Several Securities.

$S_i(t) \quad t = 1, 2 \; ; \; i = 1, \dots, n.$

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad x_i \equiv \text{amt of stock } i \text{ which is purchased.}$$

$$x^T S(0) = V(0).$$

$$w_i = \frac{x_i S_i(0)}{V(0)} \qquad w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

$$\mathbb{1} = \underbrace{\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}}_{n} \qquad w^T \mathbb{1} = w_1 + \dots + w_n = 1.$$

Portfolio set:

$$W = \{ w \in \mathbb{R}^n : w^T \mathbb{1} = 1. \}$$

$$K_i = \frac{S_i(1) - S_i(0)}{S_i(0)} \; ; \; \mu_i = \mathbb{E}(K_i). \quad m = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix}$$

$$C_{ij} = \mathrm{Cov}(K_i, K_j)$$

$$\Sigma = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & & & \\ \vdots & & & \\ c_{n1} & \cdots & & c_{nn} \end{bmatrix}$$

Assme $\det \Sigma \neq 0$

$\hookrightarrow$ all eigenvals positive.

$K_v = w^T K = w_1 K_1 + \cdots + w_n K_n$.

$\mu_v = w^T m = w_1 \mu_1 + \cdots + w_n \mu_n$.

$\sigma_v^2 = var(K_v) = w^T \Sigma w = \sum_{ij} w_i c_{ij} w_j$

Again we find mvp of σ_v^2

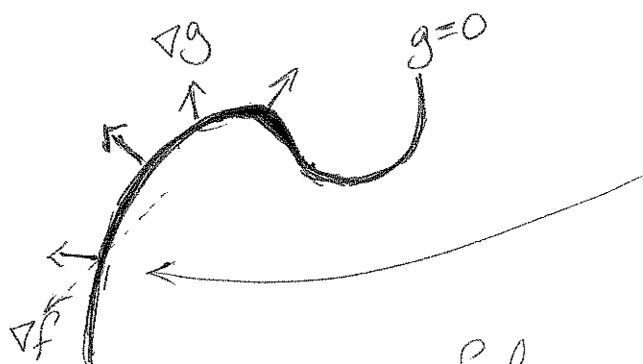
minimum of $f(w) = w^T \Sigma w$

w/ ristrictition that $g(w) = w^T \mathbb{1} - 1 = 0$.

PICTURE OF LAGRANG MULTIPLIER.

Maximize $f(x)$ w/ constraint $g(x) = 0$ $x \in \mathbb{R}^n$.

∇g always $\perp$ to tangent of curve g.



if $\nabla f \not\perp$ to g' move along $g=0$. to vary (increase) f ∴

f has max iff ∇f in direction of ∇g. ie $\lambda \nabla f = \nabla g$ for some $\lambda \in \mathbb{R}$.

Lagrange multiplier to maximize σ_v^2.

$$F = [\sigma_v^2 - \lambda(w^T \mathbb{1} - \lambda)] = [f - \lambda g]$$

$$0 = \nabla F = \nabla(\sigma_v^2) - \lambda \nabla(w^T \mathbb{1} - 1).$$

$$\nabla(\sigma_v^2) = \nabla(w^T \Sigma w) = 2\Sigma w.$$

$$\nabla(w^T \mathbb{1} - 1) = \mathbb{1}.$$

$$\therefore \quad 2\Sigma w = \lambda \mathbb{1} \quad *$$

$$w = \frac{\lambda}{2} \Sigma^{-1} \mathbb{1}$$

$$1 = \mathbb{1}^T w = \frac{\lambda}{2} \mathbb{1}^T \Sigma^{-1} \mathbb{1}.$$

$$\lambda = 2 \frac{1}{\mathbb{1}^T \Sigma^{-1} \mathbb{1}}$$

$$\hookrightarrow * \quad 2\Sigma w = \lambda \mathbb{1} = \frac{2\mathbb{1}}{\mathbb{1}^T \Sigma^{-1} \mathbb{1}}$$

<u>MVP</u> $$w_0 = \frac{\Sigma^{-1} \mathbb{1}}{\mathbb{1}^T \Sigma^{-1} \mathbb{1}}.$$

