Introduction to Arbitrage

Consider n securities over 2 time steps.

$\mathbb{T} \equiv$ time set ; $\mathbb{T} = \{0,1\}$.

$S_i(t) \equiv$ value of ith security @ time t.

$S_i(0)$ given for $i = 1, \ldots, n$

Ω = sample space. Suppose ther are m total outcomes $|\Omega| = m$.

$$\left.\begin{aligned} S_i(1) &: \Omega \to (0,\infty) \\ \omega_j &\mapsto S_i^{(\omega_j)}(1) \end{aligned}\right\} \; S_i(1) \text{ a random variable.}$$

Return for the i^{th} security: $K_i = \dfrac{S_i(1) - S_i(0)}{S_i(0)}$

Let A be the return matrix:

$$A_{ij} = K_i^{(\omega_j)} \quad ; \; A \text{ is an } n \times m \text{ matrix.}$$

Portfolio @ time $t = 0$

$$x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

x_i = \$ amount of i^{th} security purchased.

(We do not require - $x_1 + \cdots + x_n = 1$)

For portfolio x, the return is:

$$K_x = x_1 K_1 + \cdots + x_n K_n$$

for outcome ω:

$$K_x^{\omega} = x_1 K_1^{\omega} + \cdots + x_n K_n^{\omega}$$

In vector form:

$$K_x^{\omega_j} = (xA)_j \rightarrow$$ xA is row vector of outcomes of portfolio x.

Random variable: $K_x : \Omega \rightarrow (0, \infty)$

$$\omega_j \mapsto K_x^{\omega_j} = (xA)_j$$

Given Probability distribution of Ω

$\mathbb{P}(\omega_j) = q_j$, $q_i \in [0,1]$ so that $q_1 + \cdots + q_n = 1$.

Another probability distribution $\tilde{\mathbb{P}}$

we say $\tilde{\mathbb{P}}$ & $\mathbb{P}$ are equivalent if $\mathbb{P}(\omega) > 0$ if & only if $\tilde{\mathbb{P}}(\omega) > 0$

~~The property of Arbitrage~~

If $\mathbb{P}$ is a probability measure w/ arbitrage then all equivalent measures will have arbitrage.

* We have not introduced interest for the sake of simplicity.

If interest is considered, we modify the return variable:

$$\tilde{K}_i = \frac{(1+r)^{-1} S_i(1) - S_i(0)}{S_i(0)}$$

$$= \frac{S_i(1) - (1+r) S_i(0)}{(1+r) S_i(0)}$$

$$= \frac{(1+K_i) S_i(0) - (1+r) S_i(0)}{(1+r) S_i(0)}$$

$$\therefore \quad \tilde{K}_i = \frac{K_i - r}{1+r}$$

We say there exists an Arbitrage opportunity if there is a portfolio $x \in \mathbb{R}^n$ so that $\mathbb{P}(K_x \geq 0) = 1$ and $\mathbb{P}(K_x > 0) > 0$.

ie the probability of losing money is zero + the probability of gaining money is positive.

there exists a strategy so that

ARBITRAGE DICHOTOMY.

Exactly one of the two following possibilities occur for the ~~[illegible]~~ matrix A.

(i) The Arbitrage opportunity:

$\exists\, x \in \mathbb{R}^n$ (a portfolio) so that

$$(xA)_j \geq 0 \quad \forall j$$

$$+ \ \exists\, j_0 \text{ so that } (xA)_{j_0} > 0 .$$

(ii) There is a positive probability vector q

$$(\text{ie } q_1 + \cdots + q_m = 1 \ + \ q_j > 0 \ \forall j)$$

$$\text{and} \quad Aq = 0$$

* That is, there is a probability distribution on Ω so that

$$E K_i = K_i^{\omega_1} q_1 + \cdots + K_i^{\omega_m} q_m = 0$$

that is a $\mathbb{P}$ ~~which~~ for which each security is "Risk Neutral".

PROOF

(i) $\Rightarrow$ not (ii)

If (i) is true, let x be st

$$\begin{cases} (xA)_j \geq 0 & \forall j \\ \text{\& } \exists j_0 \text{ st } (xA)_{j_0} > 0. \end{cases}$$

Let q be a positive probability vector:

$$xAq = \sum_{ij} x_i A_{ij} q_j$$

$$= \sum_j \left(\sum_i x_i A_{ij}\right) q_j$$

$$= \sum_j (xA)_j q_j \geq (xA)_{j_0} q_{j_0} > 0$$

So $xAq \neq 0$

But then $Aq \neq 0$. $\blacksquare$

"Proof"

not(i) $\Rightarrow$ (ii)

Let R_A be the row space of A: $\{(xA) : x \in \mathbb{R}^n\} \subset \mathbb{R}^m$.

