PRICING OPTIONS FOR THE BINOMIAL STOCK MODEL.

Let time set be $\mathbb{T} = \{0, 1\}$.

for $t \in \mathbb{T}$, $S(t)$ is value of stock at time t.

$S(0)$ is given & $S(1)$ is a random variable

$$S(1) : \Omega \to \mathbb{R}^+$$

Let us suppose $S(1)$ takes on only 2 values

so set $\Omega = \{\omega_1, \omega_2\}$

& Let $\mathbb{P}$ is the probability measure on Ω

$p_i = \mathbb{P}(\omega_i)$

Return K^ω for $\omega \in \Omega$ is a function $K : \Omega \to (-1, \infty)$

so that

$$S^{\omega_i}(1) = S(0)\left(1 + K^{\omega_i}\right)$$

Let the interest be constant r,

$A(t) \equiv$ Value of bond @ time t $\Rightarrow$ $A(1) = A(0)(1+r)$.

For simplicity let us set $A(0) = 1$.

Let $G : \Omega \to \mathbb{R}$ be the value of the option at $t=1$.

~ eg $G(\omega) = \left(S^{\omega}_{(1)} - X\right)^{+}$ for a Euro Call.

Let x_0 be # of stocks purchased at $t=0$

— y_0 be # of bonds ——————

We attempt to 'replicate' the value of the option @ $t=1$

w/ some portfolio (x_0, y_0)

$\therefore$ At $t=1$. given portfolio (x_0, y_0)

$$x_0 S^{\omega}(1) + y_0 A(1) = G(\omega) \quad \text{for } \omega \in \Omega.$$

this forms a 2×2 matrix:

$$\cancel{\begin{pmatrix} S(0)(1+K^{\omega_1}) & A(1) \\ S(0)(1+K^{\omega_2}) & A(1) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} G(\omega_1) \\ G(\omega_2) \end{pmatrix}}$$

$$\begin{pmatrix} S^{\omega_1}_{(1)} & A(1) \\ S^{\omega_2}_{(1)} & A(1) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} G(\omega_1) \\ G(\omega_2) \end{pmatrix}$$

Solving for $\binom{x_0}{y_0}$ we have

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} S^{\omega_1}_{(1)} & A(1) \\ S^{\omega_2}_{(1)} & A(1) \end{pmatrix}^{-1} \begin{pmatrix} G(\omega_1) \\ G(\omega_2) \end{pmatrix}$$

$$= \frac{1}{A(1)\left(S^{\omega_1}_{(1)} - S^{\omega_2}_{(1)}\right)} \begin{pmatrix} A(1) & -A(1) \\ -S^{\omega_2}_{(1)} & S^{\omega_1}_{(1)} \end{pmatrix} \begin{pmatrix} G(\omega_1) \\ G(\omega_2) \end{pmatrix}$$

$$x_0 = \frac{G(\omega_1) - G(\omega_2)}{S^{\omega_1}_{(1)} - S^{\omega_2}_{(1)}} \; ; \; y_0 = \frac{G(\omega_2) S^{\omega_1}_{(1)} - G(\omega_1) S^{\omega_2}_{(1)}}{A(1)\left(S^{\omega_1}_{(1)} - S^{\omega_2}_{(1)}\right)}$$

The time zero value of the portfolio (x_0, y_0) is

$$V_0 = x_0 S(0) + y_0 A(0).$$

On the other hand, by design, $V_{(1)}^{\omega} = G(\omega)$ for $\omega \in \Omega$

$\therefore V_0 = G_0 \equiv$ value of the option @ $t=0 \longrightarrow$

$$G_0 = V_0 = x_0 S(0) + y_0 A_0$$

$$= \frac{S(0)\,G(\omega_1) - G(\omega_2)\,S(0)}{S^{\omega_1}_{(1)} - S^{\omega_2}_{(2)}} + \frac{G(\omega_2)\,S^{\omega_1}_{(1)} - G(\omega_1)\,S^{\omega_2}_{(1)}}{(1+r)\ \left(S^{\omega_1}_{(1)} - S^{\omega_2}_{(1)}\right)}$$

$$= \frac{1}{1+r}\left\{ G(\omega_1)\ \frac{S(0)(1+r) - S^{\omega_2}_{(1)}}{S^{\omega_1}_{(1)} - S^{\omega_2}_{(2)}} + G(\omega_2)\ \frac{S^{\omega_1}_{(1)} - S(0)(1+r)}{S^{\omega_1}_{(1)} - S^{\omega_2}_{(2)}} \right\}$$