<u>MVP</u>:

$$w_0 = \frac{\Sigma^{-1}\mathbb{1}}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}}$$

$$\begin{cases} \sigma_v^2(w_0) = w_0^T \Sigma w_0 = \left(\dfrac{\mathbb{1}^T\Sigma^{-1}}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}}\right)\Sigma\left(\dfrac{\Sigma^{-1}\mathbb{1}}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}}\right) \\ \qquad = \dfrac{\mathbb{1}^T\Sigma^{-1}\mathbb{1}}{(\mathbb{1}^T\Sigma^{-1}\mathbb{1})^2} = \dfrac{1}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}} \\ \sigma_v^2(w_0) = \dfrac{1}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}} \end{cases}$$

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$$\mu_v(w_0) = w_0^T m = \frac{\mathbb{1}^T\Sigma^{-1} m}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}}.$$

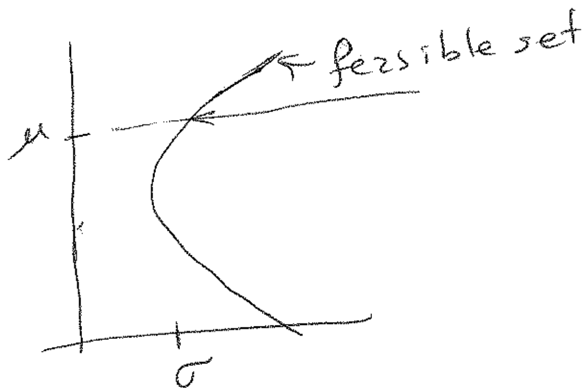
Definition: given portfolio $w^{(1)}$ & $w^{(2)}$.

if $\begin{cases} \mu_1(w^{(1)}) \geq \mu_2(w^{(2)}) \\ \sigma_1(w^{(1)}) \leq \sigma_2(w^{(2)}) \end{cases}$

ie $w^{(1)}$ has higer expected returns and lower risk than $w^{(2)}$

so $w^{(1)}$ dominates $w^{(2)}$
is preferable to ...

Recall 2 security picture:



we want to find smallest σ

for each fixed μ_0.

$\therefore$ 2nd Lagrange multiplier

minimize σ_v^2 w/ restriction $\begin{cases} \mu_v = \mu \\ w^T\mathbb{1} = 1 \end{cases}$

Lagrang multiplier:

$$F = \sigma_v^2 - \lambda_1 \mu_v - \lambda_2 (w^T \mathbb{1} - 1)$$

$$0 = \nabla_w F = 2\Sigma w - \lambda_1 m - \lambda_2 \mathbb{1}.$$

$$w^T m = \mu$$

$$w^T \mathbb{1} = 1$$

$$w = \begin{pmatrix} \frac{1}{2}\Sigma^{-1} m & \frac{1}{2}\Sigma^{-1}\mathbb{1} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$\begin{pmatrix} \mu \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} m \Sigma^{-1} m & \frac{1}{2} m \Sigma^{-1} \mathbb{1} \\ \frac{1}{2}\mathbb{1}\Sigma^{-1} m & \frac{1}{2}\mathbb{1}\Sigma^{-1}\mathbb{1} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$= A \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{ad - cb} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$a = \frac{1}{2} m \Sigma^{-1} m, \quad b = c = \frac{1}{2} m \Sigma^{-1} \mathbb{1}$$

$$d = \frac{1}{2}\mathbb{1}\Sigma^{-1}\mathbb{1}$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = A^{-1} \begin{pmatrix} \mu \\ 1 \end{pmatrix}$$

Finally: w minimizing σ_V^2

$$2\Sigma w = \lambda_1 m + \lambda_2 \mathbb{1}.$$

$$w_\mu = \frac{1}{2}(\Sigma^{-1} m)\lambda_1 + \frac{1}{2}(\Sigma^{-1}\mathbb{1})\lambda_2$$

moreover, A is independent of μ.

$\therefore$ A^{-1} independent of μ.

$$\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = A^{-1}\begin{pmatrix}\mu\\ 1\end{pmatrix} \Rightarrow \quad \lambda_1, \lambda_2 \text{ are linear in } \mu.$$

$\therefore$ w_μ is linear in μ.