Note: (i) $\Leftrightarrow$ $\exists\, x \in \mathbb{R}^n$ st $(xA)_j \geq 0$; $(xA)_{j_0} > 0$

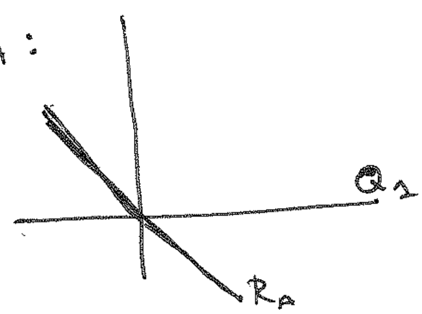
$\Leftrightarrow$ $xA \in Q_1 \equiv$ the first quadrant of $\mathbb{R}^m$.

$\therefore$ not(i) $\Rightarrow$ $xA = 0$ or

if there is i st $(xA)_i > 0$

$\underline{\qquad}$ j st $(xA)_j < 0$.

Visualize R_A:



R_A intersects w/ Q_1 only @ $\{0\}$.

Extend R_A to an $m-1$ dimensional hyperplane which preserves this property: $R'_A \supset R_A$;

R'_A is $m-1$ hyperplane

so that $R'_A \cap Q_1 = \{0\}$.

Then: some $p \in Q_1$ so that $R'_A = \{y \in \mathbb{R}^m : y^T p = 0\}$.

Thus: for all x, $xAp = 0 \Rightarrow Ap = 0$

$\blacksquare$

EXAMPLE:

GIVEN the following system of bets determine whether or not there is an arbitrage oportunity:.
(It is permitted to take "negative bets").

Suppose MSU is playing UM,
there are the ~~following~~ 3 bets you can purchase for \$1 with the following payoffs:

Ω	win	lose	draw
Bet 1	0	1	3/2
Bet 2	2	2	0
Bet 3	1/2	3/2	0
$\mathbb{P}$	P_1	P_2	P_3

Is there an arbitrage opportunity?

Payoff matrix:

$$A = \begin{pmatrix} -1 & 0 & 1/2 \\ 1 & 1 & -1 \\ -1/2 & 1/2 & -1 \end{pmatrix}$$

Arbitrage $x = (x_1, x_2, x_3)$ st $xA \geq 0$ & has at least 1 strictly positive term.

In this case $\det A \neq 0 \therefore$
we can choose any payoff vector, say $\begin{matrix} 1 \\ 1 \\ 1 \end{matrix} = \mathbb{1}$.

And solve $x = A^{-1}\mathbb{1}$.

GIVEN SAME EXAMPLE BUT SUPPOSE

Bet 3 is not avzilzble:

$$A = \begin{pmatrix} -1 & 0 & 1/2 \\ 1 & 1 & -1 \end{pmatrix} \quad \therefore \quad \exists p \text{ st } Ap = 0 \text{ ?}$$

~~find Row Space:~~ Row Reduce

$$\begin{pmatrix} -1 & 0 & 1/2 \\ 1 & 1 & -1 \end{pmatrix} p = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \iff \begin{pmatrix} -1 & 0 & 1/2 \\ 0 & 1 & -1/2 \end{pmatrix} p = \begin{matrix} y_1 \\ y_1 + y_2 \end{matrix}$$

wznt $y_1 = y_2 = 0$

$$-p_1 + p_3 \tfrac{1}{2} = 0$$

$$p_2 - \tfrac{1}{2} p_3 = 0$$

$$p_1 = \tfrac{1}{2} p_3$$

$$p_2 = \tfrac{1}{2} p_3.$$

$$\longrightarrow p = (2, 2, 1)$$

or $p = (2/5,\ 2/5,\ 1/5)$

$\exists p$ st $Ap = 0$

$\therefore$ there is no arbitrage oportunity.

Now consider 2 BETS Both costing \$10 w/ payoff

	win	Lose	draw.
Bet 1	17	0	16
Bet 2	18	16	7
π	q_1	q_2	q_3.

Normalize:

$$A = \begin{pmatrix} .7 & -1 & .6 \\ .8 & .6 & -.3 \end{pmatrix}$$

$\exists p$ st $Ap = 0$?

reduce A:

$$A' = \begin{pmatrix} 2.3 & 1.2 & 0 \\ .8 & .6 & -.3 \end{pmatrix} \quad \begin{matrix} r_1 + 2r_2 \\ r_2 . \end{matrix}$$

If $A'p = 0$ then $2.3p_1 + 1.1p_2 = 0$

$$p_1 = -\frac{1.2}{2.3} p_2 \Rightarrow p_1 = p_2 = 0$$

But then $-.3p_3 = 0 \Rightarrow p_3 = 0$

So $Ap = 0 \Rightarrow p = 0$ $\therefore$ Arbitrage exists... find arbitrage

$$\begin{pmatrix} .7 & .8 \\ -1 & .6 \\ .6 & -.3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \rightarrow \begin{pmatrix} .7 & .8 \\ 1.2 & 0 \\ .6 & -.3 \end{pmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix} \quad \begin{matrix} y_1 \\ y_2 + 2y_3 \\ \\ y_3 \end{matrix}$$

$$2x_1 = x_3$$