = \frac{-2.5}{\sqrt{5.625 \times 1.125}} \doteq -\frac{2.5}{2.5156}$$

$= -.9938.$

Now consider a portfolio of $w_1 = w_2 = 1/2$.

$$\Sigma_K = \Sigma_{K_1 K_2} = \begin{pmatrix} 5.625 & -2.5 \\ -2.5 & 1.125 \end{pmatrix} \times 10^{-3}$$

$$\sigma_V^2 = \mathrm{Var}\, K_V = \frac{1}{4} c_1 + \frac{1}{4} 2c_{12} + \frac{1}{4} c_2$$

$$= \left\{ \frac{1}{4} 5.625 + \frac{1}{4} 2(-2.5) + \frac{1}{4}(1.125) \right\} \times 10^{-3}$$

$$= 4.375 \times 10^{-4}.$$

$$\mu_V = \mathbb{E} K_V = \frac{1}{2}(.125) + \frac{1}{2}(.1) = .1125$$

We have

$$\mu_2 < \mu_V < \mu_1$$

$$\sigma_V^2 < \underset{\substack{\| \\ c_2}}{\sigma_2^2} < \underset{\substack{\| \\ c_1}}{\sigma_1^2}$$

Here eg $V(0) = 200$ then $x_1 = x_2 = 1$.

$$V_0 = x_1 S_1(0) + x_2 S_2(0) \longrightarrow W_i = \frac{x_1 S_1(0)}{V_0} = 1/2.$$