$\therefore$ Let us use 2 choices of μ say $\mu = 0, 1$.

$$\frac{1}{ad-cd}\begin{pmatrix} d & -b\\ -c & a\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}$$

$$\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = \frac{1}{ad-cd}\begin{pmatrix} -b + \mu(d-b)\\ a + \mu(a-c)\end{pmatrix}$$

$\therefore$ Line in $\mathbb{R}^n$ is given by

$$w^{(1)} = w_{\mu=0} = \frac{1}{2}\left(\Sigma^{-1} m\right)\lambda_1(0) + \frac{1}{2}\left(\Sigma^{-1}\mathbb{1}\right)\lambda_2(0)$$

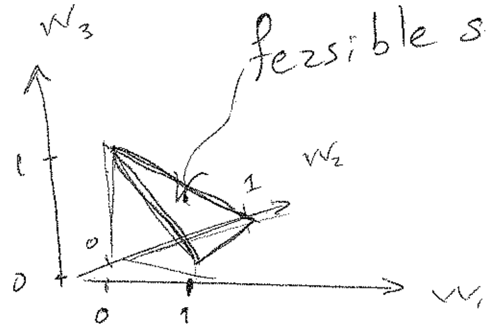
$$w^{(2)} = w_{\mu=1} = \frac{1}{2}\left(\Sigma^{-1} m\right)\lambda_1(1) + \frac{1}{2}\left(\Sigma^{-1}\mathbb{1}\right)\lambda_2(1)$$

$$w = s\,w^{(1)} + (1-s)\,w^{(2)} \quad ; \quad w^{(1)}, w^{(2)} \in \mathbb{R}^n.$$

this line in in $\mathbb{R}^n \supset \{w : w^T\mathbb{1} = 1\}$.
is called the minimal variance line.

Now consider the example of 3 securities.

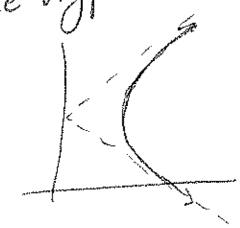
feasible set w/o short selling



Consider 2 security combinations.

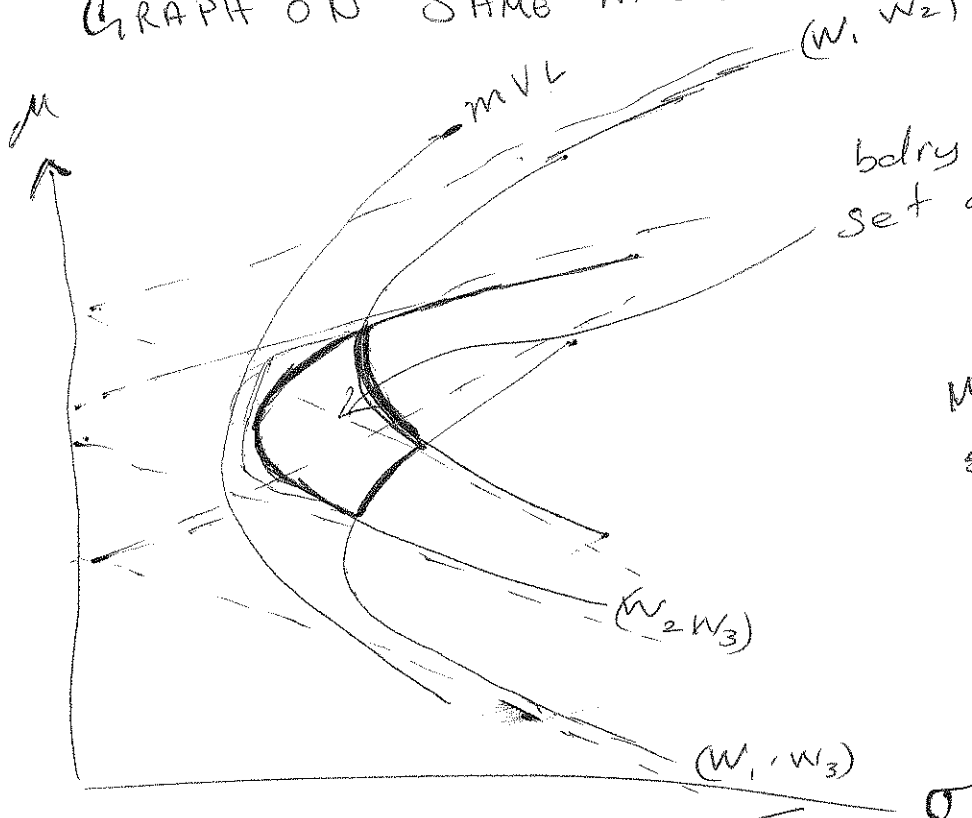
(W_1, W_2)
(W_1, W_3)
(W_2, W_3)

each look like hyperbolas:



GRAPH ALL three on the same axes:

GRAPH ON SAME AXES:



bdry of set w/o

MVL outside set feasible w/o short selling

MVL goes throug set feasible w/ short selling