$\therefore$ for $S^{\omega_2}_{(2)} < S(0)(1+r) < S^{\omega_1}_{(1)}$

we can define new probability measure:

$$\tilde{P}_1 = \frac{S(0)(1+r) - S^{\omega_2}_{(1)}}{S^{\omega_1}_{(1)} - S^{\omega_2}_{(2)}} \ ; \ \tilde{P}_2 = \frac{S^{\omega_1}_{(1)} - S(0)(1+r)}{S^{\omega_1}_{(1)} - S^{\omega_2}_{(2)}}$$

So that $G_0 = \frac{1}{1+r}\,\tilde{\mathbb{E}}(G) = \frac{1}{1+r}\left\{\tilde{P}_1 G^{\omega_1} + \tilde{P}_2 G^{\omega_2}\right\}.$

$\tilde{\mathbb{P}}$ is the Risk-neutral measure.

* Write $\tilde{\mathbb{P}}$ in terms of $\mathbb{P}$:

for $\omega \in \Omega$,

$$\tilde{\mathbb{P}}(\omega) = f(\omega)\,\mathbb{P}(\omega)$$

where $f(\omega_i) = \dfrac{\tilde{p}_i}{p_i}$ for $i = 1, 2$.

$\tilde{\mathbb{P}}$ + $\mathbb{P}$ are <u>equivalent</u> measures ~ that is

for $A \subset \Omega$; $A \in \mathcal{F}$: $\mathbb{P}(A) > 0$ iff $\tilde{\mathbb{P}}(A) > 0$.

f is the Radon-Nikodym Derivative.

Notice, model only applies to the case that

$$\min \frac{S(1)}{S(0)} < \frac{A(1)}{A_0} < \max \frac{S(1)}{S(0)}.$$

If $\dfrac{A(1)}{A(0)} < \min \dfrac{S(1)}{S(0)}$ then an arbitrage opportunity is possible.

Example. Call option.

$$S(0) = 100 \; ; \; X = 110 \; ; \; S^{\omega_1}(1) = 115 \; ; \; r = .1$$

$$S^{\omega_2}(1) = 105$$

$$\tilde{p}_1 = \frac{110\cancel{5} - 105}{115 - 105} = \frac{1}{2} \; ; \; \tilde{p}_2 = \frac{115 - 110}{115 - 105} = \frac{1}{2} .$$

$$G^{\omega_1} = \left(S^{\omega_1}(1) - X\right)^+ = 5$$

$$G^{\omega_2} = \left(S^{\omega_2}(1) - X\right)^+ = 0$$

$$G_0 = \frac{1}{1.10} \tilde{E}(G) = \frac{1}{1.10}\left(\frac{1}{2} \cdot 5 + \frac{1}{2} \cdot 0\right)$$

$$= 25/11 = 2.\overline{27}$$

It is less risky to invest in a Call than in the security itself.

(i) You only invest the ~~#~~ price of the call
- so you only stand to lose that as opposed to the entire ~~#~~ value of stock.

(ii) ~~#~~ variance is less.

Consider (A) ≡ buy stock @ $t=0$ + liquidate @ $t=1$
(B) buy call @ $t=0$ + liquidate @ $t=1$.

(A) Price of stock @ $t=0$ is $100 \Rightarrow$ value in $t=1$ dollars is \$110.

Value @ $t=1$ of portfolio

$$V = S(1) - S(0)(1+r) = S(1) - 110$$

$$\tilde{E}\, V = 0$$

$$\widetilde{Var}(V) = 25$$

$$\tilde{\sigma} = 5$$

(B) Price of call @ $t=0$ is $\frac{2.5}{1.1} \Rightarrow$ value in $t=1$ dollars is \$2.5

Value of portfolio @ $t=1$

$$V = C(1) - (1+r)\, C(0)$$

$$= C(1) - 2.5$$

$$\tilde{E}\, V = 0$$

$$\widetilde{Var}(V) = (2.5)^2$$

$$\tilde{\sigma} = 2.5$$